

Matroid Representations and Free Arrangements

Günter M. Ziegler

Department of Mathematics
Augsburg University
D-8900 Augsburg
West Germany

Mathematics Subject Classification (1985):

05B35 (Matroids, geometric lattices)

32C40 (Singularities)

Abstract

We show that Terao's Conjecture ("Freeness of the module of logarithmic forms at a hyperplane arrangement is determined by its abstract matroid") holds over fields with at most four elements. However, an example demonstrates that the field characteristic has to be fixed for this.

1. Free Arrangements

The present study continues an investigation of the connection between algebraic and combinatorial structures of hyperplane arrangements [15] [16] [23] [24].

Specifically, the question ("Terao's Conjecture") is studied whether freeness of the module of logarithmic vector fields at an arrangement is determined by the underlying matroid.

Here we apply representation theory for matroids to give positive answers in several important cases (binary matroids, arrangements over $\text{GF}(2)$, $\text{GF}(3)$ and $\text{GF}(4)$) where the arrangements are essentially projectively unique.

However, it is shown that freeness of the arrangements corresponding to certain matroids *does* depend on the field characteristic: freeness cannot be recognized from the matroid alone.

For this the technique of *supersolvable resolutions* is introduced, which allows freeness proofs (and disproofs) that depend on the embedding of the arrangement and thus on the specific representation of the underlying matroid.

This leaves the feeling that Terao's Conjecture may well be wrong, although for several reasons hard to disprove: on the one hand freeness is a "rigidity property", and if a free arrangement is so rigid that it is projectively unique, then it cannot give rise to a counterexample; on the other hand the insight seems to emerge that the only arrangements that, combinatorially and philosophically speaking, have "a right to be free" (in the sense of Terao) are the supersolvable ones.

Our hope is that the present note provides new insight into the combinatorial structure governing the algebraic properties of hyperplane arrangements.

Definition and Construction 1.1:

An *arrangement* (over \mathbf{k}) is a finite set $X = \{H_1, \dots, H_m\}$ of hyperplanes through the origin in a finite dimensional vector space $V = \mathbf{k}^n$. We call $n = \dim(V)$ the *dimension* and $m = |X|$ the *order* of X . The *rank* of an arrangement is the codimension of the intersection of all its hyperplanes, $r = r(X) = \text{codim}(\bigcap X)$. The arrangement is *essential* if $r(X) = n$, that is, if $\bigcap X = \{0\}$. The rank of a set $Y \subseteq X$ of hyperplanes is $r(Y) = \text{codim}(\bigcap Y)$.

The *direct sum* of two arrangements X in V and X' in V' is the arrangement $X \oplus X' = \{H \oplus V' : H \in X\} \cup \{V \oplus H' : H' \in X'\}$ in $V \oplus V'$.

For every hyperplane $H \in X$ we select a linear form $l_H \in V^*$ that defines H (that is, such that $H = \ker(l_H)$).

The abstract matroid $M = M(X)$ defined on X by linear independence on the corresponding vectors in $\{l_H : H \in X\}$ is the (*abstract*) *matroid of X* . This matroid M is *simple*: it does not have loops or multiple points. Its lattice of flats is the lattice L of intersections of the hyperplanes in X , ordered by reverse inclusion; it is called the *intersection lattice* of X [22]. Its rank is $r(L) = r(X)$. The matroid rank function of M coincides with the rank function defined above on the corresponding sets of hyperplanes.

Thus every hyperplane arrangement canonically determines a represented matroid. Conversely, for every represented simple matroid $S \subseteq V - \{0\}$ there is a canonical arrangement $X_S = \{H_v : v \in S\}$ in V^* given by $H_v = \{l \in V^* : l(v) = 0\}$.

With this Construction hyperplane arrangements are equivalent to *represented simple matroids (represented geometries)*. This will be an important point of view for much of this paper: we will study arrangements arising from representations of matroids over different fields, their uniqueness and “freeness”.

A special case is that of binary arrangements: a matroid is *binary* if it is representable over $\text{GF}(2)$. An arrangement (over an arbitrary field) is *binary* if its abstract matroid is binary. It is well known that if $\text{char}(\mathbf{k}) \neq 2$, then in this case the matroid is in fact *regular* (unimodular), that is, representable over every field.

The characteristic polynomial of L is denoted by $\chi(t) = t^r - mt^{r-1} + \dots$ and referred to as the *characteristic polynomial of X* [19], [21].

A coatom $h \in L$ ($r(h) = r - 1$) is *modular* if it intersects every line (flat of rank 2). In this case the characteristic polynomial of L factors as $\chi_L(t) = \chi_h(t) \cdot (t - e_r)$, where $\chi_h(t)$ is the characteristic polynomial of the flat h (of the interval $[\hat{0}, h]$), and e_r is the number of atoms of L that do not lie below h , that is, $e_r = |X| - |h|$. [11]

Recursively, a geometric lattice is defined to be *supersolvable* [12] if it has rank at most 1 or if it contains a modular hyperplane $h \in L$ such that $[\hat{0}, h]$ is supersolvable. In this case the characteristic polynomial factors over \mathbf{Z} as $\chi(t) = \prod_{i=1}^r (t - e_i)$, where $e_1 = 1$ and $\sum_{i=1}^r e_i = |X|$. The integers e_1, \dots, e_r are the *exponents* of L .

Definition 1.2:

An invariant of a class of arrangements is *combinatorial* if it is determined by the abstract matroids of the arrangements (equivalently, by the intersection lattices).

