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Bounds for Lattice Polytopes Containing A Fixed Number of Interior Points in a Sublattice

Jeffrey C. Lagarias

AT&T Bell Laboratories
Murray Hill, New Jersey

Günter M. Ziegler

Universität Augsburg
Augsburg, West Germany

ABSTRACT

A lattice polytope is a polytope in $\mathbb{R}^n$ whose vertices are all in $\mathbb{Z}^n$. The volume of a lattice polytope $\mathbf{P}$ containing exactly $k \geq 1$ points in $d\mathbb{Z}^n$ in its interior is bounded above by $kd^n\,(7(kd+1))^{n2^{n+1}}$. Any lattice polytope in $\mathbb{R}^n$ of volume V can after an integral unimodular transformation be contained in a lattice cube having side length at most $n \cdot n!V$. Thus the number of equivalence classes under integer unimodular transformations of lattice polytopes of bounded volume is finite. If $\mathbf{S}$ is any simplex of maximum volume inside a closed bounded convex body $\mathbf{K}$ in $\mathbb{R}^n$ having nonempty interior, then $\mathbf{K} \subseteq (n+2)\,\mathbf{S} - (n+1)\mathrm{s}$, where $m\mathbf{S}$ denotes a homothetic copy of $\mathbf{S}$ with scale factor m, and s is the centroid of $\mathbf{S}$.

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1. Introduction

A *lattice polytope* in $\mathbb{R}^n$ is a convex polytope all of whose vertices are lattice points, i.e. points in $\mathbb{Z}^n$. A *rational polytope* $\mathbf{P}$ is a convex polytope with all vertices in $\mathbb{Q}^n$. The *denominator* of a rational polytope $\mathbf{P}$ is the smallest integer $d \geq 1$ such that $d\mathbf{P}$ is a lattice polytope.

For each $n \geq 2$ there are lattice polytopes in $\mathbb{R}^n$ of arbitrarily large volume containing no interior lattice points, and for $n \geq 3$ there are lattice simplices of arbitrarily large volume whose vertices are their only lattice points. However D. Hensley [5] proved that any lattice polytope $\mathbf{P}$ in $\mathbb{R}^n$ containing *exactly* $k \geq 1$ interior lattice points has volume bounded by a finite bound $V(n, k)$, and furthermore the total number of lattice points in the interior and on the boundary of such $\mathbf{P}$ is bounded by a finite bound $J(n, k)$.

The main purpose of this paper is to sharpen Hensley's upper bounds for $V(n, k)$ and $J(n, k)$, and to extend his results to apply to lattice polytopes containing a fixed number $k \geq 1$ of interior points in a given sublattice Λ of $\mathbb{Z}^n$. We also prove finiteness of the number of equivalence classes of such polytopes under lattice-point preserving affine maps. Finally, we prove that any closed convex body $\mathbf{K}$ in $\mathbb{R}^n$ contains a simplex $\mathbf{S}$ such that $\mathbf{K} \subseteq (-n)\mathbf{S} + (n+1)\mathbf{s}$ and $\mathbf{K} \subseteq (n+2)\mathbf{S} - (n+1)\mathbf{s}$, where $\mathbf{s}$ is the centroid of $\mathbf{S}$, and if $\mathbf{K}$ is a lattice polytope then one can choose $\mathbf{S}$, $(-n)\mathbf{S} + (n+1)\mathbf{s}$, and $(n+2)\mathbf{S} - (n+1)\mathbf{s}$ to all be lattice simplices.

In extending Hensley's bounds, we treat first the special case $\Lambda = d\mathbb{Z}^n$. This case arises in considering rational polytopes of denominator d containing k interior lattice points in $\mathbb{Z}^n$, after rescaling to clear the denominator.

Theorem 1. Let $V(n, k, d)$ denote the maximal volume of a lattice polytope in $\mathbb{R}^n$ that contains exactly $k \geq 1$ points in $d\mathbb{Z}^n$ in its interior, and let $J(n, k, d)$ denote the maximum number of lattice points $J(n, k, d)$ inside or on the boundary of such a polytope. Then $V(n, k, d)$ and $J(n, k, d)$ are finite, with

$$V(n, k, d) \leq kd^n \,(7(kd+1))^{2^{n+1}} , \tag{1.1}$$

and

$$J(n, k, d) \leq n + n!kd^n \,(7(kd+1))^{n2^{n+1}} . \tag{1.2}$$

The proof follows the general approach of Hensley's proof, obtaining an improvement by sharpening his basic Diophantine approximation lemma. (Hensley's bound for $V(n, k, 1)$ is roughly $k(4k)^{n!+1}$.)

Any bound on $V(n, k, d)$ must have double exponential dependence on n. In §2 we generalize examples of Zaks, Perles and Wills [10] to show that for $n \geq 2$,

$$V(n, k, d) \geq \frac{k+1}{n!} \,(d+1)^{2^{n-1}-1} ,$$
$$J(n, k, d) \geq k \,(d+1)^{2^{n-1}} .$$

The bound (1.1) is probably far from the truth in its dependence on k, however, and conjectured extremal examples (see Proposition 2.6) suggest that $V(n, k, d)$ grows linearly in k as $k \to \infty$ with n and d fixed.

Exact formulae for $V(n, k, d)$ are known in a few cases. One has

$$V(1, k, d) = (k+1)\, d ,$$

and a result of Scott [9] gives

$$V(2, k, 1) = \begin{cases} \dfrac{9}{2} & \text{for } k = 1 , \\ 2(k+1) & \text{for } k \geq 2 . \end{cases}$$

The bounds of Theorem 1 immediately yield bounds applicable to a general (full rank) sublattice Λ of $\mathbb{Z}^n$. Let d be the smallest positive integer such that $d\mathbb{Z}^n \subset \Lambda$. If $\lambda_i = \min \{\lambda \in \mathbb{N} : \lambda \mathbf{e}_i \in \Lambda\}$, then $\Lambda_0 = <\lambda_1 \mathbf{e}_1, \cdots, \lambda_n \mathbf{e}_n>$ is a sublattice of Λ, and $d\mathbb{Z}^n \subseteq \Lambda$ requires $d\mathbb{Z}^n \subseteq \Lambda_0$ so that $d =$ l.c.m.

