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Some minimal non-orientable matroids of rank three

by

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Abstract.

We construct an infinite family of minor-minimal rank three matroids that are not orientable.

Contents.

1. Introduction.

"Orientability of matroids" was the title of the original paper [2] by Bland & Las Vergnas that laid the foundations of oriented matroid theory. In that paper, many examples of orientable and some non-orientable matroids were given. It was proved that orientability is preserved under taking minors, and a sequence of rank r matroids on $2r$ points ($r \geq 4$) was provided to show that the family of minor-minimal non-orientable matroids is infinite. Binary oriented matroids were identified as the regular ones.

The subsequent development of oriented matroid theory (see [1]) has not made much progress in the discussion of orientability. Even in the fundamental case of rank 3 no effective criteria are known to prove the non-orientability of a matroid. It seems safe to conjecture that the orientability problem for rank 3 matroids is NP-complete, and that no short certificate for non-orientability exists (that is, orientability is not in co-NP). We refer the reader to [3] and [6] for two recent algebraic approaches to the orientability problem, where Guedes de Oliveira's [3] concept of "final polynomials" describes certain certificates for non-orientability.

In the case of rank 3, some sporadic necessary conditions for orientability are known [1, Section 6.6]. In particular, orientable matroids satisfy the Sylvester-Gallai theorem: they have at least three 2-point lines. This implies that finite projective or affine planes are non-orientable. However, they are not minimal with this property, except for the projective plane of order 2 ("Fano plane") $F_7 = \mathtt{PG}(3, 2)$. The only other minimal non-orientable matroid that was known up to now is the MacLane matroid ML_8, which is obtained if one point is deleted from the affine plane of order 3: $ML_8 = \mathtt{AG}(3, 3)\backslash p$. However, even for this matroid, the non-orientability proof was done by computer (see [2]) and is not published.

In this paper, we will demonstrate how simple non-orientability proofs for certain rank 3 matroids follow from the topological representation theorem of Folkman & Lawrence [4], which identifies oriented matroids of rank 3 with pseudoline arrangements.

In the following, we construct an infinite family of rank 3 matroids H_n which are non-orientable, and whose minimal non-orientable submatroids G_n are easily identified. The "key to success" is that the matroids H_n are very symmetric – this makes the analysis simple, and even allows to show representability over $\mathbb{C}$. (Compare this to Ingleton's [7] difficulties with his examples for representability.) This implies that we have found an infinite family of minimal non-orientable matroids in the finite projective planes $\mathtt{PG}(3, p)$ over prime fields.

As an undeserved extra, we get that $G_3 = ML_8$ – so we finally have a simple proof for the non-orientability of the MacLane matroid.

2. Construction.

For $n \geq 3$, let H_n be the rank 3 matroid on the point set

$$\{1, 2, \ldots, n, 1', 2', \ldots, n', 1'', 2'', \ldots, n'', p, q\}$$

of size $3n + 2$, whose lines are given (with numbering modulo n) as

$$\{1, 2, \ldots, n, q\},\ \{1', 2', \ldots, n', q\} \text{ and } \{1'', 2'', \ldots, n'', q\} \qquad (3 \text{ “horizontal lines”})$$

$$\{1', 1, 1'', p\},\ \{2', 2, 2'', p\}, \ldots, \{n', n, n'', p\} \qquad (n \text{ “vertical lines”})$$

$$\{1'', 2, 3'\},\ \{2'', 3, 4'\}, \ldots, \{n'', 1, 2'\} \qquad (n \text{ “diagonal lines”})$$

together with the pairs not contained in any of these as 2-point lines. As a typical example, H_6 is depicted in Figure 1. We will, in the following, talk about *horizontal, vertical* and *diagonal* lines of H_n, having that figure in mind.

Furthermore, we will consider the submatroids G_n obtained from H_n by deleting one of the diagonal 3-point lines. By symmetry, we may assume that G_n is the matroid $G_n = H_n \backslash \{(n-1)'', n, 1'\}$ on $3n - 1$ points ($n \geq 3$).