Let x_1, \dots, x_n be a basis of V^* , such that $S := \mathbf{k}[x_1, \dots, x_n]$ is the ring of polynomial functions on V . We call $Q := \prod_{H \in X} l_H \in S$ a *defining equation* of X (considering X as a reduced hypersurface in V). Q is a homogeneous polynomial of degree m . Let $\Omega_{alg}^1(V) = S \otimes \Lambda^1(V)$ be the S -module of algebraic differential 1-forms on V , with its natural grading by polynomial degree, decomposing as

$$\Omega_{alg}^1(V) = \bigoplus_{k \geq 0} \Omega_{alg}^{1,k}(V).$$

The S -module of *logarithmic differential 1-forms at X* [8] is defined by

$$\begin{aligned}\Omega^1(X) &= \left\{ \omega \in \frac{1}{Q}\Omega_{alg}^1(V) : d\omega \in \frac{1}{Q}\Omega_{alg}^1(V) \right\} \\ &= \left\{ \frac{\omega}{Q} : \omega = \sum_{i=1}^n q^i dx_i, q^i \in S, l_H \mid dl_H \wedge \omega \text{ for all } H \in X \right\}.\end{aligned}$$

$\Omega^1(X)$ is a \mathbf{Z} -graded module with (cf. [24, Sec. 3])

$$\Omega^1(X) = \bigoplus_{k \geq -m} \Omega^{1,k}(X),$$

where

$$\Omega^{1,k}(X) = \Omega^1(X) \cap \frac{1}{Q}\Omega_{alg}^{1,k+m}(V)$$

is the \mathbf{k} -vector space of homogeneous logarithmic forms at X of degree k .

Dually (cf. Theorem 1.5(i)), the S -module of *logarithmic vector fields at X* is

$$Der(X) = \left\{ \theta = \sum_{j=1}^n p_j \frac{\partial}{\partial x_j} : p_j \in S, l_H \mid \theta(l_H) \text{ for all } H \in X \right\}.$$

Definition 1.3: [15]

The arrangement X is *free* if $\Omega^1(X)$ is a free S -module.

The question whether freeness of an arrangement is a combinatorial property ([15, p. 293] [16, p. 565] [10] [13, p. 167]) has become known as **Terao's Conjecture**. We will see in Example 4.1 that at least *some* restrictions are necessary for the validity of this conjecture. Thus a more specific formulation of Terao's Conjecture will be needed.

Definition 1.4:

Terao's Conjecture holds ("TC holds") for a class of arrangements if for arrangements in this class, freeness is a combinatorial property.

In particular, *Terao's Conjecture holds for a matroid M* if the freeness of the arrangements corresponding to M does not depend on the specific representation of M . It *holds for a class \mathcal{M} of matroids* if it holds for the class of arrangements whose abstract matroid is in \mathcal{M} .

Terao's Conjecture holds over \mathbf{k} if it holds for the arrangements over \mathbf{k} .

In view of Example 4.1 below the strongest plausible conjecture seems to be that Terao's Conjecture holds over every field, or, equivalently, in every characteristic.

Some special cases will be treated in the following section.

Now we review some algebraic facts about the modules $\Omega^1(X)$ of logarithmic forms at arrangements X and their bases.

Theorem 1.5: [8] [15] [17] [18]

(i) $\Omega^1(X)$ is a module of rank n . Thus the size of a minimal set of generators is given by $\dim_{\mathbf{k}} \mathbf{k} \otimes \Omega^1(X) \geq n$, with equality if and only if $\Omega^1(X)$ is free.

Thus an arrangement $X_{\mathbf{K}}$ over \mathbf{K} obtained from X by extension of the ground field is free if and only if X is free, since $\Omega^1(X_{\mathbf{K}}) = \mathbf{K} \otimes \Omega^1(X)$.

(ii) There is a nondegenerate pairing

$$\text{Der}(X) \times \Omega^1(X) \longrightarrow S$$

sending (θ, ω) to $\omega(\theta)$; thus $\text{Der}(X)$ and $\Omega^1(X)$ are reflexive S -modules, dual to each other. In particular $\Omega^1(X)$ is a free module if and only if $\text{Der}(X)$ is free.

Dual bases $\{\theta_1, \dots, \theta_n\}$ with $\theta_i = \sum_j p_{ij} \frac{\partial}{\partial x_j}$ and $\{\omega_1, \dots, \omega_n\}$ with $\omega_i = \sum_j q^{ij} dx_j$ are characterized by $\omega_i(\theta_j) = \delta_{ij}$, that is, $(\underline{Q}^T)^{-1} = \underline{P}$ for the coefficient matrices $\underline{P} = (p_{ij})$ and $\underline{Q} = (q^{ij})$.

Thus if X is free and $r(X) = n$, then the elements of a homogeneous basis of $\Omega^1(X)$ all have negative degree.

(iii) $\{\omega_1, \dots, \omega_n\} \subseteq \Omega^1(X)$ is a basis of $\Omega^1(X)$ if and only if $\omega_1 \wedge \dots \wedge \omega_n = c \frac{1}{Q} dx_1 \wedge \dots \wedge dx_n$ for some $c \in \mathbf{k} - \{0\}$, that is, $\det(\underline{Q}) = \frac{c}{Q}$ in the notation of (i).

(iv) If $\Omega^1(X)$ is free, then it has a homogeneous basis $\{\omega_1, \dots, \omega_n\}$, where ω_i is homogeneous of degree e_i for $1 \leq i \leq n$.

The nonnegative integers e_i ($1 \leq i \leq n$) do not depend on the homogeneous basis chosen. They are called the exponents of X .

If X is free, then $\chi(t) = \prod_{i=1}^n (t - e_i)$. (This is Terao's celebrated "Factorization Theorem".)

There are two large classes of free arrangements "occurring in nature" that are known to be free: Coxeter arrangements (and more generally arrangements corresponding to groups generated by reflections in good characteristic, see [8], [18]) and supersolvable arrangements [10] [5]. In fact, the following result holds:

Theorem 1.6: [24]

An arrangement is supersolvable if and only if in suitable coordinates it admits a triangular basis $\{\omega_1, \dots, \omega_n\}$ with $\omega_i = \frac{1}{Q} \sum_{j \geq i} q^{ij} dx_j$.

