Broken Circuit Complexes: Factorizations and Generalizations

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Abstract

Motivated by the question of when the characteristic polynomial of a matroid factorizes, we study join-factorizations of broken circuit complexes and rooted complexes (a more general class of complexes). Such factorizations of complexes induce factorizations not only of characteristic polynomial but also of the Orlik-Solomon algebra of the matroid.

The broken circuit complex of a matroid factors into a multiple join of zero-dimensional subcomplexes for some linear order of the ground set if and only if the matroid is supersolvable. Several other characterizations of this case are derived. It is shown that whether a matroid is supersolvable can be determined from the knowledge of its 3-element circuits and its rank alone. Also, a supersolvable matroid can be reconstructed from the incidences of its points and lines.

The class of rooted complexes is introduced, and much of the basic theory for broken circuit complexes is shown to generalize. Complete factorization of rooted complexes is however possible also for non-supersolvable matroids, still inducing factorization of the characteristic polynomial.

Table of Contents

1. Introduction

Let $\mathcal{M}$ be a loop-free matroid of rank r with a linear order ω on its ground set E. The *broken circuits* of $(\mathcal{M}, \omega)$ are the sets $C - \min_\omega(C)$, formed by deleting the smallest element from a circuit C of $\mathcal{M}$. The *broken circuit complex* $\mathtt{BC}(\mathcal{M}, \omega)$ is the collection of all subsets of E that do not contain a broken circuit. The broken circuit complex is known to be a pure $(r-1)$-dimensional subcomplex of the matroid complex, that is, all its *facets* (maximal faces) are bases of the matroid.

Broken circuit complexes have been studied as a tool to understand important combinatorial and homological matroid properties [4,9,10,11,38]. In particular, the following identity due to Whitney and Rota [25, p. 359] shows that the f-vector $f = (f_0, \ldots, f_r)$ of the broken circuit complex (where f_i is the number of $(i-1)$-dimensional faces) encodes the coefficients of the characteristic polynomial:

$$\chi(t) = \sum_{i=0}^{r} (-1)^i f_i t^{r-i}. \tag{1.1}$$

In order to explain the factorization (over the integers) of the characteristic polynomial for many "well-behaved" matroids, Brylawski and Oxley [9,10,11] initiated a study of the join-decompositions (factorizations) of broken circuit complexes. They showed that every modular element induces a factorization of the broken circuit complex for a suitable linear order, thus giving a deeper structural interpretation of Stanley's celebrated modular factorization theorem for the characteristic polynomial [26]. The conjectured converse for this result [11] is still open.

This paper has two purposes. The first is to treat the case of *complete* factorizations of the broken circuit complex, corresponding to the case that the characteristic polynomial has only integer roots. Such complete factorization of the characteristic polynomial was first proved by Stanley for the class of *supersolvable* matroids introduced by him [27]. Our main result includes a converse, on the level of broken circuit complexes, to this case of complete factorizations. Specifically, we prove in Theorem 2.8 that the following conditions are equivalent for a loop-free matroid $\mathcal{M}$ of rank r:

(i) $\mathcal{M}$ is supersolvable.
(ii) For some linear order ω on E, the broken circuit complex $\mathtt{BC}(\mathcal{M}, \omega)$ factors completely.
(iii) For some ω, the 1-skeleton $\mathtt{BC}(\mathcal{M}, \omega)^{[1]}$ is a (complete) r-partite graph.
(iv) For some ω, the minimal broken circuits (with respect to inclusion) all have size 2.
(v) There exists an ordered partition $(X_1, X_2, \ldots, X_r)$ of E such that if $x, y \in X_i$, $x \neq y$, then there exists $z \in X_j$ with $j < i$ such that $\{x, y, z\}$ is a circuit.

The second purpose of this paper is to introduce a generalization of broken circuit complexes which is motivated by the study of factorization: the *rooted complexes* generated by *rooting maps* on the geometric lattice of flats of a matroid. A rooting map π selects a base point ("root") for each flat in $\mathcal{M}$ in a consistent way, and a simplex in the rooted complex $\mathtt{RC}(\mathcal{M}, \pi)$ is a set that contains no circuit minus the basepoint of the flat it spans. When there is a linear order on the ground set such that the base point of each flat is its earliest element, the construction specializes to the broken circuit complex. A rooted complex of $\mathcal{M}$ may, however, be non-isomorphic to every broken circuit complex $\mathtt{BC}(\mathcal{M}, \omega)$. Rooted complexes are the hereditary closures of certain neat base families as studied by Björner [5].