$(\lambda_1, \cdots, \lambda_n)$. Since for each i there is a basis of Λ whose first vector is $\lambda_i \mathbf{e}_i$, one has $\lambda_i | \det(\Lambda)$, so that $d | \det(\Lambda)$. If the columns of the integer matrix M are a basis of Λ then $\det(\Lambda) = |\det(M)|$, and $\mathrm{adj}(M) = |\det(M)| M^{-1}$ is an integer matrix. Furthermore $\tilde{M} = \frac{d}{\det(\Lambda)} \mathrm{adj}(M)$ is also an integer matrix, because $M\tilde{M} = dI$, and the columns of $\tilde{M}$ express a basis of the sublattice $d\mathbb{Z}^n$ of Λ in terms of the basis M of Λ, hence are integral. The linear map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ given by $\Phi(\mathbf{x}) = \tilde{M}\mathbf{x}$ has $\Phi(\mathbb{Z}^n) \subseteq \mathbb{Z}^n$ and $\Phi(\Lambda) = d\mathbb{Z}^n$, and its determinant is $d^n(\det(\Lambda))^{-1}$. If a lattice polytope $\mathbf{P}$ contains exactly $k \geq 1$ interior lattice points in Λ, then $\Phi(\mathbf{P})$ is a lattice polytope containing exactly k interior lattice points in $d\mathbb{Z}^n$, hence

$$\mathrm{Vol}\,(\Phi(\mathbf{P})) \leq V(n, k, d) \ ,$$

so that

$$\mathrm{Vol}(\mathbf{P}) \leq (\det(\Lambda)) d^{-n} \ V(n, k, d) \ , \tag{1.3}$$

and one also obtains

$$\#(\mathbf{P} \cap \mathbb{Z}^n) \leq J(n, k, d) \ . \tag{1.4}$$

The second question we study concerns the finiteness of the number of integral equivalence classes of such polytopes. The group of *lattice point preserving maps* $\mathcal{L}_n(\mathbb{Z})$ consists of those affine maps L with $L(\mathbb{Z}^n) = \mathbb{Z}^n$. They are exactly the maps $L(\mathbf{x}) = G\mathbf{x} + \mathbf{m}$ with $G \in GL(n, \mathbb{Z})$ and $\mathbf{m} \in \mathbb{Z}^n$. The subgroup $\mathcal{L}_{n,d}(\mathbb{Z})$ contains all such maps which also have $L(d\mathbb{Z}^n) = d\mathbb{Z}^n$; they consist of those maps $L \in \mathcal{L}_n(\mathbb{Z})$ having $\mathbf{m} \in d\mathbb{Z}^n$. Two polytopes $\mathbf{P}_1$ and $\mathbf{P}_2$ are *integrally equivalent* if $L(\mathbf{P}_1) = \mathbf{P}_2$ for $L \in \mathcal{L}_n(\mathbb{Z})$. Integrally equivalent polytopes have the same number of lattice points in each corresponding k-dimensional face. Two polytopes are *d-integrally equivalent* if $L(\mathbf{P}_1) = \mathbf{P}_2$ for $L \in \mathcal{L}_{n,d}(\mathbb{Z})$; such polytopes have the same number of lattice points in both $\mathbb{Z}^n$ and $d\mathbb{Z}^n$ on corresponding faces.

We establish the finiteness of the number of integral equivalence classes of lattice polytopes of bounded volume, as a consequence of the following result. A *lattice cube* is a cube with sides parallel to the coordinate axes whose vertices are lattice points.

Theorem 2. Any lattice polytope in $\mathbb{R}^n$ *of volume* $\leq V$ *is integrally equivalent under a map* $\mathbf{x} \to U\mathbf{x}$ *with* $U \in GL(n, \mathbb{Z})$ *to a lattice polytope contained in a lattice cube of side length at most* $n \cdot n!V$.

The bound of Theorem 2 is reasonably tight since the lattice simplex $\mathbf{S}_n$ with vertices $\mathbf{v}_0 = \mathbf{0}$ and $\mathbf{v}_i = \mathbf{e}_i$ for $1 \leq i \leq n-1$ and $\mathbf{v}_n = [n!V]\, \mathbf{e}_n$ has volume $\mathrm{Vol}(\mathbf{S}_n) \leq V$ and for any $L \in \mathcal{L}_n(\mathbb{Z})$ the simplex $L(\mathbf{S}_n)$ is not contained in any lattice cube of side length $\frac{1}{\sqrt{n}}\ (n!)V$.

The finiteness of the number of integral equivalence classes of lattice polytopes of volume $\leq V$ follows immediately from Theorem 2. By a translation in $\mathbb{Z}^n$ we may move the cube inside $\{(x_1, \ldots, x_n) : 0 \leq x_i \leq n \cdot n!V\}$. Since there are only finitely many lattice points in this cube, there are at most finitely many integral equivalence types of such polytopes. If we wish to preserve membership in $d\mathbb{Z}^n$ as well, this translation must be in $d\mathbb{Z}^n$ and we can move the cube into $\{(x_1, \ldots, x_n) : 0 \leq x_i \leq n \cdot n!V + d\}$. The finiteness of integral equivalence classes for lattice simplices for $n = 3$ was previously established by Reznick [8, Section 3].

We also prove several properties of maximal volume simplices contained in a convex body $\mathbf{K}$, some of which are used in the proof of Theorem 2.

Theorem 3. (a) Suppose $\mathbf{K}$ *is a closed bounded convex body in* $\mathbb{R}^n$ *with nonempty interior. Let* $\mathbf{S}$ *be any simplex of maximal volume contained in* $\mathbf{K}$, *and let* $\mathbf{s}$ *be its centroid. Then*

$$\mathbf{K} \subseteq (-n)\mathbf{S} + (n+1)\mathbf{s}\ , \tag{1.5}$$

and

$$\mathbf{K} \subseteq (n+2)\,\mathbf{S} - (n+1)\mathbf{s}\ . \tag{1.6}$$

(b) Any convex polytope $\mathbf{K}$ *contains a maximal volume simplex* $\mathbf{S}$ *whose vertices are vertices of* $\mathbf{K}$. *In particular if* $\mathbf{K}$ *is a lattice polytope then this* $\mathbf{S}$ *is a lattice simplex, and both* $(-n)\mathbf{S} + (n+1)\mathbf{s}$ *and* $(n+2)\mathbf{S} - (n+1)\mathbf{s}$ *are lattice simplices.*

The study of maximal volume simplices in a convex body goes back at least to Rado [7, pp. 242-244], who showed that the centroid $\mathbf{s}$ of a maximal volume simplex in a convex body $\mathbf{K}$ as in part (a) has the property that any chord in $\mathbf{K}$ through $\mathbf{s}$ is divided into two segments of radio $k : l$ satisfying $\frac{1}{n} \leq \frac{k}{l} \leq n$. The inclusion $\mathbf{K} \subseteq (-n)\mathbf{S} + (n+1)\mathbf{s}$ is a well-known result traceable back to

Mahler [6], pp. 111-116, and appears in Andrews [1], Lemma 2. The inclusion $\mathbf{K} \subseteq (n+2)\mathbf{S} - (n+1)\mathbf{s}$ is apparently new.

