# Some Almost Exceptional Arrangements

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**Abstract.** We introduce and study a family of real hyperplane arrangements that includes the reflection arrangements of types $A_n$ and $D_n$ as well as an infinite family of arrangements $E_n'$ (constructed by Orlik, Solomon & Terao) that is related to the *exceptional* arrangements of type $E_n$, $n \le 8$.
We investigate freeness (in the sense of Terao) of these arrangements, disproving the conjecture that the arrangements $E_n'$ are free in general.

## 1. Introduction.

Consider the hyperplane arrangements $E_n'$ in $\mathbb{R}^n$ defined (for $n \ge 2$) by

$$Q_n = \prod_{1\le i<j\le n}(x_i - x_j) \cdot \prod_{1\le i<j<k\le n}(x_i + x_j + x_k).$$

These arrangements are very interesting because of their close relation to the exceptional root systems of type $E$. In fact, they were constructed by Orlik, Solomon and Terao [St, Exercise 3.56(d), p. 167] in an attempt to clarify the algebraic structure of the exceptional reflection arrangements $E_n$ ($n \le 8$).

Specifically, Orlik, Solomon and Terao conjectured that $E_n'$ is *free* in the sense of Terao's theory [T1] of free arrangements, which we briefly review below. In this paper we show that this conjecture is **false**.

**Theorem.**
*$E_n'$ is free for $n \le 7$, but not free for $n \ge 9$.*

We will proceed as follows. In Section 2 we will briefly sketch the theory of free arrangements and review the necessary facts. In Section 3 we construct a family $A_{n,k}$ of arrangements in $\mathbb{R}^n$ (for $1 \le k \le n$) which contain $E_n' = A_{n,3}$ and also the reflection arrangements of types $A$ and $D$. Then (in Section 4) we determine free arrangements among the $A_{n,k}$, using their close connections to reflection arrangements. In the next section, we do the non-freeness proofs. We close with some comments on the remaining case of $E_8'$.

## 2. Free Arrangements.

We refer to [O] or [OT] for introductions to free arrangements, their geometry and combinatorics. The necessary facts and background material can be found there, as well as further references. We only list the results that will be needed in this note.

In this paper we only consider linear, real arrangements of hyperplanes. So an *arrangement* is a finite set $X = \{H_1, \ldots, H_m\} \subseteq \mathbb{R}^n$ of hyperplanes through the origin.

We will need some data from the combinatorics of $L$, and its *matroid* (see [W]). The set of intersections of the hyperplanes in $X$, ordered by reverse inclusion, forms a geometric lattice $L = L(X)$, the *intersection lattice* of $X$. Its minimal element is $\hat{0} = \mathbb{R}^n$, its maximal element is $\hat{1} = \cap_{i=1}^m H_i$. The *rank* of $X$ is $r = r(L) = n - \dim(\cap_{i=1}^m H_i)$; the *rank of a flat* $Y \in L$ is similarly given by $r(Y) = n - \dim(Y)$. The arrangement $X$ is *essential* if $\cap_{i=1}^m H_i = \{0\}$, that is, if $r = n$. In particular we need the *characteristic polynomial* $\chi(t) = \sum_{Y \in L} \mu(\hat{0}, Y) t^{r-r(Y)} = t^r - mt^{r-1} \pm \ldots$ of $L$, where $\mu$ denotes the *Möbius function* of $L$, see [W] [St].

Every hyperplane $H_i$ of the arrangement is the kernel of a linear function, $H_i = \ker(l_i)$. Now let $S := \mathbb{R}[x_1, \ldots, x_n]$ be the ring of polynomial functions on $\mathbb{R}^n$. Then we can consider the $l_i$ as elements of $S$, and their product $Q := \prod_{i=1}^m l_i \in S$ is a polynomial of degree $m$ which *defines* $X$. Now let $S^n$ be the set of all polynomial functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. We will consider the set $\mathrm{Der}(X) := \{\mathbf{p} \in S^n : \mathbf{p}(H) \subseteq H_i\}$ of those polynomial functions that map every hyperplane of $X$ into itself. The set $\mathrm{Der}(X)$ of "derivations" has the structure of a module over $S$ of rank $n$.