It is possible to generalize much of the basic theory of broken circuit complexes to rooted complexes. In particular, their f-vectors satisfy (1.1), which means that factorization of rooted complexes induces factorization of the characteristic polynomial. The converse (ii) $\Longrightarrow$ (i) of the Theorem above fails, however, in this generality. The rooted complex $\mathtt{RC}(\mathcal{M}, \pi)$ is a cone over a subcomplex which has the homotopy type of a wedge of $\beta(\mathcal{M})$ copies of the $(r-2)$-sphere.

The characteristic polynomial $\chi_{\mathcal{M}}(t)$ of a matroid $\mathcal{M}$ is (up to an invertible substitution) the Hilbert series of a certain finite-dimensional anticommutative algebra $\mathcal{A}(\mathcal{M})$, which was introduced by Orlik and Solomon [21]. Factorizations of $\chi_{\mathcal{M}}(t)$ are sometimes related to tensor product factorizations of $\mathcal{A}(\mathcal{M})$, as shown by Terao [31]. We discuss such factorizations of $\mathcal{A}(\mathcal{M})$ from the point of view of factorizations of rooted complexes. The key to this connection is the fact that any rooted complex $\mathtt{RC}(\mathcal{M}, \pi)$ induces a basis in the algebra $\mathcal{A}(\mathcal{M})$. For the case of broken circuit complexes this was previously shown by Gel'fand and Zelevinskii [17], Jambu and Terao [19], and in an equivalent form by Björner [5]. Our proof (specialized to the broken circuit complex case) differs from the earlier ones and appears to be somewhat simpler.

We remark that the maximal generality for the type of arguments used is given by the complexes on E whose restriction to every flat X is a cone of dimension $r(X) - 1$. These form a class of complexes which is slightly more general than rooted complexes.

The structural results of this paper do *not* depend on finiteness of the ground set. Thus the matroids and geometric lattices considered are of finite rank r, but do not necessarily have finite ground set respectively set of atoms E. If E is infinite, the orderings ω of E considered for the construction of broken circuit complexes are assumed to be well-orderings (as in [5]). Only the enumerative corollaries (on characteristic polynomials, f-vectors etc.) depend on finiteness.

In conclusion, let us mention that whereas broken circuit complexes as usually defined are specific to matroids, the definition of rooted complexes is applicable to any finite atomic lattice. For instance, a well known construction of a minimal simplicial subdivision of a convex polytope amounts to taking the rooted complex induced by a certain rooting map on the face lattice of the polytope (see Remark 4.4).

2. Supersolvable Lattices and Complete Factorization

For the following, let $\mathcal{M}$ be a matroid of rank r on a (not neccessarily finite) ground set E. With the usual canonical reduction we may assume that $\mathcal{M}$ is a combinatorial geometry (that is, loop free and without multiple points). With this, $\mathcal{M}$ is completely determined by its geometric lattice L, whose set of atoms we identify with E. The closure operator of $\mathcal{M}$ will be denoted by $\sigma : A \longrightarrow \overline{A}$, and L is the lattice of its flats (closed sets), ordered by inclusion. For much of what follows we will discuss matroids in their geometric lattice version, sometimes switching freely and translating without special notice between various matroid axiomatizations. The reader is assumed to be familiar with basic matroid theory, as developed in [12], [34] or [35-37]. In particular, we use basic properties of the Möbius functions and characteristic polynomials of geometric lattices and the Whitney-Rota formula (1.1). We refer to [4] and [9] for detailed treatments of broken circuit complexes from slightly different angles.

The following notions of "factorization" of simplicial complexes will be used:

Definition 2.1:

Let Δ be a simplicial complex of dimension $r-1$ on a finite ground set E. We say that Δ factors if E has a partition $E = X_1 \cup X_2$ ($X_1, X_2 \neq \emptyset$) such that $\Delta = \Delta_1 * \Delta_2$, where $\Delta_i = \Delta|_{X_i} = \{S \in \Delta : S \subseteq X_i\}$ is the restriction of Δ to X_i ($i = 1, 2$), and the join of Δ_1 and Δ_2 is

$$\Delta_1 * \Delta_2 = \{S_1 \cup S_2 : S_1 \in \Delta_1, S_2 \in \Delta_2\}\,.$$