In this case, by Theorem 1.5(iv), the exponents of X coincide with the exponents of L defined as the zeroes of the characteristic polynomial.

Note that in particular arrangements in dimension 2 are always supersolvable and thus free.

Furthermore two important constructions preserve freeness: localization at a subspace, forming sums of arrangements, and addition/deletion of a hyperplane under suitable numerical conditions.

Theorem 1.7: [15]

(i) Let X be a free arrangement in V and let W be a subspace of V . Then the localization of X at W , given by $X^W = \{H \in X : W \subseteq H\}$, is again a free arrangement in V . The direct sum $X \oplus X'$ of two arrangements over \mathbf{k} is free if and only if both X and X' are free.

(ii) Let X be an arrangement in V and let $H \notin X$ be a hyperplane. Then for $1 \leq i \leq n$ every two of the following statements imply the third:

- * X is free with exponents $e_1, \dots, e_i, \dots, e_n$.
- * $X \cup \{H\}$ is free with exponents $e_1, \dots, e_i + 1, \dots, e_n$.
- * $X|_H$ is free with exponents $e_1, \dots, \hat{e}_i, \dots, e_n$,

where $X|_H = \{K \cap H : K \in X\}$ is the restriction of X to H .

Furthermore, when X and $X \cup \{H\}$ are both free, then for some i the above three statements hold.

The proof of part (i) is simple commutative algebra. See [9] for the statement about direct sums. Observe that localization of an arrangement corresponds to a restriction of the underlying matroid to a flat. The question (“Orlik’s Conjecture”) whether freeness is always preserved under restriction to a hyperplane in the arrangement is still open, although some positive evidence exists.

Part (ii) of Theorem 1.7 is Terao’s “Addition/Deletion Theorem”. Its original proof required heavy commutative algebra tools. For simpler proof variants see [24, Section 5]. The following proposition is a basic technical fact used in the following. Its special case $|X_1| = 1$ is [24, Lemma 6.1], which was a direct Corollary to the Strong Preparation Lemma.

The general case uses the notion of “line closure” from [4]. We define a subset $Y \subseteq X$ to be *line closed in X* if with two points of a line Y always contains the whole line.

Proposition 1.8:

Let $X = X_0 \cup X_1$, where X_0 and X_1 are line closed in X . Then

$$\Omega^1(X) = \Omega^1(X_0) + \Omega^1(X_1).$$

Proof.

We use induction on $|X_1|$, where the case $X_1 = \emptyset$ is trivial.

Let $\omega \in \Omega^1(X)$ and $H \in X_1 - X_0$. Then $l_H \omega \in \Omega^1(X - \{H\})$ can by induction be written as $l_H \omega = \omega_0 + \omega_1$ with $\omega_0 \in \Omega^1(X_0)$ and $\omega_1 \in \Omega^1(X_1 - \{H\})$.

Now ω is logarithmic at H , and thus $\omega_0 + \omega_1|_H = 0$. But this implies

$$\omega_0|_H = -\omega_1|_H \in \Omega^1(X_0|_H) \cap \Omega^1(X_1|_H) = \Omega^1_{alg}(H),$$

because $X_0|_H \cap X_1|_H = \emptyset$: there is no 3-circuit $\{H_0, H_1, H\}$ in X with $H_0 \in X_0$, $H_1 \in X_1 - \{H\}$.

Now choose $\omega' \in \Omega_{alg}^1(V)$ such that $\omega' \Big|_H = \omega_0 \Big|_H = -\omega_1 \Big|_H$.
Then $\omega_1 + \omega' \in \Omega^1(X_1)$ with $\omega_1 + \omega' \Big|_H = 0$ and thus

$$\frac{1}{l_H}(\omega_1 + \omega') \in \Omega^1(X_1).$$

Similarly, we get

$$\frac{1}{l_H}(\omega_0 - \omega') \in \Omega^1(X_0 + \{H\}),$$

and hence

$$\Omega^1(X) = \Omega^1(X_0 \cup \{H\}) + \Omega^1(X_1).$$

But H is not contained in a 3-circuit of $X_0 \cup \{H\}$, hence by [24, Lemma 6.1] we have

$$\Omega^1(X_0 \cup \{H\}) = \Omega^1(X_0) + \Omega^1(\{H\}),$$

and from $\Omega^1(\{H\}) \subseteq \Omega^1(X_1)$ the claim follows. \square

Lemma 1.9:

If X is free and $H \in X$, then $\Omega^1(X)$ has a basis that contains $\frac{dl_H}{l_H}$.

Proof.

We choose coordinates such that $l_H = x_1$.

Let $\theta_1, \dots, \theta_n$ be a basis for the dual module $\Omega^1(X)^* = \text{Der}(X)$ (see [8], [15]) with $\theta_i = \sum_{j=1}^n p_{ij} \frac{\partial}{\partial x_j}$ (for $1 \leq i \leq n$, $p_{ij} \in S$).

Then by [15, p. 296] we may assume that θ_1 is the Euler vector field $\theta_1 = \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}$.

Now $\theta_i \in \text{Der}(X)$ implies $l_H \Big|_{\theta_i(l_H)}$, that is, $x_1 \Big|_{p_{i1}}$ for $1 \leq i \leq n$. Thus, putting $\theta'_1 := \theta_1$ and $\theta'_i = \theta_i - \frac{p_{i1}}{x_1} \theta_1$ for $1 < i \leq n$, we get a basis

$$\{\theta'_i = \sum_{j=1}^n p'_{ij} \frac{\partial}{\partial x_j} : 1 \leq i \leq n\}$$

with $p'_{i1} = 0$ for $i > 1$.

The dual basis $\{\omega'_i : 1 \leq i \leq n\}$ then contains $\omega'_1 = \frac{dx_1}{x_1} = \frac{dl_H}{l_H}$.

