Supersolvable and Modularly Complemented Matroid Extensions

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Revised version, 13. November 1990

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Abstract

Finite point configurations in projective spaces are combinatorially described by matroids, where full (finite) projective spaces correspond to connected modular matroids.

Every representable matroid can in fact be extended to a *modular* one. However, we show that some matroids do not even have a *modularly complemented* extension.

Enlarging the class of "ambient spaces" under consideration, we show that every matroid has a finite (but huge) *supersolvable* extension.

In rank three, we prove that every matroid can be extended to a modularly complemented one — it is conjectured that one can even construct an extension that is modular (a finite projective plane).

Contents

1 Introduction

Matroids are a combinatorial model for finite point configurations in vector spaces and their linear or affine dependence. (Throughout this paper, all matroids are assumed to be finite and simple. See [1], [7], [16] or [17] for background and basics in matroid theory.)

To test the quality of this model one is led to study *representable matroids* which actually correspond to such point configurations. The corresponding representation problem has to a large extent been treated as an *algebraic problem*, namely the "coordinatization problem". Motivated by the origin of matroids as a combinatorial abstraction of linear independence for vector configurations, it asks for the conditions under which a matroid can be represented over a given field.

For matroids as *geometric configurations* the representation problem asks for the conditions under which it can be embedded into a projective space. Projective spaces, however, are just matroids with particularly nice structure: *connected modular matroids.* Thus the "embedding problem" studies the extension of arbitrary matroids to the highly structured modular matroids (and their relatives).

We will try to study this problem in a very geometric language. So, matroids will mainly be treated in terms of their flats, corresponding to subspaces of the ambient space. The essential ingredient of projective geometry is here translated and distilled into the fundamental concept of *modularity*: two flats of a matroid form a *modular pair* if they satisfy the modular law, that is, if their intersection has the full dimension suggested by linear algebra or projective geometry intuition.

Definition 1.1 *Two flats $x, y \in L(M)$ form a* modular pair *if*

$$r(x \vee y) + r(x \wedge y) = r(x) + r(y) \ ,$$

where $L(M)$ denotes the lattice of flats of M. A matroid M is modular *if any two flats in $L(M)$ form a modular pair.*

Let us recall some fundamental facts about modular flats that will be needed later (cf. [13] and [4]):

- $\emptyset$, S and the points (i.e., the flats of rank 0, r and 1 in $L(M)$) of a matroid M on the ground set S are always modular flats — called the *trivial flats* of M. All other modular flats in $L(M)$ will be called *non-trivial*.
- The intersection (meet) of any two modular flats is again modular.
- In a connected matroid every modular flat is connected.
- A flat $x \in L(M)$ is modular if and only if every complement y of x has complementary rank, i.e., if $x \wedge y = \emptyset$ and $x \vee y = S$ imply $r(y) = r(M) - r(x)$. (This is often the most effective criterion to check whether a given flat is modular.)
- If x is a modular flat of M, then the deletion of points in $S \setminus x$ or the extension of M by points on x keeps x modular.

It is an interesting and important open question (cf. [16, p.198] or [1, p.329]) whether every finite matroid of rank 3 can be embedded into a finite projective plane or, equivalently, into a modular geometric lattice.

In higher ranks, however, there are simple matroids that cannot be embedded into a projective geometry/modular lattice. The notorious Vamos cube V_8, depicted in Figure 1, is the simplest example for this: V_8 is a matroid of rank 4 on the set $\{a, b, c, d, e, f, g, h\}$. The circuits of V_8 are the five 4-element subsets $abef$, $bcfg$, $cdgh$, $adeh$, $aceg$ together with all 5-element subsets that do not contain any of these 4-element circuits. It is easy to check (with Theorem 2.2 below) that the line bf cannot intersect the plane $aceg$ in any extension of V_8. This implies that V_8 has no modular embedding.

This fact suggests extending the class of ambient spaces under consideration. If we are not able to embed an arbitrary matroid into a modular matroid, then, what is the "smallest" class of matroids into which every finite matroid can be embedded?

Two famous classes of matroids suggest themselves: *modularly complemented* and *supersolvable* matroids.

Let us first consider modularly complemented matroids. These matroids were introduced by Stonesifer [15].

Definition 1.2 *A matroid $M = M(S)$ on a ground set S is called* modularly complemented *if for every flat x of M there is a* modular complement*, i.e., a modular flat $y \in L(M)$ with $x \vee y = S$ and $x \wedge y = \emptyset$.*

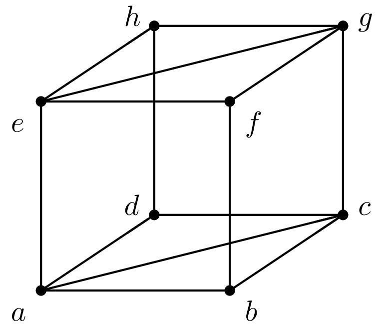

Figure 1: The Vamos Cube V_8

Obviously, every modular matroid is modularly complemented — but in general, a modularly complemented matroid M contains only a few modular flats. More specifically, M has to contain only a "frame" of modular flats: according to Definition 1.2, for every point p of M, there is a modular complement, i.e., a modular hyperplane not containing p. Inductively, we can find $r := r(M)$ modular hyperplanes $H_1, \ldots, H_r$ such that $H_1 \cap \ldots \cap H_r = \emptyset$. Since the intersection of any $r-1$ of these hyperplanes is a point, we can define

$$p_i := H_1 \cap \ldots \cap H_{i-1} \cap H_{i+1} \cap \ldots \cap H_r$$

for $i = 1, \ldots, r$. Of course, $\{p_1, \ldots, p_r\}$ is a basis of M. Since the intersection of modular flats is always modular, each proper subset of $\{p_1, \ldots, p_r\}$ determines a modular flat in M — and these flats constitute the above-mentioned "frame of modular flats".

On the other hand, a matroid with the property that every point has a modular complement (or equivalently, a matroid containing such a frame of modular flats) is modularly complemented — for a given flat x choose a basis of x and augment it to a basis of M from $\{p_1, \ldots, p_r\}$.

So, at first glance, the class of modularly complemented matroids seems to be far less restrictive than the class of modular matroids. However, Kahn and Kung [10] have shown that this is not really true for matroids of rank at least four: their main result states that a connected modularly complemented matroid of rank ≥ 4 is either a Dowling lattice (Definitions may be found in [8]), or a certain restriction of a modular matroid. This classification allows us to show that the class of modularly complemented matroids is not rich enough to embed every finite matroid (Proposition 4.2).

Therefore, we weaken our requirements once again and arrive at the class of supersolvable matroids:

Definition 1.3 *A matroid M is* supersolvable *if its lattice of flats contains a maximal chain of modular elements.*

Equivalently, M is supersolvable if and only if $r(M) \leq 2$, or $r(M) \geq 3$ and M contains a modular hyperplane H such that the restriction $M|H$ is supersolvable.

The class of these matroids, introduced by Stanley [13, 14], contains the class of modularly complemented matroids: in fact, $x_i := \overline{\{p_1, \ldots, p_i\}}$ defines a maximal chain of modular flats. And although this new class really is less restrictive, supersolvable matroids are still highly structured and have many special properties:

- every non-empty supersolvable matroid has points whose deletion preserves supersolvability (this allows proofs by induction on the ground set, which are impossible in the modular case),
- restricting a supersolvable matroid to a flat again yields a supersolvable matroid — thus we can restrict ourselves to extensions of the same rank (The analogous statement holds for the modularly complemented case, too.),
- closure and line-closure coincide, and thus supersolvable matroids are determined by their truncation to rank 3 (Halsey [9]),
- the characteristic polynomial factorizes over $\mathbb{Z}$ as $\chi(t) = (t - e_1) \cdots (t - e_r)$, where the roots e_i are easy to compute (Stanley [13]),
- broken circuit complexes factorize completely for supersolvable matroids, and this property characterizes those (Björner & Ziegler [3]).

Summing up, in this paper we consider the following hierarchy of "ambient spaces":

$$\begin{array}{c}\text{MODULAR MATROIDS}\\ \cap\\ \text{MODULARLY COMPLEMENTED MATROIDS}\\ \cap\\ \text{SUPERSOLVABLE MATROIDS}\end{array}$$

First, we will study matroids of rank 3. Since — as already mentioned — it is still an open question whether every rank 3 matroid can be embedded into a finite modular matroid of the same rank, we start with the class of supersolvable matroids as ambient spaces: in Section 2 we prove that every rank 3 matroid can be extended to a supersolvable matroid of the same rank (Proposition 2.3). Since this proof turns out to be really simple, we try more: design-theoretic constructions will show in Section 3 that even the class of modularly complemented matroids is rich enough to allow an embedding of arbitrary rank 3 matroids into it (Theorem 3.1).

Then we will turn our attention to matroids of rank at least four. In Section 4 we show that this time the class of modularly complemented matroids is too small to allow embeddings of arbitrary matroids — Proposition 4.2 presents a rank 4 matroid on 8 points that cannot be extended to a modularly complemented matroid of the same rank. After an extensive analysis of Dilworth truncations in Section 5, we will prove in Theorem 6.5 that every matroid of rank at least four can be embedded into a supersolvable matroid.

If one "raison d'être" for matroids is to provide a geometric model for vector configurations, then the quest for the optimal class of "enveloping spaces" is a natural test for the quality of this model. As described above, the present paper gives answers in this direction.