Δ factors completely if E has a partition $E = X_1 \cup \ldots \cup X_r$ into r nonempty sets, such that Δ is a multiple join of the induced subcomplexes

$$\Delta = \Delta_1 * \Delta_2 * \ldots * \Delta_r$$

(as above), where the Δ_i are discrete (0-dimensional), i. e.,

$$\Delta_i = \{\emptyset\} \cup \{\{x\} : x \in X_i\} \ \text{ for } 1 \leq i \leq r.$$

The key point is that because of the Whitney-Rota formula (1.1), factorization of $\Delta = \mathtt{BC}(L, \omega)$ implies a factorization of the characteristic polynomial, and complete factorization of Δ implies that the characteristic polynomial factors completely as

$$\chi(t) = \prod_{i=1}^{r} (t - e_i), \tag{2.1}$$

where e_i is the number of points of Δ_i.

The simplest type of factorization of broken circuit complexes is induced by distributive elements, corresponding to product factorizations of geometric lattices.

Definition 2.2: (see [1] or [12])

Let L be a geometric lattice. A flat $X_1 \in L$ is distributive if it has a unique complement $X_2 \in L$. This is the case exactly if $L \cong L_1 \times L_2$, where E partitions into $E = X_1 \cup X_2$ so that X_i is the set of atoms of L_i $(i = 1, 2)$.

Lemma 2.3:

A set $X_1 \subseteq E$ is a distributive flat if and only if for every linear ordering ω on E, the broken circuit complex $\Delta = \mathtt{BC}(L, \omega)$ factors as

$$\Delta = \Delta|_{X_1} * \Delta|_{E-X_1}.$$

Proof:

Exercise. Compare with Lemma 3.8. □

Now we need some combinatorial facts about modular elements and supersolvable geometric lattices:

Definition 2.4: [26,27]

Let L be a geometric lattice of rank $r(L) = r$. An element $M \in L$ is called modular if $r(M \wedge M') + r(M \vee M') = r(M) + r(M')$ for every $M' \in L$. L is supersolvable if it has a maximal chain $\hat{0} = M_0 < M_1 < \ldots < M_r = \hat{1}$ of modular elements, (called an M-chain of L).

An element $M \in L$ is modular if and only if its complements form an antichain [26]. This is the case exactly if all complements of M have the same rank $r - r(M)$.

A basic result about the factorization of broken circuit complexes is the following theorem of Brylawski and Oxley.

Theorem 2.5: [11, Theorem 2.6]

Let $\Delta = \mathtt{BC}(L, \omega)$ as above. Then Δ factors $\Delta = \Delta_1 * \Delta_2$ on $E = X_1 \cup X_2$ such that $x_1 \prec x_2$ for all $x_1 \in X_1$, $x_2 \in X_2$ if and only if X_1 is a modular flat of L.

Since atoms of geometric lattices are always modular, this yields the factorization $\mathtt{BC}(L, \omega) = \{x_0\} * \mathtt{BC}'(L, \omega)$, where $x_0 = \min_\omega(E)$ and $\mathtt{BC}'(L, \omega)$ is the *reduced broken circuit complex* of L. Brylawski and Oxley conjectured in [11] that in Theorem 2.5 the assumption about compatibility of linear order and partition can be dropped, that is, that $\mathtt{BC}'(L, \omega)$ factors for some linear order ω if and only if L has a modular flat M with $1 < r(M) < r$ (in other words, there is a nontrivial factorization of a broken circuit complex of L if and only if L has a nontrivial modular flat). In [11], this is proved for $r \leq 4$.

Iterated application of Theorem 2.5 gives the following result which was also independently obtained by Björner [4, Exercises] and by Garsia and Wachs [16].

Corollary 2.6:
The broken circuit complex $\mathtt{BC}(L,\omega)$ *factors completely for a partition* $E = X_1 \cup \ldots \cup X_r$ *and a compatible linear order* ω *on* E *such that* $x_i \prec x_j$ *for all* $x_i \in X_i$, $x_j \in X_j$ *and* $i < j$, *if and only if the sets* $M_i = X_1 \cup \ldots \cup X_i$ *are flats and form an* M*-chain in* L.

Via (2.1) this leads to Stanley's factorization theorem for the characteristic polynomial of supersolvable lattices.