(The essentially same argument was before used in [25].) \square

2. Projective uniqueness

For large classes of matroids, Teramo's Conjecture over a fixed field turns out to be true for the geometric reason that they are (essentially) projectively unique. In this section, we will treat three main cases of this: binary matroids (represented over arbitrary fields), matroids over $\text{GF}(3)$ and matroids over $\text{GF}(4)$.

Definition 2.1:

- (i) A representation of a simple matroid of rank r is an order preserving injection $\phi : L \rightarrow \text{PG}(r - 1, \mathbf{k})$ of (the lattice of flats of) the matroid L to the subspace lattice of \mathbf{k}^r .
- (ii) A matroid L is projectively unique over \mathbf{k} if every two such representations of L over \mathbf{k} coincide up to an automorphism of $\text{PG}(r - 1, \mathbf{k})$, that is, for any two representations $\phi_1, \phi_2 : L \rightarrow \text{PG}(r - 1, \mathbf{k})$ there is an automorphism ρ of $\text{PG}(r - 1, \mathbf{k})$ such that $\rho \circ \phi_1 = \phi_2$.

Observe that the condition in (ii) is vacuous if the matroid considered is not representable over \mathbf{k} : for ease of notation we will always consider a matroid that is not representable over a given field as projectively unique over that field.

To proceed, we need some information about the automorphisms of $\text{PG}(r - 1, \mathbf{k})$. This is provided by the following corollary to the “(First) Fundamental Theorem of Projective Geometry.”

Lemma 2.2: [1, p. 88] [2, p. 44]

The automorphism group of $\text{PG}(r - 1, \mathbf{k})$ is generated by the linear automorphism group $\text{PGL}(r - 1, \mathbf{k})$ (induced from the canonical action of the general linear group on \mathbf{k}^r) together with the automorphisms induced by field automorphisms of \mathbf{k} .

From this we get the following statement.

Proposition 2.3:

If L is projectively unique over \mathbf{k} , then Terao’s Conjecture holds for L over \mathbf{k} .

Proof.

Automorphisms of $\text{PG}(r - 1, \mathbf{k})$ (as described in Lemma 2.2) induce semilinear automorphisms of $\Omega^1(X)$ and do not affect freeness of this module over S . \square

Thus statements about the projective uniqueness of matroids translate into proofs of Terao’s Conjecture for special classes of matroids.

In particular, the classical

Proposition 2.4: [3]

- (i) Binary matroids are projectively unique over every field.
- (ii) Matroid representations over $\text{GF}(3)$ are projectively unique.

translates into

Theorem 2.5:

- (i) Terao’s Conjecture holds for binary arrangements over fields of fixed characteristic.
- (ii) Terao’s Conjecture holds over $\mathbf{k} = \text{GF}(3)$.

For part (i) we use the facts that freeness is stable under field extension (Theorem 1.5(i)) and that any two fields of the same characteristic have isomorphic extension fields (this is a well-known algebraic fact: the tensor product of the two fields over the prime field modulo any maximal ideal will do).

Now, as the four point line shows, the representation of matroids is not unique over any field other than $\text{GF}(2)$ and $\text{GF}(3)$.

However, over $\text{GF}(4)$ enough uniqueness survives. In fact, we can use the following highly non-trivial result recently obtained by J. Kahn.

Theorem 2.6: [7]

3-connected matroids are projectively unique over $\text{GF}(4)$.

As before, this immediately implies Terao's Conjecture for 3-connected matroids over $\text{GF}(4)$. However, we can prove more.

Theorem 2.7:

Terao's Conjecture holds over $\text{GF}(4)$.

Proof.

We can assume that $r(X) = n$ and that X is not a direct sum of arrangements, (that is, the matroid M is connected), because a direct sum of two arrangements is free if and only if both summands are free, by Theorem 1.7(i).

If L is 3-connected, then we are done by Theorem 2.6. Otherwise, we get a 2-separation $X = X_0 \cup X_1$, with $r(\overline{X_0} \cap \overline{X_1}) \leq 1$.

Case 1: If $r(\overline{X_0} \cap \overline{X_1}) = 0$, then $\overline{X_0} \cap \overline{X_1} = \emptyset$, and we get from Proposition 1.8 that

$$\Omega^1(X) = \Omega^1(X_0) + \Omega^1(X_1).$$

Now assume that X was free. Then for $i = 0, 1$ the module $\Omega^1(X_i)$ is free, thus the negative degree part of $\mathbf{k} \otimes \Omega^1(X_i)$ has dimension $r(X_i)$. With $\Omega^1(X_0) \cap \Omega^1(X_1) = \Omega_{alg}^1(V)$ this implies

$$\dim \Omega^1(X) \geq r(X_0) + r(X_1) > r(X) = n.$$

Hence $\Omega^1(X)$ can never be free, and thus TC holds for L .

Case 2: If $r(\overline{X_0} \cap \overline{X_1}) = 1$, we can assume $\overline{X_0} \cap \overline{X_1} = \{H\}$ with $H \in X_1$, and put $X_2 := X_0 \cup \{H\}$. Now TC holds for X_1 and X_2 by induction (on the rank).

Now if X is free, then so are X_1 and X_2 : they are localizations of X (Theorem 1.7(i)). Conversely, if X_1 and X_2 are free, we may by Lemma 1.9 choose bases \mathcal{B}_i for $\Omega^1(X_i)$ with $\frac{dH}{H} \in \mathcal{B}_i$, for $i = 1, 2$. Then $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for $\Omega^1(X)$ (Basis Criterion of Theorem 1.5(iii)). With this we know that X is free iff X_1 and X_2 are both free, and thus TC holds for X . \square

The argument in this last proof is in fact more general. It shows that for every class of matroids that is closed under deletions (in particular, for arrangements over a fixed field), it is sufficient to prove Terao's Conjecture for 3-connected matroids, or, equivalently, that every minimal counterexample to Terao's Conjecture is 3-connected.