These two inclusions in part (a) are both sharp for all $n \geq 2$, in the sense that the minimal $c_n > 0$ such that $\mathbf{S} \subseteq K \subseteq c_n \mathbf{S} + (c_n - 1)\mathbf{s}$ is $c_n = n + 2$, and the minimal $|c_n|$ with $c_n < 0$ is $c_n = -n$, see the end of §4.

2. Proof of Theorem 1.

We first consider a lattice simplex $\mathbf{S}$ in $\mathbb{R}^n$ and let $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ denote the barycentric coordinates of an interior point $\mathbf{w} \in d\mathbb{Z}^n$ in $\mathbf{S}$. The basic idea (due to Hensley [5]) is to show that $\mathbf{w}$ cannot be too close to a face of $\mathbf{S}$, i.e. that its barycentric coordinates are bounded away from 0 and 1. This bounds the coefficient of asymmetry of $\mathbf{S}$ around the lattice point $\mathbf{w}$, which leads to a bound on its volume by a generalization of Minkowski's convex body theorem due to Mahler.

The lower bound in the following one-sided Diophantine approximation lemma provides the basic ingredient in the proof. This result sharpens Lemma 3.1 in Hensley [5]. (Hensley's lemma yields roughly the bound $\delta(n,d) \geq (4d)^{-n!-1}$.)

Lemma 2.1. For $d \geq 1$ let $\delta(n,d)$ be the largest constant such that for all positive real numbers $\alpha_1, \ldots, \alpha_n > 0$ satisfying

$$1 \geq \sum_{i=1}^{n} \alpha_i > 1 - \delta(n,d)$$

there exist integers $Q, P_1, \ldots, P_n$ with $Q > 0$, all $P_i \geq 0$, such that

$$(1) \qquad \sum_{i=1}^{n} \frac{P_i}{Q} = 1,$$

$$(2) \qquad \alpha_i > \frac{d P_i}{dQ + 1} \text{ for } 1 \leq i \leq n,$$

$$(3) \qquad 1 \leq dQ + 1 \leq \delta(n,d)^{-1}.$$

Then

$$\frac{d}{t_{n+1,d}-1} \geq \delta(n, d) \geq (7(d+1))^{-2^{n+1}} , \tag{2.1}$$

where $t_{n,d}$ *is determined by* $t_{1,d} = d + 1$ *and the recursion* $t_{n,d} = t_{n-1,d}^2 - t_{n-1,d} + 1$.

One can easily prove by induction on n that

$$(d+1)^{2^{n-1}} \geq t_{n,d} \geq (d+1)^{2^{n-2}} ,$$

where the lower bound is derived using $u_{n,d} = t_{n,d} - 1$, which satisfies $u_{n,d} = u_{n-1,d}^2 + u_{n-1,d}$. These inequalities show that the lower bound in (2.1) is similar in order of magnitude to the upper bound.

Proof. The upper bound in (2.1) is obtained on choosing $\alpha_i = \frac{d}{t_{i,d}}$ for $1 \leq i \leq n$. One can easily prove by induction on n that $t_{n+1,d} - 1 = d \prod_{i=1}^{n} t_{i,d}$ and

$$\sum_{i=1}^{n} \alpha_i = 1 - \frac{d}{t_{n+1,d} - 1} .$$

Now there is no approximation satisfying (1)–(3), for if there were then (2) would give $dQ + 1 > P_i t_{i,d}$ for all i. This implies that $dQ \geq P_i t_{i,d}$ since $t_{i,d} \in \mathbb{Z}$, hence

$$\frac{d}{t_{i,d}} \geq \frac{P_i}{Q} , \quad 1 \leq i \leq n .$$

Consequently

$$1 - \frac{d}{t_{n-1,d} - 1} = \sum_{i=1}^{n} \alpha_i \geq \sum_{i=1}^{n} \frac{P_i}{Q} = 1 ,$$

a contradiction.

The main content of the lemma is the lower bound in (2.1). The proof is by induction on n, holding d fixed. It's true for all d in the base case $n = 1$, on taking $\delta(1, d) = \frac{1}{d+1}$ with $Q = P_1 = 1$. The upper bound in (2.1) holds with equality for this case.

Now suppose $n \geq 2$ and that the lower bound in (2.1) is true for all values smaller than n. Reorder the α_i so that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n > 0$, and since $\sum_{i=1}^{n} \alpha_i \geq \frac{1}{2}$ (using the upper bound in (2.1)) we have $\alpha_1 \geq \frac{1}{2n}$. Let $\frac{1}{\Delta_{n,d}}$ denote a lower bound for $\delta(n,d)$, which will be determined in the proof (by (2.11) below), and choose $\Delta_{1,d} = d + 1$. We set $\sum_{i=1}^{n} \alpha_i = 1 - \mu$ with $0 < \mu < \frac{1}{\Delta_{n,d}}$.

If there is some $j < n$ such that

$$\alpha_1 + \cdots + \alpha_j > 1 - \frac{1}{\Delta_{j,d}} ,$$

then by the induction hypothesis there exists $(Q, P_1, \ldots, P_j)$ satisfying (1)–(3) for $(\alpha_1, \ldots, \alpha_j)$, and on setting $P_{j+1} = \cdots = P_n = 0$ we obtain a solution to (1)–(3) for $(\alpha_1, \ldots, \alpha_n)$. Thus we need only consider the case that

$$\alpha_{j+1} + \cdots + \alpha_n \geq \frac{1}{\Delta_{j,d}} , \quad 1 \leq j \leq n - 1 , \tag{2.2}$$

holds. Now the ordering of the α_i's gives

$$(n - j)\ \alpha_{j+1} \geq \alpha_{j+1} + \alpha_{j+2} + \cdots + \alpha_n ,$$

which with (2.2) yields

$$\alpha_{j+1} \geq \frac{1}{n\Delta_{j,d}} , \quad 1 \leq j \leq n - 1 . \tag{2.3}$$