Figure 1: The matroid H_6.

(The vertices of G_n are marked black.)

In the following, we will show that G_n is not orientable, but every submatroid of G_n is. For this, we use the topological representation theorem: orientability is equivalent to the existence of a pseudoline arrangement whose pseudolines correspond to the points, and whose vertices correspond to the lines of the matroid. (We refer to [5] or [1, Chapter 6] for introductions to pseudoline arrangements, and to [4] or [1, Chapter 5] for topological representation.)

The crucial observation now is that we know the pseudoline representations of the submatroid on $\{1'2',\dots,n',1'',2'',\dots,n'',q\}$ of H_n: they are all representable, and isomorphic to the line arrangements given by

i': $x=\sigma(i)$ for $i=1,2,\dots,n$,

i'': $y=\tau(i)$ for $i=1,2,\dots,n$,

q: $z=0$ ("line at infinity")

where σ and τ are arbitrary permutations of $\{1,2,\dots,n\}$.

The *square grid* so obtained now has to be extended to give a pseudoline representation of H_n or the submatroids of H_n we want to consider.

For example, for $\sigma=123\dots n$ and $\tau=n12\dots n{-}1$ we obtain a pseudoline representation of $H_n\backslash p$: see Figure 2, where this is illustrated for the case $n=6$.

Similarly, for $\sigma=\tau=123\dots n=\mathrm{id}$ we get a pseudoline representation for $H_n\backslash q$: see Figure 3 for a typical example. (The situation looks slightly differently if n is odd.) Note that in both cases we cannot extend to a pseudoline representation of H_n.

With this, we have seen that $H_n\backslash p$ and $H_n\backslash q$ are orientable for all n.

Now call a point in a rank 3 matroid M *reducible* if it is contained in at most 2 "long lines" (lines of M with at least 3 points). Applying the Levi's enlargement lemma (see [5, Theorem 3.4]) we find that if p is a reducible point of M, then M is orientable if and only if $M\backslash p$ is orientable.

In particular, if from H_n any one point of the "diagonal line" $\{(i{-}1)'',i,(i{+}1)'\}$ is deleted, then the other two become reducible. So, for example, the following are equivalent:

- $H_n\backslash\{(i{-}1)'',i,(i{+}1)'\}$ is orientable,
- $H_n\backslash\{i\}$ is orientable,
- $H_n\backslash\{(i{-}1)''\}$ is orientable,
- $H_n\backslash\{(i{+}1)'\}$ is orientable,

(for $1\le i\le n$, numbering modulo n), and similarly for G_n (where $i\ne n$). Also, we know that $H_n\backslash\{(i{-}1)'',i,(i{+}1)'\}$ is orientable if and only if G_n is orientable, because they are isomorphic.

With this, it only remains to show that

- $H_n\backslash\{i\}$ is not orientable ($1\le i\le n$),
- $H_n\backslash\{i,n\}$ is orientable, for $i\ne n$.

To see that $H_n\backslash\{i,n\}$ is orientable, we again define a suitable "grid" and extend that to a pseudoline arrangement that represents $H_n\backslash\{i,n\}$: this time, for example the choice

Figure 2: Pseudoline representation of $H_6 \backslash p$.

(The little circles correspond to vertical lines, the little stars to diagonal lines of $H_6 \backslash p$. The element q is represented by the line at infinity.)

Figure 3: Pseudoline representation of $H_6 \backslash q$.

$\sigma = \tau = i(i{-}1)\ldots 21(i{+}1)\ldots(n{-}1)n$ works. We present a small example in Figure 4, and leave further details to the reader.

Figure 4: Pseudoline representation of $H_6 \backslash \{3, 6\}$.

(q is represented by the line at infinity.)