Following [T1], we call $X$ *free* if $\mathrm{Der}(X)$ is a free $S$-module, that is, if it has a basis. This is the case [Sa] if and only if there are homogeneous elements $\mathbf{p}_1, \ldots, \mathbf{p}_n$ in $\mathrm{Der}(X)$ such that $\det(\mathbf{p}_1, \ldots, \mathbf{p}_n) = Q$. The multiset of degrees $e_i := \deg(\mathbf{p}_i)$ does not depend on the specific basis of $\mathrm{Der}(X)$. They are called the *exponents* of $X$. The exponents of a free arrangement satisfy $e_1 + \ldots + e_n = m$. In $(e_1, \ldots, e_n)$ the exponent 0 appears $n - r$ times. So if $X$ is essential, then the exponents are positive integers. If the arrangement is irreducible (not a direct sum), then the exponent 1 appears exactly once.

Terao's Factorization Theorem [T2] states that if $X$ is a free arrangement, the characteristic polynomial has a special form: it is given by $\chi(t) = \prod_{i=1}^r (t - e_i)$, where the $e_i$ are the positive exponents of $X$.

A *localization* of $X$ (at $\mathbf{x} \in \mathbb{R}^n$) is the subarrangement of all hyperplanes $H_i \in X$ that contain a fixed point $\mathbf{x}$. The localizations of $X$ are in bijection with the flats $Y \in L$, via $Y \leftrightarrow \cap_{H_i \supseteq Y} H_i$. If $X$ is free, then all the localizations of $X$ are free as well [T1].

Terao's Addition-Deletion Theorem [T1] states that if two of the following facts are true for some $i$, then so is the third:

– the arrangement $X$ is free with exponents $[e_1, \ldots, e_n]$,

– the *deletion* $X \backslash H_i$ is free with exponents $[e_1, \ldots, e_i - 1, \ldots, e_n]$,

– the *restriction* $X|_{H_i}$ is free with exponents $[e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n]$.

A *Coxeter arrrangement* is the set of reflecting hyperplanes of a finite reflection group. Coxeter arrrangements are free [Sa]: this follows from Chevalley's theorem [B] with $\mathbf{p}_i := (\frac{\partial f_i}{\partial x_1}, \ldots, \frac{\partial f_i}{\partial x_n})$. The corresponding exponents are $d_i - 1$, where $d_i$ is the degree of the $i$-th fundamental invariant $f_i$ of the group. The classification of Coxeter groups, and lists of their exponents, can be found in [B].

Orlik, Solomon & Terao [OST] show that if $X_G$ is a Coxeter arrrangement with exponents $e_1 \leq \ldots \leq e_n$, then for $H \in X_G$, the deletion $X_G \backslash H$ is also free, with exponents $[e_1, \ldots, e_{n-1}, e_n - 1]$, and thus the restriction $X_G|_H$ is free with exponents $[e_1, \ldots, e_{n-1}]$.

An arrangement is *uniform* if its hyperplanes are in general position (so that the matroid is uniform). Uniform arrangements are not free if $m > r > 2$ [Z1]. We will only need the special case where $m - 1 = n = r$: in this case non-freeness can also be seen from the characteristic polynomial $\chi(t) = (1 - \frac{1}{t})((t-1)^{n-1} + (-1)^n)$.

## 3. More Arrangements.

To understand the arrangements of type $E'$, we have to clarify their relation to Coxeter arrangements. We will interpret the arrangements $E'_n$ as part of a two-parameter family of arrangements $A_{n,k}$ in $\mathbb{R}^n$, which for $1 \leq k < n$ are given by

$$Q_{n,k} := \prod_{1 \leq i < j \leq n} (x_i - x_j) \cdot \prod_{1 \leq i_1 < \ldots < i_k \leq n} (x_{i_1} + \ldots + x_{i_k}).$$

Defining $x(I) := \sum_{i \in I} x_i$, this can also be written as

$$Q_{n,k} = \prod_{1 \leq i < j \leq n} (x_i - x_j) \cdot \prod_{|I| = k} x(I).$$

This formula defines a family of arrangements of cardinality $\binom{n}{2} + \binom{n}{k}$ in $\mathbb{R}^n$. It is interesting, because teh following Proposition will show that

– it nearly contains all the Weyl arrangements with a simply laced diagram, and

– it contains the arrangements $E'_n$,

– it supports an interesting duality.