The observations by Kahn [7] about representations over larger fields also apply to our problem: projective uniqueness breaks down over $\text{GF}(q)$ for $q \geq 5$, and sufficiently high connectivity of the matroid is not sufficient to guarantee it.

In fact, although free arrangements seem to be “rigid” in some vague sense, they are by no means projectively unique in general. For example, for supersolvable arrangements, arbitrary generic points on the modular line can be added without destroying supersolvability.

We believe that Terao’s conjecture over larger fields is in fact false, and it should be possible to construct counterexamples that have a finite representation space. If the construction of representations for such a matroid uses the existence of zeroes of a quadratic polynomial, and if the zeroes of this polynomial lie in the prime field of \mathbf{k} , then it is conceivable that *one* of the representations is free by supersolvable resolution, whereas the other is non-free. The same could as well happen over \mathbf{Q} , \mathbf{R} , or \mathbf{C} .

3. Supersolvable Resolutions

Definition 3.1: [15, p. 307]

An arrangement X in V is *inductively free* if there is a sequence

$$\emptyset = X_0 \subseteq X_1 \subseteq \dots \subseteq X_m = X$$

of subarrangements such that X_i is free and $|X_i| = i$ for all $0 \leq i \leq m$.

Being inductively free is a combinatorial property for arrangements of dimension $n \leq 3$: this is clear with Terao’s Addition/Deletion Theorem 1.7(ii) and the fact that exponents are combinatorial (Theorem 1.5(iv)). If Orlik’s Conjecture is true, then the restriction $n \leq 3$ can be dropped.

Definition 3.2:

Let X be an arrangement in V . A *supersolvable resolution* of X is a finite sequence

$$X = Y_0 \subset Y_1 \subset \dots \subset Y_k = Y$$

such that $|Y_i| = |X| + i$ for $0 \leq i \leq k$, and Y is supersolvable.

Trivially every supersolvable arrangement has a supersolvable resolution. On the other hand, the lattice of all subspaces of a finite dimensional vector space V is modular and hence supersolvable; however, the set of all (central) hyperplanes in V is not finite unless V itself is finite. This motivates the following lemma:

Lemma 3.3:

Every arrangement has a supersolvable resolution.

Proof.

Let X be an arrangement with $r(X) = n$. It is sufficient to show that X is contained in a finite supersolvable arrangement Y . Given X , choose an arbitrary line (1-dimensional subspace) l in V , and let X' be the arrangement in V obtained from X by adding all the hyperplanes in V through l and an intersection of two planes in X which do not contain l , that is,

$$X' = X \cup \{ \text{span}\langle l, H' \cap H'' \rangle : H' \neq H''; l \not\subseteq H' \cap H''; H', H'' \in X \}.$$

In X' , l is a modular line. Now, by induction on the dimension, we can add a finite number of planes containing l to X' , such that in the resulting arrangement Y , the localization Y/l is supersolvable. But with Y/l supersolvable and l a modular coatom of $L(Y)$, Y is supersolvable – this follows from [12]. \square

This proof with suitable choice of l sometimes produces *small* supersolvable resolutions, which can be used to prove or disprove freeness of an arrangement.

Definition 3.4:

An arrangement X in V is called free by supersolvable resolution if it has a supersolvable resolution consisting of free arrangements.

With this, we get the following technique to check freeness that works especially well for arrangements of rank 3. Let X be an arrangement with characteristic polynomial $\prod_{i=1}^n (t - e_i)$. If X is free, then these e_i are the exponents of X . Now we attempt to find a supersolvable resolution $X = Y_0 \subset \dots \subset Y_k = Y$ such that for every i , the exponents of the restriction $Y_{i-1}|_{H_i}$ ($Y_i = Y_{i-1} \cup \{H_i\}$, $1 \leq i \leq k$) form a submultiset of the exponents of Y_{i-1} , that is, $\chi(Y_{i-1}|_{H_i}) \mid \chi(Y_{i-1})$. This is easy to check when $Y_{i-1}|_{H_i}$ is 2-dimensional. By Addition Theorem 1.7(ii), we are then done. Conversely, if $X \subseteq Y$, Y is free (e.g., supersolvable) and $\chi_X(t)$ factors over \mathbf{Z} , then we know that $e_i(X) \leq e_i(Y)$, in some ordering of the exponents, by the argument used to prove [24, Proposition 5.2]. This sometimes allows to see that X is not free. We demonstrate this technique with a sequence of examples.

Example 3.5: [6] [5]

The arrangements D_n^k defined over a field \mathbf{k} with $\text{char}(\mathbf{k}) \neq 2$ by

$$Q_n^k = \prod_{i=1}^k x_i \prod_{1 \leq i < j \leq n} (x_i^2 - x_j^2)$$

are free by supersolvable resolution with exponents $[1, 3, \dots, 2n-3, n+k-1]$. For this we note that D_n^n is supersolvable with exponents $[1, 3, \dots, 2n-3, 2n-1]$ – this is the Coxeter arrangement B_n . For arbitrary k , we hence get a supersolvable resolution $D_n^0 \subset D_n^1 \subset$

... $\subset D_n^n$. We observe that $D_n^k|_{\{x_k=0\}} \cong D_{n-1}^{n-1}$ has exponents $[1, 3, \dots, 2n-3]$ for all k by induction on n , thus we are done.

[5] proved the slightly stronger fact that the arrangements D_n^k are actually inductively free, but our argument for freeness is simpler. D_n^k is not supersolvable in general, as can be seen from the Coxeter arrangements $D_n = D_n^0$ for $n \geq 4$ and from D_3^1 : this arrangement corresponds to the non-Fano matroid. These examples show that over every field with characteristic different from 2 there are free arrangements that are not supersolvable.