By Minkowski's convex body theorem ([3], p. 71] there exists a nonzero lattice point in the open symmetric convex body $\mathbf{K} = \mathbf{K}(Q, P_2, \ldots, P_n)$ in $\mathbb{R}^n$ defined by

$$|Q| < R , \tag{2.4a}$$

$$|Q\alpha_i - P_i| < \min\left[\frac{1}{d}\ \alpha_i , \frac{1}{2n^2(d + 1)}\right] , \qquad i \geq 2 , \tag{2.4b}$$

provided that $\mathrm{Vol}(\mathbf{K}) > 2^n$, that is provided

$$R \prod_{i=2}^{n} \min \left[\frac{1}{d}\ \alpha_i , \frac{1}{2n^2(d + 1)}\right] > 1 . \tag{2.5}$$

Using the fact that $\alpha_i < 1/2$ for $i \geq 2$ and (2.3) we obtain, for $i \geq 2$,

$$\min\left[\frac{1}{d}\alpha_i\,,\ \frac{1}{2n^2(d+1)}\right] > \frac{\alpha_i}{n^2(d+1)} \geq \frac{1}{n^3(d+1)\Delta_{i-1,d}}\ .$$

Thus (2.5) is certainly satisfied whenever

$$R \geq n^{3n-3}(d+1)^{n-1}\prod_{i=1}^{n-1}\Delta_{i,d}\ . \tag{2.6}$$

Take a nonzero solution $(Q, P_2, \ldots, P_n)$ in **K**, and observe that $Q \neq 0$ because $Q = 0$ implies by (2.4b) that all $P_i = 0$, a contradiction. We may suppose that $Q > 0$ since $(-Q_1 - P_1, \ldots, -P_n)$ is also in **K**, and (2.4b) then shows that all $P_i \geq 0$ for $i \geq 2$.

Now define P_1 by

$$P_1 = Q - \sum_{j=2}^{n} P_j\ ,$$

which makes (1) hold. We also have by (2.4b) that

$$\begin{aligned}(dQ+1)\alpha_i &= dP_i + \alpha_i + d(Q\alpha_i - P_i)\\ &> dP_i\end{aligned} \tag{2.7}$$

for $2 \leq i \leq n$, which verifies (2) except for i=1. Next we show that $P_1 \geq 0$. If $\tilde{\alpha}_1 = \alpha_1 + \mu = 1 - \sum_{i=2}^{n}\alpha_i$, then

$$\begin{aligned}Q\tilde{\alpha}_1 - P_1 &= Q\left[1 - \sum_{i=2}^{n}\alpha_i\right] - \left[Q - \sum_{i=2}^{n}P_i\right]\\ &= -\sum_{i=2}^{n}(Q\alpha_i - P_i)\ .\end{aligned}$$

Hence using $\tilde{\alpha}_1 \geq \alpha_1 \geq \frac{1}{2n}$,

$$|Q\,\tilde{\alpha}_1 - P_1| \leq \sum_{i=2}^{n}|Q\,\alpha_i - P_i| \leq \sum_{i=2}^{n}\frac{1}{2n^2(d+1)} < \frac{1}{d+1}\tilde{\alpha}_1\ . \tag{2.8}$$

Thus P_1 is the nearest integer to $Q\,\tilde{\alpha}_1$, hence $P_1 \geq 0$.

We claim that (2) and (3) will hold provided $\Delta_{n,d}$ and R are suitably chosen. To check (2) we need only treat the case $i = 1$, by (2.7). We have, using (2.8) and (2.4a),

$$\begin{aligned}(dQ+1)\alpha_1 &= (dQ+1)\,\tilde{\alpha}_1 - (dQ+1)\mu \\ &= dP_1 + \tilde{\alpha}_1 + d(Q\tilde{\alpha}_1 - P_1) - (dQ+1)\mu \\ &\geq dP_1 + \tilde{\alpha}_1 - \frac{d}{d+1}\,\tilde{\alpha}_1 - (dR+1)\mu \\ &> dP_1 + \frac{1}{d+1}\,\tilde{\alpha}_1 - (dR+1)\,\frac{1}{\Delta_{n,d}}\,.\end{aligned}$$

This shows that (2) holds provided that

$$dR + 1 \leq \frac{1}{2n\,(d+1)}\,\Delta_{n,d}\,, \tag{2.9}$$

since $\tilde{\alpha}_1 \geq \frac{1}{2n}$. Also the inequality (2.9) guarantees that (3) holds, since $1 \leq Q \leq R$.

Thus to prove existence it suffices to choose $\Delta_{n,d}$ large enough that an R exists satisfying (2.6) and (2.9). Now (2.9) holds if

$$R \leq \frac{1}{2n\,(d+1)^2}\,\Delta_{n,d}\,.$$

This condition will allow an R for which (2.6) holds to exist provided that

$$\frac{1}{2n\,(d+1)^2}\,\Delta_{n,d} \;\geq\; n^{3n-3}\,(d+1)^{n-1}\prod_{i=1}^{n-1}\Delta_{i,d}\,. \tag{2.10}$$

Now the choice

$$\Delta_{n,d} = n^{3n}(d+1)^{n+1}\prod_{i=1}^{n-1}\Delta_{i,d}\,, \tag{2.11}$$

for $\Delta_{n,d}$ makes (2.10) hold for $n \geq 2$ and this choice completes the induction step.

To complete the proof, we show that

$$\Delta_{n,d} \leq (7(d+1))^{2^{n+1}}\,.$$

Indeed (2.11) for $n \geq 2$ gives the recursion

$$\log \Delta_{n,d} = 3n\log n + (n+1)\log(d+1) + \sum_{i=1}^{n-1}\log(\Delta_{i,d})$$

with $\Delta_{1,d} = d+1$. This recursion can be solved explicitly, yielding the following inequalities (in which the logarithms are to base 2):

$$\log \Delta_{n,d} = 3n \log n + 3\sum_{i=2}^{n-1} 2^{n-i-1}\, i \log i + (5 \cdot 2^{n-2} - 1) \log (d + 1)$$

$$< 3 \cdot 2^{n-1} \sum_{i \geq 2} 2^{-i} (i \log i) + 5 \cdot 2^{n-2} \log (d + 1)$$

$$< 3 \cdot 2^{n-1} \sum_{i \geq 2} 2^{-i}\, i(i - 1) + 5 \cdot 2^{n-2} \log(d + 1)$$

$$= 3 \cdot 2^{n+1} + 5 \cdot 2^{n-2} \log (d + 1) < 2^{n+1} \log 7(d + 1) . \quad \blacksquare$$

Hensley conjectured that the upper bound in (2.1) holds with equality for $d = 1$ and all n, and we extend this to conjecture that it holds for all n and d. The proof showed the conjecture is true for $n = 1$ and all d, and we have also verified it in the cases $(n, d) = (2, 1), (3, 1), (2, 2)$ and $(2, 3)$.