2 Supersolvable Extensions of Rank 3 Matroids

Let M be a matroid of rank 3. Then M is supersolvable if and only if M contains a modular line, i.e., a line intersecting every other line of M (cf. Definition 1.3).

Intuitively, we should get a supersolvable extension of M by simply intersecting an arbitrary line l of M with every line that has no point in common with l, as sketched in Figure 2.

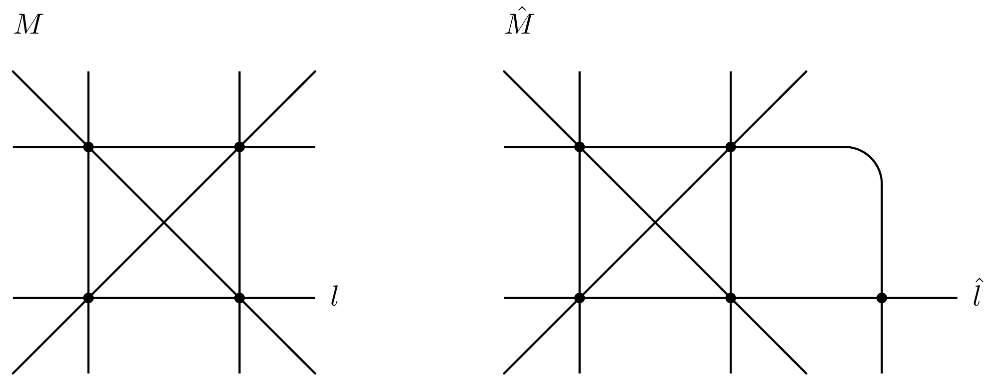

Figure 2: Constructing the Supersolvable Extension

This can be done by successive one-element extensions — one for each line not intersecting l — and this is exactly the way we will construct a supersolvable extension of an arbitrary rank 3 matroid.

For this and for further reference we recall the following characterization of one-element extensions due to Crapo [6] (cf. also [1]):

Definition 2.1 *A modular filter $\mathcal{M}$ of M is a family of flats of M such that*

- *$\mathcal{M}$ is a filter in $L(M)$, i.e., if $x \in \mathcal{M}$ and $y \in L(M)$ with $y \geq x$, then $y \in \mathcal{M}$.*
- *If $x, y \in \mathcal{M}$ form a modular pair, then $x \wedge y \in \mathcal{M}$.*

From now on, we will simply write p to denote the one-element set $\{p\}$ and use the symbol "$\uplus$" to denote a disjoint union of sets.

Theorem 2.2 (Crapo [6]) *Let $\hat{M}$ be a matroid on the set $\hat{S} = S \uplus p$ and let M denote the restriction $\hat{M}|S$.*

Then the set $\mathcal{M} = \{x \in L(M) : p \in \bar{x}\}$ — where $\bar{x}$ denotes the closure of x in $\hat{M}$ — is a modular filter of M.

Conversely, let M be a matroid on the set S and $\mathcal{M}$ be a modular filter of M. Then there is a unique one-element extension $\hat{M}$ of M, i.e., a matroid $\hat{M}$ on $\hat{S} = S \uplus p$ with $\hat{M}|S = M$, such that $\mathcal{M} = \{x \in L(M) : p \in \bar{x}\}$. Furthermore, the flats of $\hat{M}$ are the following subsets of $\hat{S}$:

- *All flats of $L(M) \setminus \mathcal{M}$.*
- *All sets $x \uplus p$ where x is an element of $\mathcal{M}$.*
- *All sets $x \uplus p$ where $x \in L(M) \setminus \mathcal{M}$ and x is not covered in $L(M)$ by an element of $\mathcal{M}$.*

Note that $\mathcal{M} = L(M)$ corresponds to the extension of M by a loop, that is $r(p) = 0$. Similarly $\mathcal{M} = \emptyset$ corresponds to the extension by a coloop, that is $r(\tilde{M}) = r(M)+1$. In all other cases M is extended by a proper point to a matroid of the same rank: $r(\tilde{M}) = r(M)$, and $r(p) = 1$.

This theorem enables us to extend an arbitrary matroid of rank 3 to a supersolvable rank 3 matroid, as described above.

Proposition 2.3 *Let M be a matroid of rank 3 on S and l be an arbitrary line of M. Then there is a supersolvable matroid $\hat{M}$ of the same rank on the set $\hat{S} \supseteq S$ with a modular line $\hat{l} \supseteq l$ and $\hat{M}|S = M$.*

Proof: Let x be a line of M not intersecting l. Then x, l and S form a modular filter in $L(M)$, thus determining a one-element extension M' of M on the set $S \uplus p$ according to Theorem 2.2. In M' the line $l' = l \uplus p$ intersects the line $x' = x \uplus p$ and all lines of the form $y \uplus p$ (where y is a point of $L(M)$ not covered by x or l). Inductively the claim of Proposition 2.3 follows. $\square$

Constructing this extension has been an almost trivial task. So it is only natural to ask, whether every rank 3 matroid can be embedded into a modularly complemented rank 3 matroid.

This requires one to construct three modular lines not intersecting in the same point. The next proposition — following the simple Lemma 2.4 — will show how we can construct at least two modular lines.

Lemma 2.4 *Let M be a rank 3 matroid on the set S containing a modular line m, and let l be an arbitrary line of M. Then, for all points $p \in S \setminus (m \cup l)$, the number of lines not intersecting l and passing through p equals $|m| - |l|$.*

Proof: Let $p \in S \setminus (m \cup l)$. Since every line passing through p has to intersect m, there are exactly $|m|$ lines through p. Now, $|l|$ of these lines intersect l, too. Thus, $|m| - |l|$ lines do not intersect l. □

Now we are able to extend two arbitrary lines of a given rank 3 matroid to modular lines:

Proposition 2.5 *Let M be a rank 3 matroid and l_1, l_2 be arbitrary lines of M. Then there is a rank 3 extension $\tilde{M}$ of M such that l_1 and l_2 determine modular lines in $\tilde{M}$.*

Proof: Due to Proposition 2.3, we may assume l_1 to be modular. Let S denote the ground set of M and suppose $k := |S \setminus (l_1 \cup l_2)| \geq 1$. (For $k = 0$ it follows immediately that l_2 is modular, too.) Now let $S \setminus (l_1 \cup l_2) = \{p_1, \ldots, p_k\}$ and let $L_1, \ldots, L_n$ be the lines of M not intersecting l_2.

We construct $\tilde{M}$ in two steps:

STEP 1: Extend M by nk "new" points placed on l_1 "in generic position" — corresponding to nk one-element-extensions with respect to the modular filter $\mathcal{M} = \{S, l_1\}$ (where by the usual abuse of notation l_1 also denotes the line it determines in extensions of M). This yields a new matroid M' with the following properties:

- The ground set of M' is $S' = S \,\dot{\cup}\, \{q_\nu^i : \nu = 1, \ldots, k \text{ and } i = 1, \ldots, n\}$.
- M' has nk^2 "new" lines that do not intersect l_2, i.e., all lines of the form $p_\mu q_\nu^i$, where $\mu, \nu = 1, \ldots, k$ and $i = 1, \ldots, n$. All other new lines — of the form pq_ν^i, $p \in (l_2 \setminus l_1)$ — intersect both l_1 and l_2. Thus l_1 is modular in M', too.
- The set of all lines not intersecting l_2 is now given by $\mathcal{N} := \{L_1, \ldots, L_n\} \,\dot{\cup}\, \{p_\mu q_\nu^i : \mu, \nu = 1, \ldots, k\ ,\ i = 1, \ldots, n\}$.

STEP 2: Now we partition $\mathcal{N}$ into subsets such that each subset $N \subseteq \mathcal{N}$ satisfies the following two conditions:

- The lines in N are pairwise disjoint.
- Every point p_μ, $\mu = 1, \dots, k$, lies on exactly one line of N.

These subsets are listed below:

- For every $i = 1, \dots, n$ we get the set $\{L_i\} \cup \{p_\mu q_\mu^i : p_\mu \notin L_i\}$.
- The second class of subsets of $\mathcal{N}$ is obtained by partitioning the set $\{p_\mu q_\nu^i : \mu \neq \nu\}$ for every $i = 1, \dots, n$. This can always be done using that the complete bipartite graph $K_{k,k}$ can be decomposed into k perfect matchings.
- Now, the lines of $\mathcal{N}$ not yet distributed are exactly the lines of the form $p_\mu q_\mu^i$, where $p_\mu \in L_i$ for $i = 1, \dots, n$. Any two of these lines are either disjoint or intersect in a point p_μ. In view of Lemma 2.4 every p_i lies on the same number of such lines, and thus the set of these lines can be partitioned in the desired way, too.

Finally, for each subset $N \subseteq \mathcal{N}$ of one of these three types we perform a one-element-extension corresponding to the modular filter $N \cup \{l_2, S'\}$, where we abuse notation as above. In the extension constructed with the aid of N, the lines of N intersect l_2 and the resulting new lines intersect both l_1 and l_2. Thus we arrive at a new matroid $\tilde{M}$ in which l_1 and l_2 determine modular lines. □

The extension $\tilde{M}$ constructed as above is *almost* modularly complemented: only the intersection point of the two modular lines has no modular complement in general. However, constructing the necessary third modular line seems to be quite hard. In fact, we do not know whether three arbitrary lines of a given matroid can always be extended to modular lines. (Of course, if this is impossible, then the corresponding rank 3 matroid cannot be embedded into a finite projective plane, either.)