For the case of $H_n \backslash n$, assume that a pseudoline representation exists. After an isomorphism of the arrangement, its restriction to $\{1'2', \ldots, n', 1'', 2'', \ldots, n'', q\}$ is given by a square grid $i' : x = \sigma(i)$, $i'' : y = \tau(i)$ $(1 \leq i \leq n)$, and q is represented by the line at infinity.

The pseudoline representing p meets a vertex on every line of the grid, so it has to be (a deformation of) the diagonal (which is given by $x + y = n + 1$) or the anti-diagonal ($x = y$). After an isomorphism, we get that p is the diagonal. The combinatorics (p, i' and i'' lie on one line of H_n, for all i) then implies that $\sigma(i) = \tau(i)$, for all i, so $\sigma = \tau$.

Now we try to position the pseudolines $1, 2, \ldots, n{-}1$. These have to be *parallel*, that is, meet at a common vertex at infinity. Furthermore, they all meet one "diagonal vertex" on p, and another "off-diagonal" one. We will now talk about the *slope* of these parallels – depending on where they meet the line at infinity.

If they have "positive slope" (like the pseudolines in Figure 4), then the "corner vertices" $(1, n)$ and $(n, 1)$ of the grid cannot be used, because a pseudoline through one of them does not meet another vertex of the grid.

Thus the pseudolines $1, 2, \ldots, n{-}1$ must have "negative slope" (like p), and by symmetry we may assume that they are "steeper than p". But then they cannot meet any off-diagonal vertex on the first or the last vertical line of the grid. But the $n{-}1$ "off-diagonal" vertices lie on different vertical lines – a contradiction.

3. Realization.

Finally, we want to see that the matroids H_n ($n \geq 3$) are representable over $\mathbb{C}$. (Being non-orientable, they cannot have an $\mathbb{R}$-representation.) For this, we show that the following coordinates correctly realize H_n:

$$\begin{array}{rll} i: & (\alpha^i, \;\; 0, 1) & \text{for } 1 \leq i \leq n, \\ i': & (\alpha^i, \;\; \alpha, 1) & \text{for } 1 \leq i \leq n, \\ i'': & (\alpha^i, -1, 1) & \text{for } 1 \leq i \leq n, \\ p: & (\;0, \;\; 1, 0), & \\ q: & (\;1, \;\; 0, 0), & \end{array}$$

where α is a primitive n-th root of unity.

In fact, with this choice of p and q, we find that the points i, i', i'' $(1 \leq i \leq n)$ are represented on a "$(3 \times n)$-grid" in affine space ("$z = 1$"), where the points on the horizontal lines through q get constant y-coordinates ($y = \alpha, 0, -1$), and the points on the vertical lines get constant x-coordinates ($x = \alpha^i$; $i = 1, \ldots, n$).

So the problem is to check under which conditions i, j' and k'' are collinear in the above representation. This amounts to

$$\begin{vmatrix} \alpha^i & 0 & 1 \\ \alpha^j & \alpha & 1 \\ \alpha^k & -1 & 1 \end{vmatrix} = \alpha^i(\alpha + 1) - \alpha^j - \alpha^{k+1} = 0,$$

or

$$\alpha + 1 = \alpha^{j-i} + \alpha^{k+1-i}.$$

Now considering the geometry of these roots of unity, it is easy to see that either $\alpha = \alpha^{j-i}$ and $1 = \alpha^{k+1-i}$, or $1 = \alpha^{j-i}$ and $\alpha = \alpha^{k+1-i}$. That is,

either $j - i \equiv 0$, $k + 1 - i \equiv 1 \bmod n$, so $i = j = k$,

or $j - i \equiv 1$, $k + 1 - i \equiv 0 \bmod n$, so $i \equiv j - 1 \equiv k - 2 \bmod n$,

as required.

Being representable over $\mathbb{C}$, the matroids H_n (and thus their submatroids G_n) are also representable over $\mathtt{GF}(p)$ for infinitely many primes p [8, Theorem 6]. In particular, the (non-orientable) finite projective planes $\mathtt{PG}(3, p)$ contain infinitely many different *minimal* non-orientable submatroids.

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