Corollary 2.7: [27]
Let L *be a finite supersolvable geometric lattice, and for some* M*-chain* $\hat{0} = M_0 < M_1 < \ldots < M_r = \hat{1}$ *let* $e_i = |M_i - M_{i-1}|$, $1 \le i \le r$. *Then*

$$\chi_L(t) = \prod_{i=1}^{r}(t - e_i).$$

In particular, the multiset $\{e_1, \ldots, e_r\}$ *does not depend on the choice of an* M*-chain for* L.

We will show that the assumption of compatibility of linear order and partition in Corollary 2.6 can be dropped. This means that the analogue of the Brylawski-Oxley conjecture for *complete* factorization is true.

First recall a few definitions. A pure simplicial complex of dimension $r-1$ (i. e., such that all the facets have size r) is *completely balanced* if the vertex set can be partitioned into r classes such that every facet of the complex has exactly one vertex in every class [28]. Thus every completely factoring complex is completely balanced. A graph is *complete* r*-partite* if its vertex set can be partitioned into r classes ("colored") such that the edges are exactly the sets of two vertices in different classes ("of different color.") An r*-partite* graph is any subgraph of a complete r-partite graph, that is, a graph with partition such that the edges join some of the vertices in different blocks. With this, a pure complex of dimension $r-1$ is completely balanced if and only if its 1-skeleton is r-partite. Also, in graph theoretic terms, a graph is r-partite if and only if its chromatic number is at most r.

Theorem 2.8:

Let L be a geometric lattice of rank r. The following are equivalent:

(1) L is supersolvable.

(2) For some linear order ω on E, the broken circuit complex $\mathtt{BC}(L,\omega)$ factors completely.

(2') For some ω, $\mathtt{BC}(L,\omega)$ is completely balanced.

(3) For some ω, the 1-skeleton

$$\mathtt{BC}(L,\omega)^{[1]} = \{S \in \mathtt{BC}(L,\omega) : |S| \leq 2\}$$

of the broken circuit complex is a complete r-partite graph.

(3') For some ω, $\mathtt{BC}(L,\omega)^{[1]}$ is r-partite.

(4) For some ω, the minimal broken circuits (under inclusion) are all of size 2.

(5) There is a partition $E = X_1 \cup \ldots \cup X_r$ such that for any two distinct $x, y \in X_i$ there is an element $z \in X_j$ with $j < i$ such that $\{z, x, y\}$ is a circuit.

Proof:

(1) $\Longrightarrow$ *(2)* follows from Corollary 2.6.

(2) $\Longrightarrow$ *(2')*, *(2')* $\Longrightarrow$ *(3')*, *(2)* $\Longrightarrow$ *(3)*, *(3)* $\Longrightarrow$ *(3')*, and *(2)* $\Longrightarrow$ *(4)* are trivial.

[*(3')* $\Longrightarrow$ *(2')* follows from the fact that $\mathtt{BC}(L,\omega)$ is pure $(r-1)$-dimensional.]

(3') $\Longrightarrow$ *(1)*: Let $E = X_1 \cup \ldots \cup X_r$ such that there is no edge of $\mathtt{BC}(L,\omega)^{[1]}$ between $x \neq y$ if x and y are in the same X_i. For $1 \leq i \leq r$, let $a_i = \min(X_i)$. Assume that the X_i are labeled such that $a_1 \prec a_2 \prec \ldots \prec a_r$ with respect to ω. For $0 \leq i \leq r$, let

$$M_i = \bigvee\big(\bigcup_{j \leq i} X_j\big).$$

In particular, this means $M_0 = \hat{0}$ and $M_r = \hat{1}$. To see that the M_i are modular flats and $r(M_i) = i$, we verify a sequence of facts.

(i) For $1 \leq i \leq r$, $M_i = M_{i-1} \vee a_i$.
This is true because for $a_i \prec y$, $y \in X_i$, $\{a_i, y\}$ is a broken circuit, and thus there is a $z \prec a_i$ such that $\{z, a_i, y\}$ is a circuit. But $z \in X_j$ for some $j < i$ (since $a_j \preceq z \prec a_i$), hence

$$y < z \vee a_i \leq M_{i-1} \vee a_i.$$

(ii) $r(M_i) = i$ for all i.
From (i), we get $r(M_i) \leq r(M_{i-1}) + 1$. But $r(M_0) = 0$ and $r(M_r) = r$, hence equality holds for all i.