Therefore, we have chosen another way of embedding an arbitrary rank 3 matroid into a modularly complemented matroid. This — using some design-theory tools — will be presented in the next section.

3 Modularly Complemented Rank 3 Extensions

As mentioned above, *extending* three lines of a given rank 3 matroid to modular lines is quite a problem.

The proof of the following theorem solves this problem by *adding* three modular lines to the given matroid. Although this is not too simple either, decomposing the situation into separate, simpler design problems yields the following result:

Theorem 3.1 *Every matroid of rank* 3 *has a modularly complemented extension.*

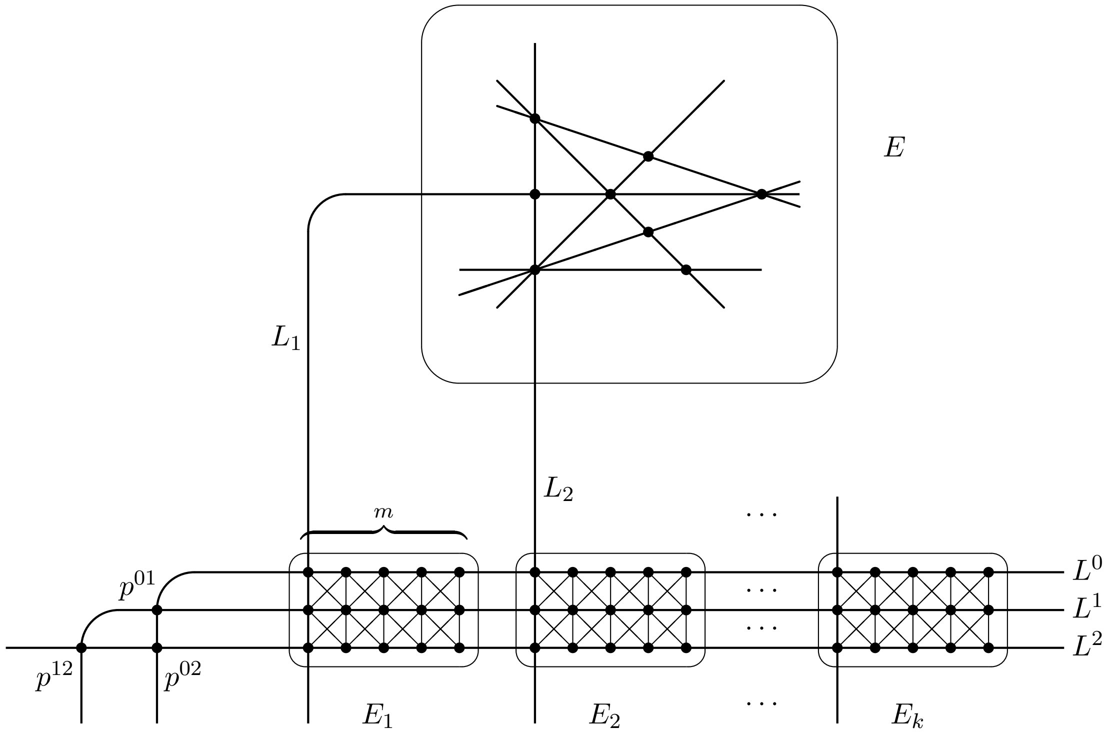

Figure 3: Construction of the Modularly Complemented Extension in Rank 3 (Schematic Sketch)

Proof: Let $M = M(E)$ be the given matroid on the ground set $E = \{e_1, \dots, e_n\}$ and let $\{L_1, \dots, L_l\}$ be its set of lines. Thus, $|E| = n$ and l is the number of lines of M. Let k be the smallest integer with $k \geq l$ that is congruent to 1 or 3 modulo 6, and let $m = 3q$, where q is the smallest prime power such that $q \geq n + 2$. We will define a new matroid $\tilde{M} = \tilde{M}(\tilde{E})$ of rank 3 on the ground set

$$\tilde{E} = E \cupdot \{p^{01}, p^{02}, p^{12}\} \cupdot E_1 \cupdot E_2 \cupdot \dots \cupdot E_k$$

with

$$E_i = \{p_{ij}^s : 1 \leq j \leq m, s \in \{0, 1, 2\}\}$$

for $1 \leq i \leq k$. Clearly $|E_i| = 3m$. The matroid $\tilde{M}$ will have the following properties:

- $\tilde{M}|E = M$,

- the sets

$$\begin{aligned} L^0 &= \{p^{01}, p^{02}\} \uplus \{p^0_{ij} : 1 \le i \le k, 1 \le j \le m\}, \\ L^1 &= \{p^{01}, p^{12}\} \uplus \{p^1_{ij} : 1 \le i \le k, 1 \le j \le m\}, \\ \text{and} \quad L^2 &= \{p^{02}, p^{12}\} \uplus \{p^2_{ij} : 1 \le i \le k, 1 \le j \le m\} \end{aligned}$$

  are modular lines,

- $\tilde{E} = E \uplus (L^0 \cup L^1 \cup L^2)$ with $L^i \cap L^j = \{p^{ij}\}$.

The construction will be given by listing all the lines of $\tilde{M}$. The concept for the construction is sketched in Figure 3. Two essential steps are collected in Lemma 3.2. The lines of $\tilde{M}$ are assembled in the following six steps:

(1) The sets L^0, L^1 and L^2 are modular lines of $\tilde{M}$.

(2) For $1 \le i \le l$, the set $L_i \uplus \{p^0_{i1}, p^1_{i1}, p^2_{i1}\}$ is a line of $\tilde{M}$, extending the line L_i of M to intersect the modular lines.

(3) Choose a Steiner triple system, i.e., a collection of triples (3-element subsets) in $\{1, \dots, k\}$ such that every pair is contained in exactly one triple. This exists if and only if $k \equiv 1 \bmod 6$ or $k \equiv 3 \bmod 6$, as we assumed (cf. Beth, Jungnickel & Lenz [2]). Now declare that $\{p^0_{ij}, p^1_{i'j'}, p^2_{i''j''}\}$ is a line of $\tilde{M}$ if $\{i, i', i''\}$ is a block of the triple system and if $j + j' + j'' \equiv 0 \bmod m$. Note that for any two points $p \in E_i \cap L^s$ and $p' \in E_{i'} \cap L^{s'}$ with $i \ne i'$ and $s \ne s'$ this defines a unique third point p'' on the third modular line $L^{s''}$ such that $\{p, p', p''\}$ is a 3-point line.

(4) To get $\tilde{M}|E_i$, choose a design such that for $p \in E_i \cap L^s$ and $p' \in E_i \cap L^{s'}$ with $s \ne s'$ there is a unique $p'' \in E_i \cap L^{s''}$ with $\{s, s', s''\} = \{0, 1, 2\}$, i.e., different blocks of the design have at most one point in common. Then declare p, p' and p'' to be collinear if and only if $\{p, p', p''\}$ is a block of that design. There are many ways to get such a design — for example, we could put p^0_{ij}, $p^1_{ij'}$ and $p^2_{ij''}$ collinear if and only if $j + j' + j'' \equiv 0 \bmod m$ — however, we will need several additional properties of this design. We will show in Lemma 3.2 below that the required design exists.

    A *factor* of the design is a partition of E_i into blocks. We require that there are n factors $F_1, \dots, F_n$ that intersect only in the block $\{p^0_{i1}, p^1_{i1}, p^2_{i1}\} =: B^*_i$ and n further factors $G_1, \dots, G_n$ which are disjoint from each other and from the F_i. Thus, $F_r \cap F_s = \{B^*_i\}$ for $r \ne s$, $F_r \cap G_s = \emptyset$ for all r, s, and $G_r \cap G_s = \emptyset$ for all $r \ne s$ between 1 and n. From this, we derive the following lines of $\tilde{M}$:

    - The set $L_i \uplus \{p^0_{i1}, p^1_{i1}, p^2_{i1}\}$ is a line of $\tilde{M}$ for $1 \le i \le k$, where $L_i = \emptyset$ for $l < i \le k$. (This includes the lines from Step 2.)

- For every point e_s of the line L_i and every block B of $F_s \setminus \{B_i^*\}$, the set $\{e_s\} \cup B$ is a line of $\tilde{M}$ $(1 \le i \le k)$.
- For every point $e_r \in E \setminus L_i$ and every block B of G_r, the set $\{e_r\} \cup B$ is a line of $\tilde{M}$.
- Every block of the design that is not contained in any factor F_s $(e_s \in L_i)$ or G_r $(e_r \in E \setminus L_i)$ forms a 3-point line in $\tilde{M}$.

(5) Then for $1 \le i \le n$ choose a 4-point line $\{e_i\} \cup B_i^\bullet$ as above with $B_i^\bullet = \{b_i^0, b_i^1, b_i^2\} \subseteq E_i$, remove it, and instead declare $B_i^\bullet$, $\{p^{01}, b_i^2, e_i\}$, $\{p^{02}, b_i^1, e_i\}$ and $\{p^{12}, b_i^0, e_i\}$ to be 3-point lines of $\tilde{M}$.