Lemma 2.2. If $\mathbf{S}$ *is a lattice simplex in* $\mathbb{R}^n$ *with* $k = \#(d\mathbb{Z}^n \cap \mathrm{Int}(\mathbf{S})) \geq 1$, *and if* $(\alpha_0, \ldots, \alpha_n)$ *are the barycentric coordinates of an interior point* $\mathbf{w}$ *in* $d\mathbb{Z}^n$ *then*

$$\delta(n, dk) \leq \alpha_i \leq 1 - n\delta(n, dk) .$$

Proof. Suppose not, so that some $\alpha_i < \delta(n, dk)$, which we may take to be α_0. Lemma 2.1 applies to $(\alpha_1, \ldots, \alpha_n)$ and the $(Q, P_1, \ldots, P_n)$ it produces satisfies

$$(jQ+1)\alpha_i > jP_i, \quad 1 \leq i \leq n$$

for $1 \leq j \leq kd$. If $\mathbf{v}_i$ are the vertices of $\mathbf{S}$ then

$$\mathbf{x}_m = (mdQ + 1)\mathbf{w} + m \sum_{i=1}^{n} dP_i \mathbf{v}_i$$

for $0 \leq m \leq k$ are distinct points in $d\mathbb{Z}^n \cap \mathrm{Int}(\mathbf{S})$, a contradiction. $\blacksquare$

Theorem 1.1 for a lattice simplex $\mathbf{S}$ follows from Lemma 2.1 and the following bound.

Lemma 2.3. Suppose that $\mathbf{S}$ *is a lattice simplex in* $\mathbb{R}^n$ *such that* $k = \#(d\mathbb{Z}^n \cap \mathrm{Int}(\mathbf{S})) \geq 1$. *Then*

$$\mathrm{vol}(\mathbf{S}) \leq \frac{1}{n!} (k + 1) d^n \delta(n, dk)^{-n} .$$

Proof: We adapt the proof of Theorem 3.4 in [5]. Let Φ be an affine map that takes $\mathbf{S}$ to the "standard simplex" $\mathbf{S}_0$ with vertices $\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_n$ in $\mathbb{R}^n$. Let $\Lambda = \Phi(\mathbb{Z}^n)$, so that Λ is a (possibly noninteger) lattice of determinant $|\det(\Phi)|$ and $\mathbf{S}$ has volume $\mathrm{vol}(\mathbf{S}) = \frac{1}{n!} |\det(\Phi)|^{-1}$.

Suppose that $\mathbf{y} \in d\mathbb{Z}^n \cap \mathrm{Int}(\mathbf{S}_0)$ and set $\mathbf{v} = \Phi(\mathbf{y}) = \sum_{i=1}^{n} \alpha_i \mathbf{e}_i$, where α_i are barycentric coordinates. The region $\mathbf{R} = \{\mathbf{v} + \mathbf{u} : |u_i| < \alpha_i \text{ for } 1 \le i \le n\}$ is centrally symmetric about $\mathbf{v}$, and $\Phi(d\mathbb{Z}^n) = \mathbf{v} + d\Lambda$ is a coset of the lattice $d\Lambda$. By van der Corput's theorem ([4], p. 51) $\mathbf{R}$ contains at least the greatest integer strictly less then $(\prod_{i=1}^{n} \alpha_i) \frac{1}{d^n} |\det(\Phi)|^{-1}$ distinct pairs of points $\mathbf{v} \pm \mathbf{u}$ where $\mathbf{u} \in d\Lambda$ is nonzero. Now let $\mathbf{u} = \sum_{i=1}^{n} u_i \mathbf{e}_i$ with $|u_i| < \alpha_i$ for all i. Then at least one of $\mathbf{v} + \mathbf{u} \in \mathrm{Int}(\mathbf{S}_0)$ or $\mathbf{v} - \mathbf{u} \in \mathrm{Int}(\mathbf{S}_0)$ if some $\alpha_i > 1/2$ and both $\mathbf{v} \pm \mathbf{u}$ are in $\mathrm{Int}(\mathbf{S}_0)$ otherwise. Thus Lemma 2.2 yields

$$k = \#(d\mathbb{Z}^n \cap \mathrm{Int}(\mathbf{S})) = \#((\mathbf{v} + d\Lambda) \cap \mathrm{Int}(\mathbf{S}_0)) \ge \frac{1}{d^n} \left\lceil \prod_{i=1}^{n} \alpha_i \right\rceil |\det(\Phi)|^{-1} - 1 ,$$

$$\ge d^{-n} \, \delta(n, kd)^n \, n! \, \mathrm{vol}(\mathbf{S}) - 1 . \qquad \blacksquare$$

To prove Theorem 1 for a general lattice polytope $\mathbf{P}$ we follow Hensley's arguments exactly. As a consequence of Lemma 2.2 one has:

Lemma 2.4. Let $\mathbf{F}$ *be a lattice polytope in* $\mathbb{R}^n$ *of dimension* $n - 1$. *Let* $\mathbf{x}_0$ *be a lattice point not in the* $(n - 1)$*-dimensional hyperplane containing* $\mathbf{F}$ *and let* $\mathbf{P}$ *be the conical lattice polytope which is the convex hull of* $\mathbf{F}$ *and* $\mathbf{x}_0$. *Suppose* $k = \#(d\mathbb{Z}^n \cap \mathrm{Int}(\mathbf{P})) \ge 1$. *If* $\mathbf{x}_1, \ldots, \mathbf{x}_m$ *are the lattice vertices of* $\mathbf{F}$ *then for any barycentric representation of* $\mathbf{y} \in d\mathbb{Z}^n \cap \mathrm{Int}(\mathbf{P})$ *as* $\mathbf{y} = \sum_{i=0}^{m} \alpha_i \mathbf{x}_i$ *with all* $\alpha_i \ge 0$, $\sum_{i=0}^{m} \alpha_i = 1$, *one has*

$$\delta(n, dk) \le \alpha_0 \le 1 - \delta(n, dk) .$$

Proof. See Hensley, Corollary 3.2. ∎

The *coefficient of asymmetry* $\sigma(\mathbf{K}, \mathbf{x})$ of a convex body $\mathbf{K}$ about a point $\mathbf{x}$ is

$$\sigma(\mathbf{K}, \mathbf{x}) = \sup_{\|y\| = 1} \frac{\max\{\lambda : \mathbf{x} + \lambda \mathbf{y} \in \mathbf{K}\}}{\max\{\lambda : \mathbf{x} - \lambda \mathbf{y} \in \mathbf{K}\}} .$$

Using Lemma 2.4 one finds that the coefficient of asymmetry $\sigma(\mathbf{P}, \mathbf{y})$ of a lattice polytope $\mathbf{P}$ having $\#(d\mathbb{Z}^n \cap \mathrm{Int}(\mathbf{P})) = k \geq 1$ about any $\mathbf{y} \in d\mathbb{Z}^n \cap \mathrm{Int}(\mathbf{P}))$ satisfies

$$\sigma(\mathbf{P}, \mathbf{y}) \leq \frac{1 - \delta(n, kd)}{\delta(n, kd)} . \tag{2.12}$$

Now we use the following extension of a theorem of Mahler (see [4], p. 52).