We start by observing some isomorphisms, and by identifying the Weyl arrangements among the arrangements $A_{n,k}$. The case of $k = 0$ will be ignored in our discussion.

**Proposition.**

(i) $A_{n,1} \cong A_n$ *for* $n \geq 1$.

$A_{n,2} \cong D_n$ *for* $n \geq 3$.

$A_{n,3} \cong E'_n$ *for* $n \geq 3$.

(ii) $A_{n,k} \cong A_{n,n-k}$ *for* $1 \leq k < n$.

(iii) $A_{n,k}$ *is a localization of* $A_{n+1,k}$*, and of* $A_{n+1,k+1}$*,* *for* $1 \leq k \leq n$.

**Proof:**

(i) is clear.

(ii) follows from the coordinate transformation

$$x'_i := x_i - \frac{1}{k}\sum_{j=1}^{n} x_j,$$

whose inverse is given by

$$x_i = x'_i - \frac{1}{n-k}\sum_{j=1}^{n} x'_j.$$

This transformation satisfies

$x'_i - x'_j = x_i - x_j$ for $i < j$ and

$x'(I) = -x([n]\backslash I)$ for $|I| = k$,

and thus transforms $A_{n,k}$ into $A_{n,n-k}$.

(iii) holds because the hyperplanes of $A_{n+1,k}$ that contain $\mathbf{e}_{n+1}$ are exactly those which do not contain the variable $x_{n+1}$; the corresponding localization is therefore isomorphic to $A_{n,k}$.

The second statement follows now from (ii). □

## 4. Freeness Proofs.

In this section, we prove that $E'_n$ is free for $n \leq 7$, and determine the corresponding exponents.

**Theorem.** (Orlik, Solomon and Terao, see [St, p. 192])

*The arrangements $E'_n$ are free for $2 \leq n \leq 7$, with the following sets of exponents:*

$E'_2$ $[0, 1]$

$E'_3$ $[1, 1, 2]$

$E'_4$ $[1, 2, 3, 4]$

$E'_5$ $[1, 3, 4, 5, 7]$

$E'_6$ $[1, 4, 5, 7, 8, 10]$

$E'_7$ $[1, 5, 7, 9, 10, 11, 13]$

**Proof:**

Coxeter arrrangements are free. From this we get that

$E'_3 \cong A_2 \times A_1$

$E'_4 \cong A_{4,3} \cong A_{4,1} \cong A_4$ and

$E'_5 \cong A_{5,3} \cong A_{5,2} \cong D_5$

are free with the exponents as stated.

[[Note that $D_5$ is not supersolvable, so neither is $E'_n$ for $n \geq 5$.]]

For $E'_6$ we observe that $E_6$ is in the non-standard coordinates given by Cartan in his thesis (“in Cartan’s coordinates”) [C] [S] given by

$$Q_{6,3} \cdot (x_1 + \ldots + x_6) = Q_{6,3} \cdot x([6]).$$

In other words, $E_6' \cong E_6 \backslash H$, where $H = \{\mathbf{x} \in \mathbb{R}^6 : x([6]) = 0\}$. Now $E_6$ is a Coxeter arrrangement with exponents $[1,4,5,7,8,11]$. With the Orlik-Solomon-Terao theorem [OST], this yields that $E_6'$ is free with the exponents as stated.
[[At this point, we can also notice that $E_6 \backslash H$ is not simplicial, and thus $E_n'$ is not simplicial for $n \geq 6$.]]
For $E_7'$, we use that $E_7$ is "in Cartan's coordinates" given by

$$Q_{7,3} \cdot \prod_{i=1}^{7} (x_1 + \ldots + x_7 - x_i) = Q_{7,3} \cdot \prod_{|I|=6} x(I),$$