(6) Finally, we get the 2-point lines of $\tilde{M}$ as

$$\begin{array}{lll} \{p^{01}, p\} & \text{for} & p \in L^2 \setminus \{p^{02}, p^{12}, b_1^2, \dots b_n^2\}, \\ \{p^{02}, p\} & \text{for} & p \in L^1 \setminus \{p^{01}, p^{12}, b_1^1, \dots b_n^1\}, \quad \text{and} \\ \{p^{12}, p\} & \text{for} & p \in L^0 \setminus \{p^{01}, p^{02}, b_1^0, \dots b_n^0\}. \end{array}$$

With the set of lines constructed for $\tilde{M}$ in this way, it is easy to verify that

- there is a unique line through any two points,
- any two lines intersect in at most one point,
- $\tilde{M}|E = M$, and
- each of L^0, L^1 and L^2 intersects every other line.

This completes the proof of the theorem. □

Now we will construct the "transversal design" used for Step 4 in the preceeding theorem. For this, we translate the problem into the language of *latin squares*.

In fact, given the design we can construct an $m \times m$ matrix $A = (a(j, j'))$ with entries in $\{1, \dots, m\}$ by putting $a(j, j') = j''$ if $\{p_{ij}^0, p_{ij'}^1, p_{ij''}^2\}$ is a block of the design. Then every row and column contains every element of $\{1, \dots, m\}$ exactly once: A is a latin square. Furthermore, every latin square determines a transversal design with the required properties.

Now a factor of the design corresponds to a *transversal* of the latin square, that is, to the choice of entries of the square such that exactly one entry is picked from every row and column, and every integer in $\{1, \dots, m\}$ is picked exactly once.

With this, the "missing piece" of our above construction boils down to the following:

Lemma 3.2 *Let $n \geq 3$ and let $m = 3q$ where q is a prime power with $q \geq n+2$. Then there exists an $m \times m$ latin square $A = (a(j,j'))$ with transversals $F_1, \ldots, F_n$ and $G_1, \ldots, G_n$ such that $F_r \cap F_s = \{(1,1)\}$ for $r \neq s$, whereas $F_r \cap G_s = \emptyset$ for all r, s and $G_r \cap G_s = \emptyset$ for $r \neq s$.*

Proof: Let A be the $q \times q$ matrix given by the addition table of the field $F = GF(q)$. Then q disjoint transversals of A are given by $\hat{G}_z = \{(x,y) : x + z_0 y = z\}$ for $z \in F$, where z_0 is a fixed element in $F \setminus \{0,1\}$. Also, $q-2$ intersecting transversals are given by $\hat{F}_z = \{(x,y) : x + zy = 0\}$ for $z \in F \setminus \{0,1\}$, where $\hat{F}_z \cap \hat{F}_{z'} = \{(0,0)\}$ for $z \neq z'$. However, the transversals $\hat{G}_z$ and $\hat{F}_{z'}$ are not disjoint.

Therefore, one passes to the $3q \times 3q$ matrix

$$\hat{A} = \begin{pmatrix} A & A' & A'' \\ A'' & A & A' \\ A' & A'' & A \end{pmatrix}$$

whose row and column indexing set is $\hat{F} = F \uplus F' \uplus F''$. Here we assume that F' and F'' are relabelled copies of F, that A' and A'' are the corresponding addition tables, etc. Now, q disjoint transversals G_z are easily found by considering

$$G_z = \left(\begin{array}{c|c|c} & \hat{G}'_z & \\ \hline \hat{G}''_z & & \\ \hline & & \hat{G}_z \end{array} \right).$$

At the same time, $q-2$ intersecting transversals F_z ($z \in F \setminus \{0,1\}$) are found in $\hat{A}$ by taking

$$F_z = \left(\begin{array}{c|c|c} \hat{F}_z & & \\ \hline & & \hat{G}'_z \\ \hline & \hat{G}''_z & \end{array} \right).$$

F_z and $F_{z'}$ only intersect in $(0,0)$ for $z \neq z'$, whereas $F_z \cap G_{z'} = \emptyset$ and $G_z \cap G_{z'} = \emptyset$ as well. Relabeling the rows, columns and entries of $\hat{A}$ by $\{1, 2, \ldots, 3q\}$ yields a latin square of size $3q \times 3q$ with the required transversals. $\square$

We observe that the construction of our theorem produces extremely large examples. Even for the 4-point uniform matroid $M = U_{3,4}$ (with $n = 4$, $l = 6$) it yields a modularly complemented extension $\tilde{M}$ with 154 elements, with $k = 7$, $q = 7$, $m = 21$.

However, the size of the modularly complemented extensions is polynomially bounded: we have $l \leq \binom{n}{2}$, and $k \leq l + 3$. Furthermore, there is a prime power between $n+2$ and $2(n+2)$ such that we know $q \leq 2n+1$. Hence $\tilde{M}$ has a polynomially bounded number of points, namely

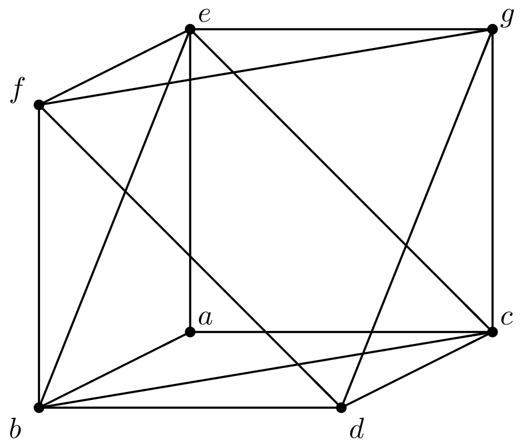

Figure 4: The Matroid $\mathcal{F}^*$ (Sketch: $adfg$ is a Plane)

$$|\tilde{E}| = n + 3 + 9qk \leq n + 3 + 9(2n+1)(3 + \binom{n}{2}) = O(n^3).$$

4 An Example of Rank 4

Now, what about matroids of arbitrary rank? The difficulties arising in rank 3 should indicate that probably it is not possible to extend an arbitrary rank r matroid, where $r \geq 4$, to a modularly complemented matroid — the following example is to confirm this presentiment.

Let $\mathcal{F}^*$ denote the dual of the Fano matroid, depicted in Figure 4, and let W_8 be the one-element extension of $\mathcal{F}^*$ — with new point h — determined by the modular filter $\mathcal{M} = \{abcdefg, bce, bdf, cdg, efg\}$. Applying Theorem 2.2, we find that the planes of W_8 are given by

$$\begin{aligned} \mathcal{H} &= \{abcd, efgh, abef, bdfh, cdgh, aceg, bcfg, cdef, bdeg, adfg, \\ &\quad bceh, abg, abh, agh, bgh, acf, ach, afh, cfh, ade, aeh, adh, deh\}. \end{aligned}$$

W_8 is illustrated in Figure 5. All 4-element planes are represented in the cube except the two “twisted” ones: $adfg$ and $bceh$.

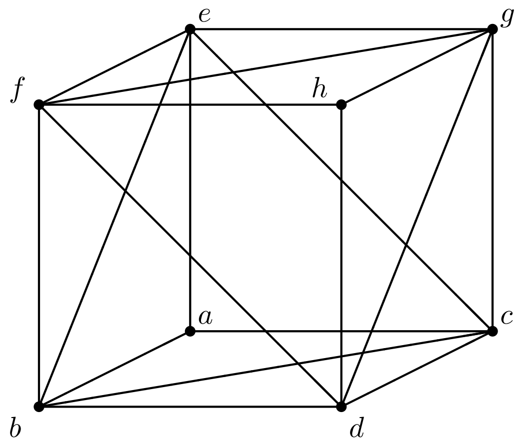

Figure 5: The Example W_8 (Sketch: $adfg$ and $bceh$ are Planes)

Note that $\mathcal{F}^*$ is isomorphic to $AG(4,2) \setminus p$, such that W_8 can be obtained from the affine geometry $AG(4,2)$ by relaxing the three 4-element planes that contain a fixed line.

The property needed to prove that W_8 cannot be embedded into a modularly complemented matroid is provided by the following lemma.

Lemma 4.1 *Let H be an arbitrary plane of W_8. Then H cannot be extended to a modular plane, i.e., there is no extension $\tilde{W}$ of W_8 such that the plane $\tilde{H}$ determined by H is modular.*

Proof: Of course, in order to verify this claim we will not examine all 23 planes, but make use of the symmetry of W_8. Consider the following four subsets of $\mathcal{H}$:

$$\begin{aligned}
\mathcal{H}_1 &= \{abef, bdfh, cdgh, aceg, bcfg, ade, aeh, adh, deh\} \\
\mathcal{H}_2 &= \{abcd, cdgh, efgh, abef, cdef, abg, abh, agh, bgh\} \\
\mathcal{H}_3 &= \{abcd, aceg, efgh, bdfh, bdeg, acf, ach, afh, cfh\} \\
\mathcal{H}_4 &= \{adfg, bceh\}.
\end{aligned}$$

Each plane of W_8 is contained in at least one of these sets. Furthermore, $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ each determine a Vamos-configuration as in Figure 1. So, by symmetry, we only have to verify our claim for $\mathcal{H}_1$ and $\mathcal{H}_4$. Applying the symmetry of the Vamos-configuration and of $\mathcal{H}_4$, it can readily be seen that we merely have to consider the four planes $abef$, $bcfg$, ade and $adfg$.