Theorem 2.5. If $\mathbf{K}$ *is any convex body having* $k = \#(d\mathbb{Z}^n \cap \mathrm{Int}(\mathbf{K})) \geq 1$, *such that the coefficient of asymmetry* $\sigma(\mathbf{P}, \mathbf{y})$ *about some* $\mathbf{y} \in d\mathbb{Z}^n \cap \mathrm{Int}(\mathbf{K})$ *satisfies* $\sigma(\mathbf{P}, \mathbf{y}) \leq \frac{1 - \delta}{\delta}$ *then*

$$\mathrm{Vol}\,(\mathbf{K}) \leq k \left[\frac{d}{\delta}\right]^n .$$

Proof. By rescaling coordinates by a factor of d we may suppose without loss of generality that $d = 1$, and by a further translation we may suppose that $\mathbf{y} = \mathbf{0}$. We argue by contradiction. If $\mathrm{Vol}(\mathbf{K}) > k\delta^{-n}$, then one can choose $\varepsilon > 0$ small enough that $\mathbf{K}' = (1 - \varepsilon)\mathbf{K}$ has $\mathrm{Vol}(\mathbf{K}') > k\delta^{-n}$. Then put $\mathbf{K}'' = (1 + \sigma)^{-1}\mathbf{K}' = \delta^{-1}\mathbf{K}'$, and $\mathrm{Vol}(\mathbf{K}'') > k$. By van der Corput's theorem ([4], p. 51) $\mathbf{K}''$ contains points $\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_{k+1}$ such that all $\mathbf{y}_i - \mathbf{x} \in \mathbb{Z}^n$. Now $-\frac{1}{\sigma}\mathbf{x} \in \mathbf{K}''$ by definition of $\sigma = \sigma(\mathbf{K}, \mathbf{0}) = \sigma(\mathbf{K}'', \mathbf{0})$. By convexity

$$\frac{1}{1 + \sigma}(\mathbf{y}_i - \mathbf{x}) = \frac{1}{1 + \sigma}\mathbf{y}_i + \frac{\sigma}{1 + \sigma}\left[-\frac{1}{\sigma}\mathbf{x}\right] \in \mathbf{K}'' ,$$

hence all $\mathbf{y}_i - \mathbf{x} \in \mathbf{K}'$. Since $\mathbf{K}' \subseteq \mathrm{Int}(\mathbf{K})$, there are $k + 1$ interior lattice points in $\mathbf{K}$, a contradiction. ∎

We have now completed all the work for Theorem 1. In fact, applying Theorem 2.5 to (2.12) yields

$$\mathrm{Vol}\,(\mathbf{P}) \leq kd^n\, \delta(n, kd)^{-n} ,$$

and (1.1) follows using Lemma 2.1. If **P** is a lattice simplex Lemma 2.3 gives a slightly stronger bound for $n \geq 2$.

A theorem of Blichfeldt ([2], [3] p. 69) asserts that any body **P** containing J lattice points spanning $\mathbb{R}^n$ has vol (**P**) $\geq \frac{J-n}{n!}$, which yields $J \leq n + n!\,\mathrm{vol}\,(\mathbf{P})$, and (1.2) follows. ∎

We give lower bounds for $V(n, k, d)$ and $J(n, k, d)$ by extending examples of Zaks, Perles and Wills [10]. These involve the sequences $t_{n,d}$ defined in Lemma 2.1.

Proposition 2.6. The lattice simplex $\mathbf{S}_{n,k,d}$ *having vertices* $\mathbf{v}_0 = \mathbf{0}$, $\mathbf{v}_i = t_{i,d}\mathbf{e}_i$ *for* $1 \leq i \leq n-1$, *and* $\mathbf{v}_n = (k+1)(t_{n,d}-1)\mathbf{e}_n$ *contains exactly k interior lattice points in* $d\mathbb{Z}^n$. *Hence*

$$V(n, k, d) \geq \frac{k+1}{n!}\left[\prod_{i=1}^{n-1} t_{i,d}\right](t_{n,d} - 1) = \frac{k+1}{n!}\,\frac{1}{d}\,(t_{n,d}-1)^2 \ , \tag{2.13}$$

and

$$J(n, k, d) \geq (k+1)(t_{n,d} - 1) \ .$$

This proposition gives the lower bounds stated in §1 using $t_{n,d} > (d+1)^{2^{n-2}}$ for $n \geq 2$.

Proof. We show that

$$\mathrm{Int}(\mathbf{S}_{n,k,d}) \cap d\mathbb{Z}^n = \{(d, d, \ldots, d, id) : 1 \leq i \leq k\} \ .$$

Let $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ denote the barycentric coordinates of a lattice point $\mathbf{w} = \sum_{i=0}^{n} \alpha_i \mathbf{v}_i \in d\mathbb{Z}^n$ in $\mathrm{Int}(\mathbf{S}_{n,k,d})$. By induction on i for $1 \leq i \leq n-1$ starting from $i = 1$ one shows that $\alpha_i = \frac{d}{t_{i,d}}$ using the relation

$$\sum_{j=1}^{i} \frac{d}{t_{j,d}} = 1 - \frac{d}{t_{i+1,d} - 1} \ , \tag{2.14}$$

because necessarily $\alpha_i = \frac{md}{t_{j,d}}$ for some $m \geq 1$, and choosing $m \geq 2$ gives $\sum_{j=1}^{i} \alpha_i > 1$, a contradiction. Next (2.14) allows only $\alpha_n = \frac{md}{(k+1)(t_{n,d} - 1)}$ with $1 \leq m \leq k$. Since $\alpha_0 = 1 - \sum_{j=1}^{n} \alpha_i$ one checks that these barycentric coordinates actually yield the k lattice points in $d\mathbb{Z}^n$ above. ∎

It is possible that equality holds in (2.13) for all $(n, k, d) \neq (2, 1, 1)$. This is however an open problem even for $n = 2$. Furthermore it is possible that the only lattice polytopes attaining equality in (2.13) are lattice simplices unless $(n, d) = (2, 1)$.