The following is an example of an arrangement that is free by supersolvable resolution, but *not* inductively free:

Example 3.6:

Let T be the arrangement in \mathbf{R}^3 sketched by [15, p. 311]. We define it by the following matrix

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ t & t+1 & 1 \\ -t & t+1 & 1 \\ t & -(t+1) & 1 \\ -t & -(t+1) & 1 \\ t^2 & 0 & 1 \\ -t^2 & 0 & 1 \\ t & 1-t & 1 \\ t & t-1 & 1 \\ -t & 1-t & 1 \\ -t & t-1 & 1 \end{pmatrix}$$

where $t^2 - 2t - 1 = 0$ has to be satisfied. Note that with this T is representable over $\mathbf{Q}[\sqrt{2}]$ (with $t_{1/2} = 1 \pm \sqrt{2}$), but *not* over \mathbf{Q} . Figure 1 is an affine representation of the real arrangement obtained for $t = 1 + \sqrt{2}$, whereas the Figure in [15, p. 311] corresponds to $t = 1 - \sqrt{2}$.

The characteristic polynomial for T is computed as $\chi_T(t) = (t-1)(t-9)(t-9)$. However, every restriction $T|_H$ is free with exponents $[1, 7]$. Thus, T is not inductively free. Now use the supersolvable resolution

$$T = Y_0 \subset Y_1 \subset Y_2 \subset \dots \subset Y_6 = Y$$

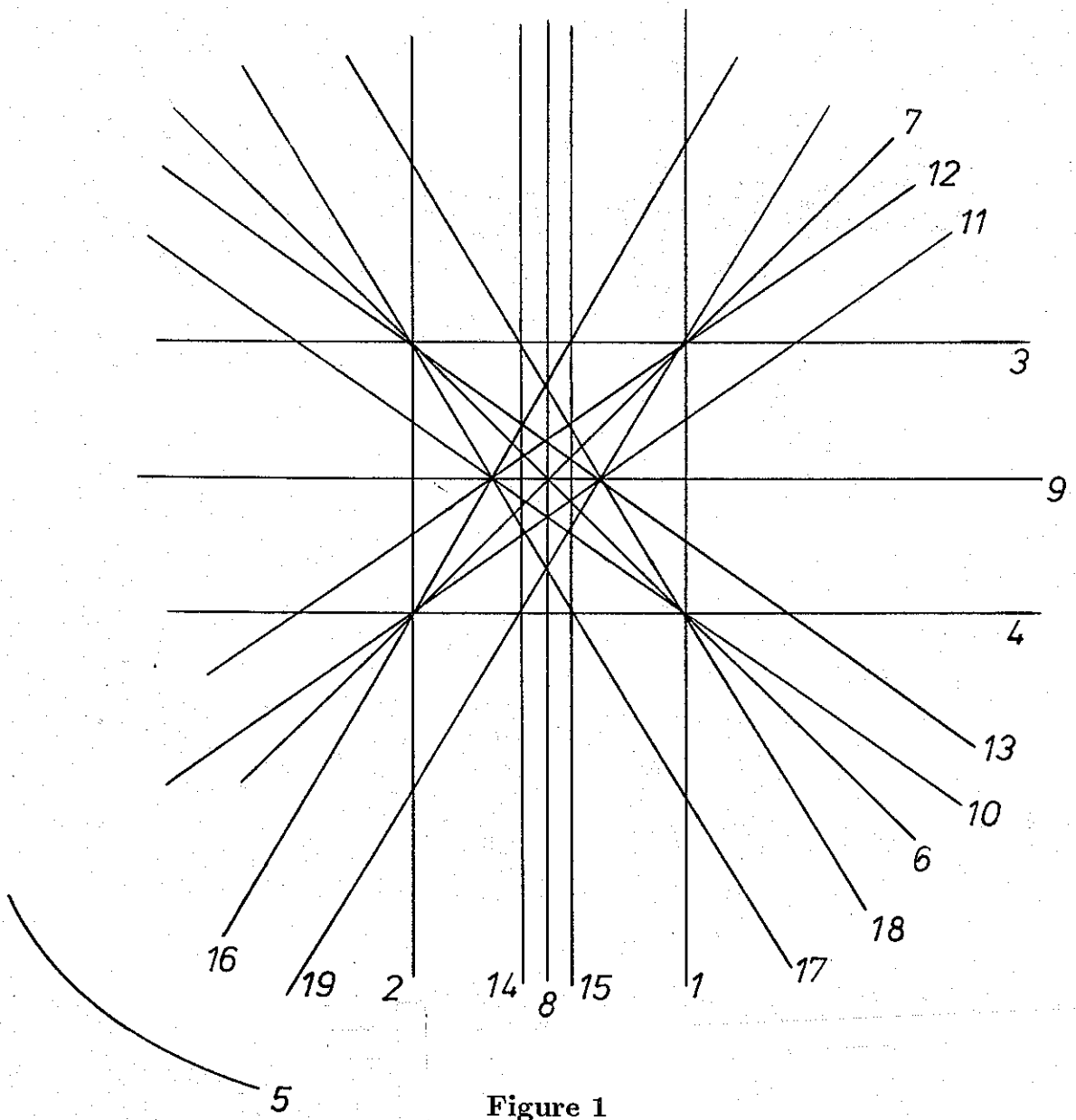


Figure 1

with $Y_i = Y_0 \cup \{H_i\}$, where the H_i are the “vertical” hyperplanes given by

$$\begin{pmatrix} t & 0 & 1 \\ -t & 0 & 1 \\ 1-2t & 0 & 1 \\ 2t-1 & 0 & 1 \\ -t & 0 & t+2 \\ t & 0 & t+2 \end{pmatrix}$$

Then Y is supersolvable (and thus free) with exponents 1, 13, 11, and in fact repeated application of the Addition/Deletion Theorem yields that T is free by supersolvable resolution with exponents 1, 9, 9.