Let us examine the fourth plane. Suppose $H = adfg$ could be extended to a modular plane. Then there would be a rank 4 matroid $\tilde{W}$ on $\tilde{S} \supseteq S = \{a, b, c, d, e, f, g, h\}$ with modular plane $\tilde{H} \supseteq H$. Thus $\tilde{H}$ would intersect the line l determined by bh. Since $adfg$ does not intersect bh in W_8, there is a point $p \in \tilde{S} \setminus S$ contained in both l and $\tilde{H}$. Now consider the restriction $W' = \tilde{W}|S \cup p$. The plane $adfgp$ intersects the line bhp. Consequently, there is a one-element extension of W_8 containing both $adfgp$ and bhp as flats. By Theorem 2.2 this determines a modular filter $\mathcal{M}$ in W_8 with $adfg \in \mathcal{M}$ and $bh \in \mathcal{M}$. Furthermore, the empty set cannot be in $\mathcal{M}$, since otherwise p would be a loop in W' according to Theorem 2.2. This observation leads to a contradiction: $bh \in \mathcal{M}$ implies that $bceh$ and $bdfh$ are contained in $\mathcal{M}$. Now, $bdfh$ forms a modular pair with $adfg$, resulting in $df \in \mathcal{M}$, $cdef \in \mathcal{M}$. Since $cdef$ and $bceh$ form another modular pair, we have $ce \in \mathcal{M}$, $aceg \in \mathcal{M}$. The intersection of $aceg$ and $adfg$ yields $ag \in \mathcal{M}$. But now, ag and bh form a modular pair, since there is no plane containing both ag and bh. Thus, the empty set, the intersection of ag and bh, is an element of $\mathcal{M}$ — a contradiction.

Thus, the plane $adfg$ cannot be extended to a modular plane. Similar arguments show that there are no one-element extensions such that $abef$ intersects cg, $bcfg$ intersects ae and ade intersects bf. Hence, $abef$, $bcfg$ and ade cannot be extended to modular planes, too — and our claim is verified. □

This lemma directly leads us to

Proposition 4.2 *The above-defined rank 4 matroid W_8 cannot be embedded into a modularly complemented matroid.*

Proof: Suppose that there is a modularly complemented extension $\tilde{W}$ of W_8. We can assume that $r(\tilde{W}) = r(W_8) = 4$, because restrictions of modularly complemented matroids to flats are again modularly complemented [15]. Since W_8 is connected, $\tilde{W}$ has to be connected, too.

$\tilde{W}$ cannot be embedded into a modular matroid, so the main result of Kahn & Kung [10] implies that there are four points p_1, p_2, p_3 and p_4 in $\tilde{W}$ such that

- these points determine six different lines in $\tilde{W}$,
- these six lines contain all points of $\tilde{W}$ and
- the four planes spanned by $p_1, \ldots, p_4$ are all modular.

The union of any three of these four planes contains the ground set $\tilde{S}$ of $\tilde{W}$, and thus the 8-point set S. Therefore, three points $x, y, z \in S$ lie on a plane $\overline{p_i p_j p_k}$ of $\tilde{W}$. Since there are no 3-point lines in W_8, x, y and z span the plane $\overline{p_i p_j p_k}$. But now, this plane is a modular extension of the plane $\overline{xyz}$ — in contrast to Lemma 4.1. □

Remark: By the same method of proof, it can easily be shown that V_8 cannot be extended to a modularly complemented matroid, too. However, whereas in W_8 no plane can be extended to a modular plane, there are planes in V_8 which can be extended to modular ones — e.g. the plane *abc*.

So, the best we can achieve in rank $r \geq 4$ is a supersolvable extension. But in order to construct such an extension, we are not able to proceed as in the rank 3 case (cf. Proposition 2.3); Lemma 4.1 shows that *extending* hyperplanes to modular hyperplanes does not work in general. Nevertheless, it is possible to obtain a supersolvable extension and the first — and crucial — step in accomplishing this is to extend the given matroid M to a matroid $\hat{M}$ of the same rank that contains a modular hyperplane by *adding* a modular hyperplane.

However, to prove this fact, we need some results concerning the Dilworth truncation of a matroid. These results will be presented in the next section.

5 Dilworth Truncations and Modular Flats

Let M be an arbitrary simple matroid of rank r and L denote the lattice of flats of M. Deleting all points from L we obtain a new lattice L' called the *lower truncation* of L. For $r(M) \geq 4$, L' is not geometric — consider for instance the lower truncation of the boolean algebra on 4 elements. However, in a natural way it is possible to embed L' into a geometric lattice by adding further flats. This geometric lattice will be called the *(first) Dilworth truncation* $D^1(M)$ of the matroid M.

Geometrically, the Dilworth truncation of M is constructed by interpreting M as a point configuration and intersecting a general position hyperplane H with every line of M. Then the intersection points in H determine the Dilworth truncation of M.

Similarly, deleting all flats $x \in L$ of rank $1 \leq r(x) \leq k$ for some $k \geq 2$ does not yield a geometric lattice in general. But in this case, too, the lattice constructed in this way can be embedded into a geometric lattice in a natural way — e.g. using Edmonds' method of semimodular functions, see [1]. The resulting lattice is called the *k-th Dilworth truncation* $D^k(M)$.

Our interest in the Dilworth truncation comes from Crapo's interpretation: the idea is to start the construction of a supersolvable extension of an arbitrary matroid M by first adding a "modular hyperplane in general position". Thus the restriction of the new matroid $\hat{M}$ to the modular hyperplane should be the Dilworth truncation of M.

The easiest way to obtain $\hat{M}$ was first discovered by Mason [11] — one can put $\hat{M} := D^1(M \oplus p)$. This $\hat{M}$ contains M as a submatroid and $D^1(M)$ as a modular hyperplane. Now Mason suggested that we get a supersolvable embedding of M by repeating this construction. This claim leads us to a careful analysis of modular flats in Dilworth truncations. The analysis is not easy, but it buys us a complete characterization theorem and perhaps a deeper understanding of the (complicated) structure of Dilworth truncations.

As a side result, we find that repeating Mason's construction $\hat{M} := D^1(M \oplus p)$ does not help. Instead, we will use Brylawski's modular join construction in Section 6.

In order to describe the construction of the k-th Dilworth truncation and to prove the needed results, we first have to introduce some notation:

In the following, M always denotes a matroid of rank r on the ground set S, L denotes the lattice of flats of M. Let S_i denote the set of all rank i flats of M, so $S = S_1$ and S_2 is the set of all lines of M. For fixed i and arbitrary $x \in L$ let $A_x = \{y \in S_i : y \le x\} \subseteq S_i$. Note that A_x is the empty set if and only if $r(x) < i$.

The k-th Dilworth truncation $D^k(M)$ of M is a matroid of rank $r - k$ on the set S_{k+1} and is characterized by the following proposition.

Proposition 5.1 (Aigner [1]) *Let M be a matroid of rank r and $1 \le k < r$. Let $\mathcal{F}$ consist of all sets $A \subseteq S_{k+1}$ satisfying one of the following two conditions:*

- *$A = A_x$ for some $x \in L$,*
- *$A = A_{x_1} \cup \ldots \cup A_{x_t}$ with $r(x_{i_1} \vee \ldots \vee x_{i_s}) > \sum_{j=1}^{s} r(x_{i_j}) - (s-1)k$ for all $\{x_{i_1}, \ldots, x_{i_s}\}$, $2 \le s \le t$. (In particular, for $s = 2$ this implies that the A_{x_i}'s are pairwise disjoint.)*

Then $(S_{k+1}, \mathcal{F})$ satisfies the flat axioms and thus determines a matroid on S_{k+1} called the k-th Dilworth truncation *$D^k(M)$.*

The flats A of the form $A = A_x$ will henceforth be called *L-flats*. The next proposition provides more information on these flats:

Proposition 5.2 (Aigner [1]) *Let x and y be arbitrary elements of $L(M)$ and let $1 \le k < r(M)$. Then we have in the k-th Dilworth truncation $D^k(M)$:*

(1) *$A_x \wedge A_y = A_{x \wedge y}$.*

(2) *$A_x \vee A_y = A_x \uplus A_y$, if $r(x \vee y) > r(x) + r(y) - k$.*

(For $k = 1$ this is equivalent to $x \wedge y = \emptyset$ and $r(x \vee y) = r(x) + r(y)$.)

Otherwise we have $A_x \vee A_y = A_{x \vee y}$.

(3) *$r(A_x) = r(x) - k$, for $r(x) \ge k$.*

(4) *Let $r(x) \ge k$. Then y covers x in L if and only if A_y covers A_x in $D^k(M)$.*

Proof: The proofs for (1), (3) and (4) can be found in Aigner [1]. To prove (2), first assume $r(x \vee y) > r(x) + r(y) - k$. Then $A_x \vee A_y = A_x \uplus A_y$ by Proposition 5.1.

For $r(x \vee y) \le r(x) + r(y) - k$ Proposition 5.1 implies that $A_x \cup A_y$ is not a flat in $D^k(M)$. But $A_x \le A_{x \vee y}$ and $A_y \le A_{x \vee y}$ are clear, and they imply $A_x \vee A_y \le A_{x \vee y}$.