3. Proof of Theorem 2.

First consider the case that the polytope is a simplex $\mathbf{S}$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{Z}^n$. Consider the lattice Λ spanned by the basis vectors $\mathbf{w}_i = \mathbf{v}_i - \mathbf{v}_0$ for $1 \leq i \leq n$. Then Λ is a sublattice of $\mathbb{Z}^n$ and

$$\det(\Lambda) = [\mathbb{Z}^n : \Lambda] = n! \operatorname{vol}(\mathbf{S}) \leq n! V .$$

Let B be the integer matrix whose i^{th} row is $\mathbf{w}_i$, so that $|\det(B)| = \det(\Lambda)$. If $\mathbf{P}_0$ is the parallelepiped $\left\{ \mathbf{y} : \mathbf{y} = \sum_{i=1}^{n} y_i \mathbf{w}_i ,\ 0 \leq y_i \leq 1 \right\}$ then $\mathbf{S}$ is contained in the translated parallelepiped $\mathbf{v}_0 + \mathbf{P}_0$. Now there is a matrix $U \in GL(n, \mathbb{Z})$ taking the basis matrix to the lower-triangular form (Hermite normal form):

$$UB = \begin{bmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \tag{3.1}$$

with $0 \leq a_{ji} < a_{ii}$ for $j > i$ and all $a_{ii} > 0$ ([2], p. 13). Now $|\det(B)| = \prod_{i=1}^{n} a_{ii} \leq n! V$, hence $1 \leq a_{ii} \leq n! V$ and the parallelepiped generated by the row vectors of UB is contained in the cube $\{\mathbf{x}: 0 \leq x_i \leq n!V \ \text{ for } \ 1 \leq i \leq n\}$. The map $\mathbf{x} \to U\mathbf{x} \in \mathcal{L}_n$ takes $\mathbf{S}$ to $U\mathbf{S}$, which is contained in this parallelepiped, and thus lies in a lattice cube of side at most $n! V$.

Now suppose that $\mathbf{P}$ is an arbitrary lattice polytope. Then by Theorem 3(b) it contains a maximal volume simplex $\mathbf{S}$ which is a lattice simplex. The argument above shows that there exists a transformation $U \in GL(n, \mathbb{Z})$ such that $\mathbf{x} \to U\mathbf{x}$ maps $\mathbf{S}$ to a lattice simplex $\mathbf{S}_1$ contained in a lattice cube $\mathbf{C}$ of side $n!V$, and maps $\mathbf{P}$ to a lattice polytope $\mathbf{P}_1$. Then $\mathbf{S}_1$ is a maximal volume simplex in $\mathbf{P}_1$, so by Theorem 3(a) $\mathbf{P}_1$ is contained in the lattice simplex $(-n)\mathbf{S}_1 + (n+1)\mathbf{s}$, where $\mathbf{s}$ is the centroid of $\mathbf{S}_1$, and $(n+1)\mathbf{s} \in \mathbb{Z}^n$. Consequently $\mathbf{P}_1$ is contained in the lattice cube $(-n)\mathbf{C} + (n+1)\mathbf{s}$ of side $n \cdot n! V$. ■

4. Proof of Theorem 3.

Let $\mathbf{S}$ be any maximal volume simplex in the bounded convex body $\mathbf{K}$, and let $\mathbf{v}_0, \ldots, \mathbf{v}_n$ be the vertices of $\mathbf{S}$. By making a translation if necessary we may assume that the centroid of $\mathbf{S}$ is $\mathbf{0}$, i.e. $\sum_{i=0}^{n} \mathbf{v}_i = \mathbf{0}$. Our object is then to show that $\mathbf{K} \subseteq (n+2)\,\mathbf{S}$. Let H_i be the hyperplane spanned by all the vertices except $\mathbf{v}_i$, and let $d_i = \operatorname{dist}(\mathbf{v}_i, H_i)$. Define H_i^+, H_i^- to be the two hyperplanes parallel to H_i such that H_i^+ contains $\mathbf{v}_i$ while H_i^- is at distance d_i from H_i with H_i separating H_i^- from $\mathbf{v}_i$. We claim that $\mathbf{K}$ is contained in the closed region $\mathbf{R}_i$ between H_i^+ and H_i^-. For if $\mathbf{y} \in \mathbf{K}$ were outside this region, then the simplex spanned by $\mathbf{y}$ and all $\mathbf{v}_j$ for $j \neq i$ would have volume bigger than vol $(\mathbf{S})$, a contradiction. Hence $\mathbf{K} \subseteq \bigcap_{i=0}^{n} \mathbf{R}_i$.

We will show that

$$\bigcap_{i=0}^{n} \mathbf{R}_i = (n+2)\mathbf{S} \cap (-n)\mathbf{S} \ , \tag{4.1}$$

which implies part (a) of the theorem. Since $\mathbf{S}$ has nonzero volume, all points in $\mathbb{R}^n$ have unique barycentric coordinates $\mathbf{y} = \sum_{i=0}^{n} \beta_i \mathbf{v}_i$ with $\sum_{i=0}^{n} \beta_i = 1$. The region $\mathbf{R}_i$ is given by the barycentric coordinates:

$$\mathbf{R}_i = \{\mathbf{y} = \sum_{j=0}^{n} \beta_j \mathbf{v}_j : \sum_{j=0}^{n} \beta_j = 1 \quad \text{and} \quad |\beta_i| \le 1\} \ .$$

This is clear since if $\mathbf{y} = \sum_{j=0}^{n} \beta_j \mathbf{v}_j$ then $\operatorname{dist}(\mathbf{y}, H_i) = |\beta_i| d_i$. Hence

$$\bigcap_{i=0}^{n} \mathbf{R}_i = \{\mathbf{y} = \sum_{j=0}^{n} \beta_j \mathbf{v}_j : \sum_{j=0}^{n} \beta_j = 1 \quad \text{and all } |\beta_j| \le 1\} \ . \tag{4.2}$$

Since $\sum_{i=0}^{n} \mathbf{v}_i = \mathbf{0}$ by hypothesis,

$$(-n)\mathbf{S} = \{y = \sum_{j=0}^{n} \alpha_j(-n\mathbf{v}_j) : \sum_{j=0}^{n} \alpha_j = 1 \quad \text{and all } \alpha_j \geq 0\}$$

$$= \{y = \sum_{j=0}^{n} \beta_j \mathbf{v}_j : \sum_{j=0}^{n} \beta_j = 1 \quad \text{and all } \beta_j \leq 1\}\ , \tag{4.3}$$

where $\beta_j = -n\alpha_j + 1$. Similarly

$$(n+2)\mathbf{S} = \{y = \sum_{j=0}^{n} \alpha_j(n+2)\mathbf{v}_j : \sum_{j=0}^{n} \alpha_j = 1 \quad \text{and all } \alpha_i \geq 0\}$$

$$= \{y = \sum_{j=0}^{n} \beta_j \mathbf{v}_j : \sum_{j=0}^{n} \beta_j = 1 \quad \text{and all } \beta_j \geq -1\} \tag{4.4}$$

where $\beta_j = (n+2)\alpha_j - 1$. The equality (4.1) follows on comparing (4.2)–(4.4).