(that is, $E_7 \cong E_7' \cup A_{7,6}$). This Coxeter arrangement is free with exponents [1,5,7,9, 11,13,17]. By the Orlik-Solomon-Terao theorem, we get that $E_7|_{H_i}$ (with $H_i = \{\mathbf{x} \in \mathbb{R}^7 : x_1 + \ldots + x_7 - x_i = 0\}$) is free with exponents $[1,5,7,9,11,13]$. Now $H_i \cap H_j \subseteq \{\mathbf{x} : x_i = x_j\}$, and thus $E_7 \backslash \{H_1, \ldots, H_{i-1}\}|_{H_i} \cong E_7|_{H_i}$. This allows for a repeated application of Terao's Addition-Deletion Theorem, to get that $E_7'$ is free with exponents $[1,5,7,9,11,13,10]$, as required. □

Note that the last argument is analogous to the proofs of freeness given in [Z2]. In fact, it constructs a *resolution* of $E_7'$ in the sense of [Z2]: however, in this case this is a *Coxeter-resolution*, analogous to the supersolvable resolutions of [Z2].

## 5. Non-freeness proofs.

In this section, we show the following result.

**Theorem.**
*The arrangements $A_{9,3} = E_9'$ and $A_{8,4}$ are not free.*

**First Proof.** We start with $E_9'$. In fact, the subarrangement of $E_9'$ defined by

$$Q^0 := (x_1+x_2+x_3)(x_4+x_5+x_6)(x_7+x_8+x_9)(x_1+x_4+x_7)(x_2+x_5+x_8)(x_3+x_6+x_9)$$

is the localization of $E_9'$ at a point like $\mathbf{x} = (0,1,-1,2,4,-6,-2,-5,7)$. However, $Q^0$ defines a generic subarrangement of six hyperplanes of rank 5, which cannot be free. Hence $E_9'$ is not free, and neither is $E_n'$ for $n \geq 9$.

In matroid theory terms, one studies the matroid $M(E_n')$ given by the set

$$S := \{\mathbf{e}_i - \mathbf{e}_j : 1 \leq i < j \leq n\} \cup \{\mathbf{e}_j + \mathbf{e}_j + \mathbf{e}_k : 1 \leq i < j < k \leq n\}$$

of vectors in $\mathbb{R}^n$. One finds that the subset

$$S^0 := \{\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_4+\mathbf{e}_5+\mathbf{e}_6, \mathbf{e}_7+\mathbf{e}_8+\mathbf{e}_9, \mathbf{e}_1+\mathbf{e}_4+\mathbf{e}_7, \mathbf{e}_2+\mathbf{e}_5+\mathbf{e}_8, \mathbf{e}_3+\mathbf{e}_6+\mathbf{e}_9\}$$

forms a flat of rank 5 of $M(E_9')$ that is a 6-circuit. In other words, $S^0$ is a flat of $M(E_9')$ that is isomorphic as a matroid to the uniform matroid $U_{5,6}$. The characteristic polynomial of the interval $[\hat{0}, E']$ of $L(E'_9)$ is $\chi(t) = t^5 - 6t^4 + 15t^3 - 20t^2 + 15t - 5$, which does not factor completely in $\mathbb{Z}[t]$.

To see that $A_{8,4}$ is not free, consider the localization of $A_{8,4}$ at the point $\mathbf{x} := (0, 1, 2, -3, 4, -5, -8, 9)$. There are only four hyperplanes in the arrangement that contain this point. They are described by

$$Q^0_{8,4} = (x_1 + x_2 + x_3 + x_4)\cdot(x_3 + x_4 + x_7 + x_8)\cdot(x_1 + x_2 + x_5 + x_6)\cdot(x_5 + x_6 + x_7 + x_8).$$

They determine a uniform subarrangement of rank 3, whose matroid is $U_{3,4}$. Thus this localization of $A_{8,4}$ is not free, and hence $A_{8,4}$ is not free. $\square$