Put $A_x \vee A_y =: A_{z_1} \uplus \ldots \uplus A_{z_t}$ and assume that A_x and A_y lie in different sets A_{z_i} — without loss of generality $A_x \subseteq A_{z_1}$, $A_y \subseteq A_{z_2}$. Now $r(x \vee y) \leq r(x) + r(y) - k$ implies $r(x \vee y) \leq r(z_1) + r(z_2) - k < r(z_1 \vee z_2)$ due to Proposition 5.1. On the other hand, $z_1 \leq x \vee y$ and $z_2 \leq x \vee y$ by construction, resulting in $z_1 \vee z_2 \leq x \vee y$ and $r(z_1 \vee z_2) \leq r(x \vee y)$ — a contradiction.

Thus, A_x and A_y are contained in an A_{z_i}, $1 \leq i \leq t$. This implies $x \leq z_i$, $y \leq z_i$ and $x \vee y \leq z_i$. Hence $A_{x \vee y} \leq A_{z_i} \leq A_x \vee A_y$ and finally $A_{x \vee y} = A_x \vee A_y$. □

Additional properties of the flats of $D^k(M)$ are contained in

Proposition 5.3 (Aigner [1]) *Let A be an arbitrary non-empty flat of $D^k(M)$ and $A_{x_1}, \ldots, A_{x_t}$ be the maximal L-flats contained in A. Then $A = A_{x_1} \uplus \ldots \uplus A_{x_t}$, i.e., the A_{x_i}'s form a partition of A. Furthermore, the rank of A equals $r(A) = r(A_{x_1}) + \ldots + r(A_{x_t}) = r(x_1) + \ldots + r(x_t) - kt$.*

In order to simplify notation, this canonical representation of an arbitrary non-empty flat of $D^k(M)$ will be abbreviated to $A = A(x_1, \ldots, x_t)$, $t \geq 1$. Furthermore, the canonical representation of the empty set in $D^k(M)$ is defined as $A(\emptyset)$.

In the following we will characterize the modular flats of an arbitrary Dilworth truncation $D^k(M)$. To begin with, we list and prove three lemmas:

Lemma 5.4 *For all $k \geq 1$ the k-th Dilworth truncation $D^k(M)$ is connected.*

Proof: If a matroid is not connected, then the bipartite graph of points versus "long lines" (lines with at least three points) is disconnected. We show that this cannot happen:

We only have to consider the case $1 \leq k \leq r(M) - 3$. Every (k+2)-flat of M covers at least $k + 2 \geq 3$ (k+1)-flats of M. Furthermore, the bipartite graph of (k+2)-flats vs. (k+1)-flats in M is connected — there are many simple proofs for this. From this the result already follows. □

Lemma 5.5 *A flat A of $D^k(M)$ is connected if and only if it is an L-flat.*

Proof: If A is not an L-flat, then it is of the form $A = A_{x_1} \uplus \ldots \uplus A_{x_t}$ with $r(A) = r(A_{x_1}) + \ldots + r(A_{x_t})$, where the A_{x_i} are disjoint flats (Proposition 5.3). Hence A is not connected. On the contrary, if A is an L-flat, then A is connected by Lemma 5.4 — because it follows easily from Proposition 5.1 that for every flat $x \in L(M)$ we have $D^k(M)|A_x \cong D^k(M|x)$, i.e., the Dilworth truncation commutes with the restriction to flats. □

Lemma 5.6 *A flat $A \subseteq D^k(M)$ of the form $A = A(x_1, \ldots, x_t)$, $t \geq 2$, is never modular.*

Proof: Brylawski proved in [4, Corollary 3.16] that modular flats of a connected matroid are always connected. By Lemmas 5.4 and 5.5, A is a flat in the connected matroid $D^k(M)$ that is not connected — hence it cannot be modular. □

Now we are ready to characterize the modular flats of an arbitrary Dilworth truncation $D^k(M)$. First, consider the case $k = 1$:

Proposition 5.7 *Let M be a matroid, L its lattice of flats and $x \in L$ a flat of rank $r(x) \geq 3$. Then A_x is modular in $D^1(M)$ if and only if all circuits of M are contained in x, i.e., if and only if $L = [\emptyset, x] \times B_{r(M)-r(x)}$.*

Proof: To begin with, assume there are circuits in M that are not contained in x. Choose a circuit $C \not\subseteq x$ such that $|C - x| > 0$ is minimal and let $y := \overline{C}$. Observe that $|C - x| \geq 2$ and $C \cap x$ is independent. We first state and prove the following two facts:

(1) If $C \cap x \subseteq B \subseteq x$ and B is independent, then $(B \cup C) \setminus p$ is independent for all $p \in C$.

Proof: Note that $(B \cup C) \cap x = B$ is independent. Thus a circuit $C' \subseteq (B \cup C) \setminus p$ would violate the minimality of C, if $p \in C \setminus x$. If $p \in C \cap x$, we could eliminate some $p' \in (C' \cap C) \setminus x$ and again get a contradiction against the minimality of C.

(2) $r(x \wedge y) = |C \cap x|$ and $r(x \vee y) = r(x) + |C \setminus x| - 1$.

Proof: Obviously $C \cap x \subseteq x \wedge y$ and a larger independent set B with $C \cap x \subseteq B \subseteq x \wedge y$ would imply $r(C) = r(y) \geq |(B \cup C) \setminus p| > |C| - 1$, where $p \in C \setminus x$. Similarly, the second assertion follows.

Now we will construct a flat $A \in D^1(M)$ such that A and A_x do not form a modular pair:

Case 1: If $|C \cap x| \geq 3$, choose $q_1, q_2 \in C \cap x$ with $q_1 \neq q_2$ and partition $C \setminus x = C_1 \,\dot\cup\, C_2$ into non-empty subsets. Now $C_1 \,\dot\cup\, C_2 \,\dot\cup\, q_1 \,\dot\cup\, q_2 \subseteq C$ is independent. Put $y_1 := \overline{C_1 \,\dot\cup\, q_1}$, $y_2 := \overline{C_2 \,\dot\cup\, q_2}$. Then $A := A(y_1, y_2)$ is a flat of $D^1(M)$ of rank $r(A) = |C_1 \,\dot\cup\, q_1| + |C_2 \,\dot\cup\, q_2| - 2 = |C \setminus x|$ by Propositions 5.1 and 5.3. Furthermore, $A \vee A_x = A_{y_1} \vee (A_{y_2} \vee A_x) = A_{x \vee y}$ by Proposition 5.2, and $r(A \vee A_x) = r(x \vee y) - 1 = r(x) + |C \setminus x| - 2$. With $r(A_x) = r(x) - 1$ and $A \cap A_x = \emptyset$, the result follows.

Case 2: If $|C \cap x| = 2$, we choose $q_1 \in C \cap x$ and $q_2 \in x \setminus C$ such that $(C \cap x) \,\dot\cup\, q_2$ is independent. Hence $(C \setminus x) \cup \{q_1, q_2\}$ is independent by (1), and we can proceed exactly as in Case 1.

Case 3: If $C \cap x = \{q_0\}$, choose $q_1, q_2 \in x$ such that $\{q_0, q_1, q_2\}$ is independent and proceed as above.

Case 4: If $C \cap x = \emptyset$, choose an independent set $B = \{q_1, q_2, q_3\}$ in x and partition $C = C_1 \cup C_2 \cup C_3$ into non-empty sets. (All matroids are assumed to be simple.) Then by (1), $(C \setminus p) \cup B$ is independent for $p \in C$. This easily yields that $A = A(\overline{C_1 \cup q_1}, \overline{C_2 \cup q_2}, \overline{C_3 \cup q_3})$ is a flat of $D^1(M)$ with $r(A) = |C|$. Furthermore, $A \vee A_x = A_{x \vee y}$, $A \cap A_x = \emptyset$, which again results in non-modularity.

Thus we have shown that if A_x is modular in $D^1(M)$, all circuits of M have to be contained in x.

Conversely, assume that all circuits of M are contained in x. Let $A = A(x_1, \ldots, x_t)$, $t \geq 1$, be a complement of A_x. Now if $L = [\emptyset, x] \times B_{r(M)-r(x)}$, then $A \cap A_x = \emptyset$ implies that every x_i satisfies $|x_i \cap x| \leq 1$, thus we have $r(x_i) \leq |x_i \setminus x| + 1$. Furthermore, the x_i are disjoint by Proposition 5.1, resulting in

$$r(A) = \sum_{i=1}^{t} (r(x_i) - 1) \leq \sum_{i=1}^{t} |x_i \setminus x| \leq r(M) - r(x).$$

Finally, $r(D^1(M)) = r(M) - 1$ and $r(A_x) = r(x) - 1$ yield $r(A) = r(D^1(M)) - r(A_x)$. Thus, A_x is modular. □

For $k \geq 2$ the modular flats of $D^k(M)$ are characterized in the following

Proposition 5.8 *The k-th Dilworth truncation has no non-trivial modular flats for $k \geq 2$.*

Proof: According to Lemma 5.6, $A(x_1, \ldots, x_t)$ is not modular for $t \geq 2$. Now let A_x be an L-flat in $D^k(M)$ with $2 \leq r(A_x) < r(M)$. By Proposition 5.2 we have $r(x) \geq k + 2$. Let $I \subseteq x$ be independent of size $k+2$ and choose a $p \in S \setminus x$, where S denotes the ground set of M.