Note that T is *projectively unique* over $\mathbf{Q}[\sqrt{2}]$, but not over \mathbf{R} or \mathbf{C} . Since freeness is stable under field extension by Theorem 1.5(i), the matroid of T cannot produce a counterexample to TC in characteristic 0. It does not produce one in positive characteristic, either: over fields like $\text{GF}(17)$, $\text{GF}(23)$, $\text{GF}(31)$ etc. that contain a root of $t^2 - 2t - 1$ and admit representations of T the supersolvable resolution described below always exists, and T is again free by supersolvable resolution.

Example 3.7

Let X be an arrangement consisting of a pencil of five planes in \mathbf{k}^3 and two extra planes whose intersection is not contained in a plane of the pencil (cf. Figure 2). Then X is not supersolvable, but its characteristic polynomial splits as $\chi(t) = (t-1)(t-3)^3$ [12, Example 4.7].

To see that X cannot be free, we observe that X has a supersolvable resolution of length 1 given by

$$X \subseteq X \cup \{H\} = Y,$$

where H is the plane spanned by the axis of the pencil and the intersection line of the two extra planes.

Now Y is supersolvable (and thus free) with $\chi_Y(t) = (t-1)(t-5)(t-2)$, and thus X cannot be free by Theorem 1.7(ii).

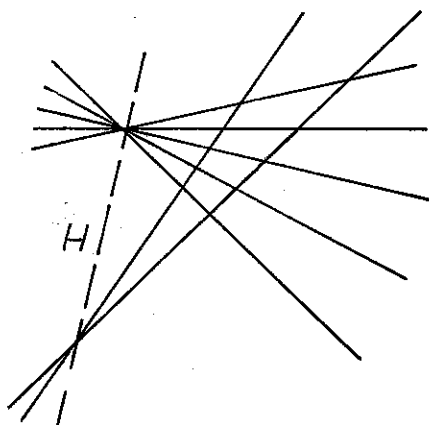


Figure 2

4. Two Examples

The following extended example illustrates several phenomena that can occur when a matroid is represented over fields of different characteristics:

- The freeness of an arrangement can depend on the field characteristic.
- The existence of specified supersolvable resolutions can depend on the characteristic of the field, even if the matroid is projectively unique over every field considered.
- There are free, non-supersolvable arrangements over fields of characteristic 2.
- There are free, not inductively free arrangements of nine respectively ten hyperplanes that are even representable in characteristic 2. (The only previously published example is the arrangement of 19 planes in Example 3.6, which is not representable over small fields)

Example 4.1:

Let $M = AG(2, 3)$ be the matroid of the affine plane of order 3. This matroid (and the corresponding hyperplane arrangement) can be represented over a field \mathbf{k} if and only if \mathbf{k} contains a root of

$$\omega^2 - \omega + 1.$$

Every such ω determines a coordinatization by (compare [14, p. 19]):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & \omega & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ \omega & \omega & 1 \\ \omega & 0 & 1 \\ 1 & \omega & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Thus M is coordinatizable in every characteristic; in particular it is representable over $GF(3)$ (with $\omega = -1$), over $GF(4)$, and over larger prime fields like $GF(13)$ (with $\omega = 4, -3$), $GF(19)$ (with $\omega = 8, -7$), etc.

Note that in these last two cases, there is no automorphism of \mathbf{k} that carries ω to ω^{-1} , and thus the representations of M over such fields as $GF(13)$ and $GF(19)$ are not projectively unique. (However, this changes if automorphisms of the matroid are admitted in the definition of projective uniqueness).

M is also realizable over $\mathbf{Q}[\sqrt{-3}]$ and thus over \mathbf{C} , but it is not realizable over \mathbf{R} or any subfield of \mathbf{R} : in fact, it would violate the Sylvester-Gallai Theorem there.

The characteristic polynomial of M is

$$\chi(t) = (t - 1)(t - 4)^2.$$

Case 1:

If \mathbf{k} is a field of characteristic 3, then M is representable over \mathbf{k} and projectively unique. The corresponding hyperplane arrangement X_M has a supersolvable resolution of length 1: in fact, adding in an extra hyperplane given by the vector $A = (1\ 1\ 0)$, we get a supersolvable arrangement with exponents 1, 3, 6. (as sketched in Figure 3a). Thus it follows from the Addition/Deletion Theorem 1.7(ii) that X_M cannot be free. For this construction to work, the characteristic of \mathbf{k} has to be 3, because otherwise the lines (123), (456) and (789) of the matroid cannot be extended by a common point.

Case 2:

If $\text{char}(\mathbf{k}) \neq 3$, then every arrangement X_M obtained from a representation of M over \mathbf{k} has a supersolvable resolution of length 2: we first add in the point $A = (1\ 1\ 0)$ (on the intersection of (123) and (456)) and then the point $B = (-\omega\ 1\ 0)$ (on the intersection of (123) and (789)), as in Figure 3b.

This new arrangement is supersolvable with exponents 1,4,6 and with the Addition/Deletion Theorem it is easily checked that X_M is in fact free by supersolvable resolution, with exponents 1,4,4. The Addition/Deletion Theorem in fact implies that $X_M - \{H\}$ is *not* free for any $H \in X_M$. Thus in particular X_M is not inductively free in the sense of Definition 3.1.

The standard example of a free, not supersolvable arrangement corresponds to the non-Fano plane F^- , which is denoted $A_1(7)$ in [15], D_3^+ in [5] (see Example 3.5), and X_0 in [23] and [24]. This arrangement of rank 3 is in fact inductively free [15] [5].

However, it is well known that F^- is realizable over a field \mathbf{k} if and only if $\text{char}(\mathbf{k}) \neq 2$. It turns out that *all* published examples of free, not supersolvable arrangements are not binary. The matroid of Example 4.1 is representable in characteristic 2, but not over $\text{GF}(2)$.