To prove part (b), let $\mathbf{P}$ be a convex polytope having nonzero volume, and we wish to show that $\mathbf{P}$ contains a maximal volume simplex whose vertices are all vertices of $\mathbf{P}$. Let $\mathbf{S}'$ be a maximal volume simplex contained in $\mathbf{P}$. If it has a vertex $\mathbf{w}'$ not a vertex of $\mathbf{P}$, consider the linear program of maximizing the (oriented) distance of a point in $\mathbf{P}$ from the hyperplane spanned by the other n vertices of $\mathbf{S}'$. Some vertex $\mathbf{w}''$ of $\mathbf{P}$ is an optimal point for this linear program, so we can replace $\mathbf{w}'$ by $\mathbf{w}''$ to obtain a new maximal volume simplex for $\mathbf{P}$ which has one fewer vertex not a vertex of $\mathbf{P}$. Continuing in this way, we eventually obtain a maximal volume simplex $\mathbf{S}$ all of whose vertices are vertices of $\mathbf{P}$.

If $\mathbf{P}$ is a lattice polytope this $\mathbf{S}$ is a lattice simplex. If its vertices are $\mathbf{v}_0, \ldots, \mathbf{v}_n$ then $(n+1)\mathbf{s} = \sum_{i=0}^{n} \mathbf{v}_i \in \mathbb{Z}^n$. Hence $(-n)\mathbf{S} + (n+1)\mathbf{s}$ and $(n+2)\mathbf{S} - (n+1)\mathbf{s}$ are lattice simplices. ■

Remarks. (1) If $\mathbf{P}$ is a lattice polytope having the maximum volume simplex $\mathbf{S}$ which is a lattice simplex, then

$$\bigcap_{i=0}^{n} \mathbf{R}_i = (n+2)\mathbf{S} \cap (-n)\mathbf{S}$$

is a lattice polytope. For (4.2) implies that its vertices are contained in the set $\{\sum_{i=0}^{n} \beta_i \mathbf{v}_i : \sum_{i=0}^{n} \beta_i = 1$ and all $\beta_i \in \{1, 0, -1\}\}$ of lattice points.

(2) The inclusion $\mathbf{K} \subset (-n)\mathbf{S} + (n+1)\mathrm{s}$ is sharp in the sense that if $\mathbf{K} \subset c_n\mathbf{S} + (1-c_n)\mathrm{s}$ for all $\mathbf{K}$ and $c_n < 0$ then $c_n \le -n$. Take $\mathbf{K}$ to be a simplex

$$\mathbf{S} = \mathrm{conv}\,(0, \mathrm{e}_1, \ldots, \mathrm{e}_n)\;.$$
$$= \{\mathrm{x} \in \mathbb{R}^n : \text{all } x_i \ge 0 \quad \text{and} \quad \sum_{i=1}^{n} x_i \le 1\}\;.$$

Then $\mathrm{s} = \left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}\right)$ and for $c_n < 0$ one has

$$c_n\mathbf{S} = \{\mathrm{x} \in \mathbb{R}^n : \text{all } x_i \le 0 \quad \text{and} \quad \sum_{i=1}^{n} x_i \ge c_n\}\;.$$

Hence

$$c_n\mathbf{S} + (1-c_n)\mathrm{s} = \{\mathrm{x} \in \mathbb{R}^n : \text{all } x_i \le \frac{1-c_n}{n+1} \text{ and } \sum_{i=1}^{n} x_i \ge \frac{1}{n+1}(n+c_n)\}\;.$$

To obtain e_1 in this region requires $c_n \le -n$.

(3) The inclusion $\mathbf{K} \subset (n+2)\mathbf{S} - (n+1)\mathrm{s}$ is sharp in the sense that if $\mathbf{K} \subset c_n\mathbf{S} + (1-c_n)\mathrm{s}$ for all $\mathbf{K}$ and $c_n > 0$ then $c_n \ge n+2$. Let

$$\mathbf{K} = \mathrm{conv}\,\{\pm\, \mathrm{e}_i : 1 \le i \le n\}$$

be the n-dimensional cross-polytope. A maximum volume simplex $\mathbf{S}$ in $\mathbf{K}$ is given by

$$\mathbf{S} = \mathrm{conv}\,\{-\mathrm{e}_1, \mathrm{e}_1, \mathrm{e}_2, \cdots, \mathrm{e}_n\}$$
$$= \{\mathrm{x} \in \mathbb{R}^n : x_2 \ge 0, \cdots, x_n \ge 0,\ \pm 1 + \sum_{i=2}^{n} x_i \le 1\}\;.$$

of volume $\frac{2}{n!}$, with centroid $\mathrm{s} = (0, \frac{1}{n+1}, \cdots, \frac{1}{n+1})$. This holds because every lattice simplex in $\mathbf{K}$ has this form after a suitable permutation of the coordinate axes, and after sending certain $x_i \to -x_i$. Now suppose $c_n > 0$ is such that $\mathbf{K} \subseteq c_n\,\mathbf{S} - (c_n - 1)\mathrm{s}$. Computation yields

$$c_n\,\mathbf{S} = \{x \in \mathbb{R}^n : x_2 \ge 0, \cdots, x_n \ge 0,\ \pm x_1 + \sum_{i=2}^{n} x_i \le c_n\}\;,$$

hence

$$c_n\mathrm{S} - (c_n - 1)\mathrm{s} = \left\{ x \in \mathbb{R}^n : x_2 \geq \frac{1 - c_n}{n+1}, \ldots, x_n \geq \frac{1 - c_n}{n+1}, \pm x_1 + \sum_{i=2}^{n} x_i \leq \frac{c_n}{n+1} + \frac{n+1}{n-1} \right\}.$$

For $n \geq 2$ the condition $-\mathbf{e}_2 \in c_n\,\mathbf{S} - (c_n - 1)\,\mathrm{s}$ requires $-1 \geq \frac{1 - c_n}{n+1}$, which is $c_n \geq n + 2$.

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