Now write $I = I_1 \cup I_2$ with $|I_1| = |I_2| = k$ and $|I_1 \cap I_2| = k - 2 \geq 0$. For $x_1 := \overline{I_1 \cup p}$ and $x_2 := \overline{I_2 \cup p}$ we get $r(x_1) = r(x_2) = k + 1$. Furthermore, $r(x_1 \vee x_2) = |I| + 1 = k + 3$ and $r(x_1) + r(x_2) - k = k + 2$. Thus $A := A(x_1, x_2)$ is a rank 2 flat of $D^k(M)$ by Propositions 5.1 and 5.3.

Since $A \leq A_{x \vee p}$ we have $A_x \vee A \leq A_{x \vee p}$ and $r(A \vee A_x) \leq r(A_{x \vee p}) = r(x) + 1 - k$. On the other hand, $r(x_i \wedge x) = |I_i| = k$, where $i = 1, 2$, implies $A \cap A_x = \emptyset$. Thus we get

$$r(A_x \wedge A) + r(A_x \vee A) \leq r(x) + 1 - k < 2 + r(x) - k = r(A_x) + r(A).$$

But then, A and A_x do not form a modular pair, i.e., A_x cannot be modular. □

Proposition 5.8 shows that for $k \geq 2$ the k-th Dilworth truncation is supersolvable only in the trivial case $r(D^k(M)) \leq 2$, i.e., $k \geq r(M) - 2$. For $k = 1$, Proposition 5.7 yields the following result:

Corollary 5.9 *Let M be a matroid of rank $r(M) \geq 3$. Then the following are equivalent:*

(1) *$D^1(M)$ is supersolvable.*

(2) *$D^1(M)$ has a modular line.*

(3) *$M = \tilde{M} \oplus F_{r(M)-3}$, where $r(\tilde{M}) = 3$, and $F_{r(M)-3}$ denotes the free matroid of rank $r(M) - 3$.*

Proof: (1) $\Rightarrow$ (2) is trivial, (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) follow directly from Lemma 5.6 and Proposition 5.7. $\square$

6 Supersolvable Extensions of Arbitrary Matroids

In this section we show how every matroid of arbitrary rank can be extended to a supersolvable matroid of the same rank. To begin with, we state and prove the following key lemma:

Lemma 6.1 (Mason [11]) *Let M be a matroid on the ground set S. Then there is a matroid $\hat{M}$ of the same rank on the set $\hat{S} \supseteq S$ satisfying:*

(1) *$\hat{S} \setminus S$ is a modular hyperplane in $\hat{M}$,*

(2) *$\hat{M}|S = M$, and*

(3) *$\hat{M}|(\hat{S} \setminus S)$ is the Dilworth truncation of M.*

Proof: Let $M' = M \oplus p$ be the one-element extension of M by a coloop, corresponding to the modular filter $\mathcal{M} = \emptyset$. According to Theorem 2.2, the lines of M' are the lines of M together with all sets $\{q, p\}$, where $q \in S$. Let us call the first set S' and identify the second one with S.

Now consider the matroid $\hat{M} = D^1(M')$. Its ground set is $\hat{S} = S \cupdot S'$. M is a flat of M' with $M' = M \oplus p$, by construction. Applying Proposition 5.7 yields that A_M is a modular flat of $\hat{M}$. Since obviously we have $A_M = S'$ the first claim is verified.

The second and third assertion follow immediately from Proposition 5.1. $\square$

The following corollary is an immediate consequence of Proposition 5.1 and explicitly describes all flats of $\hat{M}$:

Corollary 6.2 *Let M be a matroid on the set S and $\hat{M}$ denote the extension of M according to the preceeding lemma. Then the flats of $\hat{M}$ are of the following three types:*

(a) $A \subseteq S'$, *where* A *is a flat of* $D^1(M)$*;*

(b) $A = x \cupdot \tilde{A}$, *where* $\tilde{A} = A(x, x_2, \ldots, x_t)$ *is a flat of* $D^1(M)$*;*

(c) $A = \{y\} \cupdot \tilde{A}$, *where* $\tilde{A} = A(x_1, \ldots, x_t)$ *is a flat of* $D^1(M)$ *and* y *is an element of* $S \setminus (x_1 \cupdot \ldots \cupdot x_t)$.

In order to use Lemma 6.1 in an inductive proof of our main result, we need the following construction due to Brylawski [4]:

Definition 6.3 *Let* M_1 *and* M_2 *be matroids on the ground sets* S_1 *and* S_2*, respectively. Let* S *be a subset of both* S_1 *and* S_2*.*

For $T = (S_1 \setminus S) \cupdot (S_2 \setminus S) \cupdot S$ *let* $\mathcal{F}$ *denote the family of all subsets* $F \subseteq T$ *such that* $F \cap S_i$ *is a flat in* M_i *for* $i = 1, 2$*.*

If $\mathcal{F}$ *satisfies the flat axioms, we call the corresponding matroid* $J = J(T)$ *the* strong join *of* M_1 *and* M_2 *relative to* S*.*

Of course, the strong join of two matroids does not exist in general. However, the following proposition (cf. Brylawski [4, Theorem 5.3]) provides a useful sufficient criterion for existence:

Proposition 6.4 *If* $M_1(S_1)$ *and* $M_2(S_2)$ *are matroids where* $H = S_1 \cap S_2$ *is a modular flat of* M_1 *and a restriction of* M_2*, then the strong join* J *of* M_1 *and* M_2 *relative to* H *exists. Furthermore, the rank of* J *is* $r(M_1) + r(M_2) - r(H)$*. (In this special situation the strong join is called the generalized parallel connection.)*

Now we are able to state and prove our main result:

Theorem 6.5 *Let* M *be a matroid of arbitrary rank* r *on the set* S*. Then there is a supersolvable rank* r *matroid* $\mathcal{S}(M)$ *with* $\mathcal{S}(M)|S = M$*.*

Proof: The proof uses induction on the rank r. For $r \leq 2$ there is nothing to show. Hence, we may assume that $r > 2$ and the theorem holds for arbitrary matroids of rank less than r.

By Lemma 6.1, there is a rank r extension $\hat{M}$ of M containing a modular hyperplane H. Applying the inductive hypothesis, the matroid $\hat{M}|H$ can be extended to a supersolvable rank $r-1$ matroid M' on the set $S' \supseteq H$. Due to Proposition 6.4, the strong join $\mathcal{S}(M)$ of $\hat{M}$ and M' relative to H exists and has rank r. Obviously, S' is a modular hyperplane in $\mathcal{S}(M)$ with $\mathcal{S}(M)|S'$ being supersolvable. Thus, $\mathcal{S}(M)$ is supersolvable (cf. Definition 1.3). Since it is clear that $\mathcal{S}(M)|S$ equals M, this completes the proof of the theorem. □

The following three remarks are appropriate:

- Note that this construction also works for matroids with infinite ground set.
- Of course, this theorem is clear for representable matroids. However, the problem to find small supersolvable extensions is non-trivial then. Such small supersolvable extensions are put to use in [18]. Again, as in the rank 3 case (Theorem 3.1), the extensions produced by the proof of Theorem 6.5 are huge and far from minimal.
- The roots e_i of the characteristic polynomial of $\mathcal{S}(M)$ can be computed as follows: e_{r-i} equals the number of points of the iterated Dilworth truncation

$$D^{1(i)}(M) = \underbrace{D^1(\cdots D^1}_{i-times}(M)\cdots),$$

where $0 \leq i < r$. In particular $e_r = |S|$ and $e_{r-1} = |S_2|$.

After having shown that every matroid admits a supersolvable extension, we will now study the problem of coordinatizing this extension over a given field.

We start with an example for the construction of Lemma 6.1 which indicates the problems arising and is crucial for yielding the canonical coordinatization for matroids like $\hat{M}$:

(1) Let $M = F_r$ be the free matroid on r points, whose lattice of flats is the boolean algebra B_r of rank r.

Then $D^1(M)$ is the graphic matroid $M(K_r)$, whose flats are disjoint unions of cliques $K_i \quad (i \geq 2)$ in K_r (see Proposition 5.1), corresponding to the partition lattice Π_r as lattice of flats.

The matroid $\hat{M}$ constructed from M is $\hat{F}_r = D^1(F_{r+1}) = M(K_{r+1})$. Here, K_r is a subgraph of K_{r+1}, which yields the inclusion $D^1(M) \hookrightarrow \hat{F}_r$, and forms a modular hyperplane. Its complement is the star of a vertex v, with $M = F_r$ as its matroid. The flats of $M(K_{r+1})$ are again disjoint unions of cliques $K_i \quad (i \geq 2)$, where a flat $A \subseteq E(K_{r+1})$ is:

- of type (a), if v is not contained in a clique,
- of type (b), if v is contained in an s-clique K_s with $s \geq 3$,
- of type (c), if v is contained in a single edge (2-clique K_2),

according to the description of the flats of $\hat{M}$ in Corollary 6.2 (cf. Figure 6).

(2) The Dilworth truncation of $M(K_r)$ is the "geometry of triples" described by Crapo in [17, p.21]. Whereas F_r and $M(K_r)$ are regular matroids, $D^1(M(K_r))$ is not binary any more for $r \geq 4$. For example, $D^1(M(K_4))$ is a seven point line.