Our second example provides a free arrangement over $\text{GF}(2)$ that is not supersolvable. It has rank 5 (as compared to the many examples of rank 3 over other fields) and considerable size. This is necessary, because binary free arrangements of rank at most four are always supersolvable: in the case $r = 3$ this follows from [24, Corollary 7.6] together with the fact that 3-point lines in binary matroids are always modular, and in the case $r = 4$ from a detailed analysis of the possible binary critical arrangements as defined in [24, Section 8]. Compare [26] for the combinatorial structure of supersolvable binary matroids.

Example 4.2:

Let M_1 be the geometry over $\text{GF}(2)$ of rank 5 obtained by deleting from $\text{PG}(4, 2)$ the points (00001), (00011), (00101), (01001), (10001), (11001) and (10101). Since none of the deleted points is on the hyperplane $H = \{x_5 = 0\}$, we get that M_1 is supersolvable with exponents 1,2,4,8,9. In fact H is easily checked to be the only hyperplane from which no points have been deleted, and hence (in view of the exponents) the only modular hyperplane of M_1 . The corresponding arrangement X_1 over $\text{GF}(2)$ is supersolvable and thus free with exponents 1,2,4,8,9.

Now let M_2 be the geometry obtained from M_1 by deleting $x_1 = (11110)$, and X_2 the corresponding arrangement. Let $H_1 \in X_1 - X_2$ be the hyperplane corresponding to x_1 . Then since no two different points of $\text{PG}(4, 2) - M_1$ lie on the same line through x_1 , we get that $X_2|_{H_1}$ corresponds to the matroid $\text{PG}(4, 2)$ and is thus free with exponents 1,2,4,8.

Thus from Terao's Addition/Deletion Theorem 1.7(ii) we can conclude that X_2 is free with exponents 1,2,4,8,8.

However, X_2 does not have a modular hyperplane, because in view of the exponents every modular hyperplane has to be complete (with exponents 1,2,4,8), and M_2 does not contain a complete hyperplane of $\text{PG}(4, 2)$.

5. References

- [1] **E. Artin:** *Geometric Algebra*, Interscience Publishers, 1957.
- [2] **R. Baer:** *Linear Algebra and Projective Geometry*, Academic Press, 1952.
- [3] **T. H. Brylawski and D. Lucas:** *Uniquely representable combinatorial geometries*, Proc. 1973 Rome International Colloq. Combinatorial Theory I, Accademia Nazionale dei Lincei, Rome, 1976, 83-104.
- [4] **M. D. Halsey:** *Line closed combinatorial geometries*, Discrete Mathematics **65** (1987), 245-248.
- [5] **M. Jambu and H. Terao:** *Free arrangements of hyperplanes and supersolvable lattices*, Advances in Mathematics **52** (1984), 248-258.
- [6] **P. Orlik and L. Solomon:** *Coxeter arrangements*, Proc. Arcata Symposium on Singularities (ed. P. Orlik), Proc. Symp. Pure Math. **40** (1983), Part 2, 269-291.
- [7] **J. Kahn:** *On the uniqueness of matroid representations over $\text{GF}(4)$* , Bull. London Math. Soc. **20** (1988), 5-10.
- [8] **K. Saito:** *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Science, University of Tokyo, Sci. IA **27** (1980), 265-291.
- [9] **L. Solomon and H. Terao:** *A formula for the characteristic polynomial of an arrangement*, Advances in Mathematics **64** (1987), 305-325.
- [10] **R. P. Stanley:** *T-free arrangements of hyperplanes*, in: *Progress in Graph Theory* (J. A. Bondy and U. S. R. Murty, eds.), Academic Press, 1984, 539.
- [11] **R. P. Stanley:** *Modular elements of geometric lattices*, Algebra Universalis **1** (1971), 214-217.
- [12] **R. P. Stanley:** *Supersolvable lattices*, Algebra Universalis **2** (1972), 197-217.

- [13] **R. P. Stanley:** *Enumerative combinatorics, Volume I*, Wadsworth, 1986.
- [14] **B. Sturmfels:** *Computational algebraic geometry of projective configurations*, in *Symbolic Computations in Geometry* (B. Sturmfels and N. White, ed.), University of Minnesota, IMA Preprint Series **389** (1988), 17-37.
- [15] **H. Terao:** *Arrangements of hyperplanes and their freeness I*, J. Fac. Science, University of Tokyo, Sci. IA **27**(2) (1980), 293-312.
- [16] **H. Terao:** *The exponents of a free hypersurface*, Arcata Symposium on Singularities (ed. P. Orlik), Proc. Symp. Pure Math. **40**, Part 2 (1983), 561-566.
- [17] **H. Terao:** *Generalized exponents of a free arrangement of hyperplanes and Shephard-Todd-Brieskorn formula*, Inventiones math. **63** (1981), 159-179.
- [18] **H. Terao:** *Free arrangements of hyperplanes over an arbitrary field*, Proc. Japan Acad., Ser. A **59** (1983), 301-303.
- [19] **D. J. A. Welsh:** *Matroid Theory*, Academic Press, 1976.
- [20] **N. White, ed.:** *Theory of Matroids*, Cambridge University Press, 1986.
- [21] **N. White, ed.:** *Combinatorial Geometries*, Cambridge University Press, 1987.
- [22] **T. Zaslavsky:** *Facing up to arrangements: face-count formulas for partitions of space by hyperplanes*, Memoirs Amer. Math. Soc. **1** (No. 154), 1983.
- [23] **G. M. Ziegler:** *Algebraic combinatorics of hyperplane arrangements*, Ph. D. Thesis, MIT 1987.
- [24] **G. M. Ziegler:** *Combinatorial construction of logarithmic differential forms*, Advances in Mathematics, to appear.
- [25] **G. M. Ziegler:** *Multiarrangements of hyperplanes and their freeness*, to appear in: *Proceedings of the International Conference on Singularities, Iowa City 1986*, Contemporary Mathematics, 1988.
- [26] **G. M. Ziegler:** *Supersolvable binary matroids and modular constructions*, in preparation.