A different method to show that an arrangement is not free is to show that the characteristic polynomial doesn't factor. For this, it may be sufficient to compute only part of the characteristic polynomial. For example, the formula one gets for $E'_n$ is

$$\begin{aligned}\chi_n(t) = t^n\\ &- \{\binom{n}{2} + \binom{n}{3}\}t^{n-1}\\ &+ \{5\binom{n}{3} + 15\binom{n}{4} + 25\binom{n}{5} + 10\binom{n}{6}\}t^{n-2}\\ &- \{2\binom{n}{3} + 42\binom{n}{4} + 290\binom{n}{5} + 995\binom{n}{6} + 1575\binom{n}{7} + 1120\binom{n}{8} + 280\binom{n}{9}\}t^{n-3}\\ &\pm \ldots\end{aligned}$$

by enumeration of the flats of $L(E'_n)$ of rank at most 3, and computation of their Möbius functions. This can be done by hand, if one considers the orbits of the $\mathcal{S}_n$-action. In fact, it would be nice to get the next coefficient, but that seems to be beyond reach.

**Second Proof.** For $E'_9$, one gets from the above formula that

$$\chi_9(t) = t^9 - 120t^8 + 6300t^7 - 189280t^6 \pm \ldots,$$

which shows (with an enumeration by computer) that $\chi_9(t)$ does not factor completely in $\mathbb{Z}[t]$ – this is another proof that $E'_9$ is not free.

For $A_{8,4}$ the characteristic polynomial begins with

$$\chi_{8,4}(t) = t^8 - 98t_7 + 4137t^6 \mp \ldots,$$

and this implies (again by computer enumeration) that the zeroes of $\chi_{8,4}(t) \in \mathbb{Z}[t]$ are not all positive integers: so $A_{8,4}$ is not free. $\square$

## 6. Comments.

Combining our results, we can conclude whether $A_{n,k}$ is free or not for all values of $k$ and $n$, except for $E'_8 = A_{8,3} \cong A_{8,5}$.

**Corollary.**

*$A_{n,k}$ is free if $\min\{k, n-k\} \leq 2$ and if $n \leq 7$.*
*$A_{n,k}$ is not free if $\min\{k, n-k\} \geq 4$ and if $\min\{k, n-k\} = 3$, $n \geq 9$.*

What about the open case? $E'_8$ has characteristic polynomial

$$\chi_8(t) = (t-1)\cdot(t-7)\cdot(t-10)\cdot(t-11)\cdot(t-12)\cdot(t-13)\cdot(t-14)\cdot(t-16)$$

(enumeration by computer). This suggests that $E'_8$ is free, although we have no proof for this. In principle, this is a question that can be decided by a computer calculation, using the Gröbner basis methods. However, it seems to be just beyond the reach of a fast workstation (with extra memory) to even to compute the Hilbert series of $\mathrm{Der}(E'_8)$, using the method of Billera & Rose [BR, Section 5]. However, a Gröbner basis computation shows a generator in $\mathrm{Der}(E'_8)$ of degree 7 (at least computing in finite charateristic), but no generator of degree 6, killing the possible conjecture that in general $\mathrm{Der}(E'_n)$ contains a has a homogeneous generator of degree $n-2$.

Also it is not clear (and similarly hard to check) whether every restriction of $E'_8$ to one of its hyperplanes is free. There are two isomorphism classes of such. In the first case, a Hilbert series computation indicates that $E'_8|_{x_i=x_j}$ is free with exponents $[1, 7, 10, 11, 11, 13]$, the second case was again beyond reach. This might be of interest in connection with "Orlik's Conjecture" [O, p. 86] that restrictions of free arrangements to their hyperplanes are always free, which was recently disproved (using the same Gröbner basis methods on different examples) by Edelman & Reiner [ER].

We note that $E'_8$ is a localization of $E'_9$, while "in Cartan's coordinates" [C] [S] $E_8$ arises as the restriction of $E'_9$ to $\{\mathbf{x} \in \mathbb{R}^9 : \sum x_i = 0\} \notin E'_9$. However, there is no direct connection between $E_8$ and $E'_8$ that would lead to a conclusion about the freeness of $E'_8$.

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