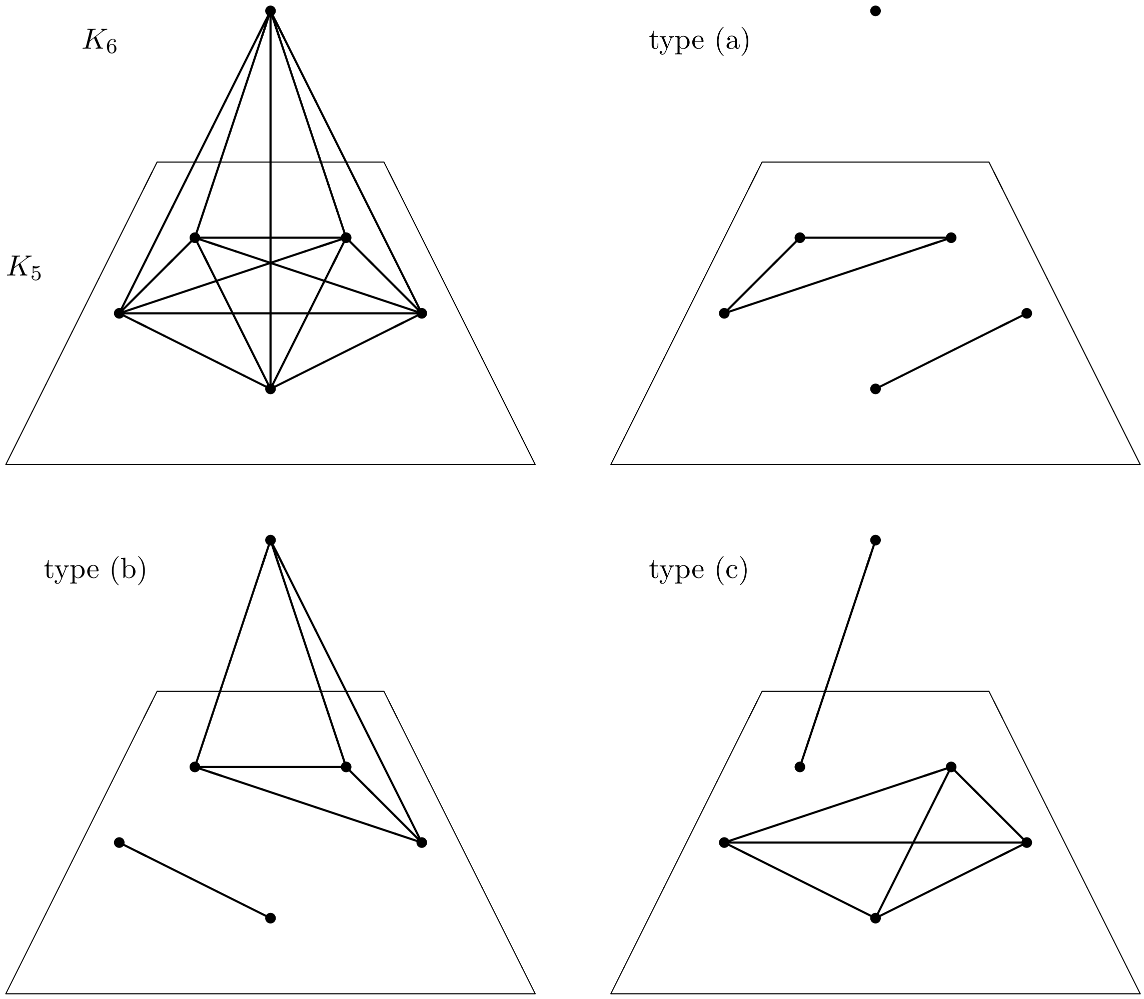

Figure 6: The Flats of $\hat{F}_5$

Part (2) of this example suggests studying the relation between coordinatizations of M, $D^1(M)$ and $\hat{M}$.

The basic results about the coordinatization of $D^1(M)$ are due to Brylawski [5], continuing work by Mason [11, 12].

In the following two propositions we will cover the coordinatization problem for $\hat{M}$, slightly extending Brylawski's results.

Proposition 6.6 *For a given field k, let M be a matroid representable over k ("k-linear"). Then $\hat{M}$ is representable over every infinite extension field of k.*

Proof: M is k-linear if and only if $M \oplus p$ is k-linear. Thus it suffices to study representability of $D^1(M)$ for every matroid M.

Let the vectors $v_1, \ldots, v_n$ represent M over the field k, and thus (with the same notation) over an extension field K of k.

Now Brylawski [5, Theorem 3.1] has shown that if $\lambda_1, \ldots, \lambda_n \in K$ are independent transcendentals over k, then $D^1(M)$ is coordinatized by

$$(\lambda^T v_i)v_j - (\lambda^T v_j)v_i$$

for $\lambda = (\lambda_1, \ldots, \lambda_n)^T$.

The λ_i, however, do not have to be independent transcendentals: they only have to be chosen from K in such a way that $\lambda^T v_i \neq 0$ and certain determinants (which are polynomials in the λ_i and coordinates of the v_i) do not vanish.

Now assume K to be an infinite extension field of k. (In particular, we can put $K := k$ if k is infinite.) Since the affine space over K is not a union of proper closed algebraic subvarieties, $\lambda_1, \ldots, \lambda_n$ can be substituted by elements of K such that still $\lambda^T v_i \neq 0$ and "Brylawski's determinants" do not vanish. This completes the proof of the proposition. $\square$

Proposition 6.7 *If $r(M) \geq 4$ and $D^1(M)$ is representable over a given field k, then so is $\hat{M}$. Furthermore, every k-representation of $D^1(M)$ extends to a representation of $\hat{M}$.*

Proof: Let r denote the rank of M, let $D^1(M)$ be coordinatized in k^{r-1} and put $V = k^r \supseteq k^{r-1}$.

Choose a basis $B \subseteq S$ of M. Then $D^1(B) \cong M(K_r)$ and $\hat{B} \cong M(K_{r+1})$ are submatroids of $\hat{M}$. Since they are unimodular (with $D^1(B) \cup B = \hat{B}$), we can choose coordinates for V such that B is coordinatized by $\{e_1, \ldots, e_r\}$, and $D^1(B)$ by $\{e_i - e_j : 1 \leq i < j \leq r\}$.

Now, Brylawski's proof for [5, Theorem 3.1 (2)] shows — in less symmetric coordinates — that the representation of $D^1(M)$ in the subspace

$$H = \{x \in V : \sum_{i=1}^{r} x_i = 0\}$$

of V determines a representation of M in V such that for v_i, v_j (representing points p_i and p_j of M), v_{ij} (representing the line $\overline{p_i p_j}$) is linearly dependent from v_i and v_j, i.e.,

$$v_{ij} = (\mathbf{1}^T v_i)v_j - (\mathbf{1}^T v_j)v_i$$

up to a scalar, where $\mathbf{1}$ is the all-ones vector.

Thus we find vectors in V such that M and $D^1(M)$ are represented correctly. To see that this yields a representation of $\hat{M}$, let B be a basis of $\hat{M}$. Then the corresponding vectors span V: this follows from Brylawski's construction for $|B \cap S| = 1$ ($B \cap S = \emptyset$ is impossible). For $|B \cap S| \geq 2$ it follows by induction on $|B \cap S|$, using that for $v_i, v_j \in B \cap S$ there is a $v_{ij} \notin S$ collinear with v_i and v_j (basis exchange).

Similarly, if C is dependent in $\hat{M}$, then the corresponding set of vectors is dependent in V: for $C \cap S = \emptyset$ this has been shown by Brylawski, $|C \cap S| = 1$ cannot happen since S_2 is a flat of $\hat{M}$, and for $|C \cap S| \geq 2$ we use induction: for $v_i, v_j \in C$ we get a $v_{ij} \in S_2$ with $span(C) = span((C \setminus v_i) \cup v_{ij})$. □

This result enables us to describe the connection between coordinatizations of M and its supersolvable extension $\mathcal{S}(M)$:

Theorem 6.8 *$\mathcal{S}(M)$ is representable over a given field k if and only if the rank 3 matroid $\hat{D}^{1(r-3)}(M)$ is representable over k, where r denotes the rank of M and*

$$D^{1(r-3)}(M) = \underbrace{D^1(\cdots D^1}_{(r-3)-times}(M)\cdots).$$

Furthermore, the characteristic sets of M and $\mathcal{S}(M)$ coincide.

Proof: $\hat{D}^{1(r-3)}(M)$ is a restriction of $\mathcal{S}(M)$, so the "only if" part is trivial.

For the "if" part, we use induction on r: $\mathcal{S}(M)$ is a strong join of $\hat{M}$ and $\mathcal{S}(D^1(M))$. Now for $r \geq 4$, $\mathcal{S}(D^1(M))$ is k-linear by induction, thus $D^1(M)$ is k-linear. By Theorem 6.7, $\hat{M}$ is k-linear, too. Furthermore, $\hat{M}$ has the *modular extension property* [4, Definition 6.1] with respect to $D^1(M)$. This implies that $\mathcal{S}(M)$ is k-linear by [4, Theorem 6.12]. The second assertion of the theorem follows immediately from Theorem 6.6. □

Acknowledgment

We wish to thank Noga Alon and Curt Lindner for (independently) providing the proof of Lemma 3.2. Furthermore, we thank Mark Haiman for helpful discussions.

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