(iii) $M_i = \bigcup_{j \leq i} X_j$ (that is, $\bigcup_{j \leq i} X_j$ is a flat.)
For this, assume $y \leq M_i$ for some $y \in X_j$, $j > i$. But then an argument such as in (i) implies that $a_j \leq M_{j-1}$ and thus $M_j = M_{j-1}$, contradicting (ii).

(iv) If $|S \cap X_i| = 1$ for all i, then $S \subseteq E$ is a basis of L.
If S satisfies the condition, then $|S| = r$ and $\bigvee S = \hat{1}$ by (ii) and (iii).

(v) For $1 \leq i \leq r$, M_i is modular.
Let M^* be a complement of M_i. As is easy to see, the lexicographically smallest basis B for M^* cannot contain a broken circuit. Hence, $|B \cap X_j| \leq 1$ for all j. Also, $B \cap X_j = \emptyset$ for all $j \leq i$, because $M^* \wedge M_i = \hat{0}$. Take $A = \{a_1, \ldots, a_i\}$ as a basis for M_i, then $A \cap B = \emptyset$ and $A \cup B$ is independent by (iv). Thus $A \cup B$ is a basis for $M^* \vee M_i = \hat{1}$, and $r(M^*) + r(M_i) = |A \cup B| = r$.

(4) $\Longrightarrow$ *(2):* For this, we define $a \sim b$ for $a, b \in E$ whenever $\{a, b\}$ is a broken circuit or $a = b$. With *(4)* and the fact that $\mathtt{BC}(L, \omega)$ is pure $(r-1)$-dimensional, it suffices now to show that '$\sim$' is an equivalence relation on E. Reflexivity and symmetry are trivial. To verify transitivity, we show that '$\sim|_T$' is transitive for all triples $T \subseteq E$, by induction on the lexicographic order '$<_L$' on the set of all triples. To start the induction, it is clear that '$\sim|_T$' is transitive on all triples that contain $\min_\omega(E)$. Now, let $a \sim b$, $b \sim c$, where a, b and c are distinct. Choose $x, y \in E$ minimal such that $\{x, a, b\}$ and $\{y, b, c\}$ are circuits. We can assume that $x \prec y$: if $x = y$, then x, a, b, c all lie on a line and $\{x, a, c\}$ is a circuit. But $\{x, a, y, c\}$ is easily checked to be a circuit, which makes $\{a, y, c\}$ into a broken circuit. Now, $y \not\sim c$ and $y \not\sim b$ since y was chosen minimal, and since $a \sim b$ and $y \not\sim b$ we get from $\{a, y, b\} <_L \{a, b, c\}$ that $a \not\sim y$. Thus from *(4)* applied to $\{a, y, c\}$, we get $a \sim c$.

(5) $\Longrightarrow$ *(3'):* For this choose ω such that $x \prec y$ whenever $x \in X_i$, $y \in X_j$ and $i < j$.

(1) $\Longrightarrow$ *(5):* Let $X_i = \{x \in E : x \leq M_i, x \not\leq M_{i-1}\}$ for $1 \leq i \leq r$ and M-chain $\hat{0} = M_0 < M_1 < \ldots < M_r = \hat{1}$. Then for $x, y \in X_i$ we can choose $z = M_{i-1} \wedge (x \vee y)$. □

We remark that if a particular linear ordering ω of E satisfies one of *(2), (2'), (3), (3')* or *(4)*, then the same ω satisfies all the others.

Corollary 2.9:

Whether a geometric lattice L is supersolvable can be decided from the rank $r(L)$ and the 3-truncation $L^{[3]} = \{X \in L : r(X) \leq 2\} \cup \{\hat{1}\}$ alone. If L is supersolvable, then it is determined by $L^{[3]}$, that is, any supersolvable matroid can be reconstructed from the incidences of its points and lines.

Proof:

The first part is clear from either of *(3), (3')* or *(5)*. The second statement follows from the fact that the supersolvability of L allows the reconstruction of all bases in L from knowledge of the 3-circuits. To see this, choose a linear ordering ω of E such that *(3)* is satisfied and then argue as in the last proof to identify the facets of $\mathtt{BC}(L, \omega)$. Since the minimal broken circuits with respect to ω have size 2, any basis of L which contains a broken circuit can be shifted into a basis in $\mathtt{BC}(L, \omega)$ by a sequence of exchanges determined by 3-circuits, and this way all bases of L can be identified. □

Wilf [38, p.325] computes the characteristic polynomial for the graphs that admit a broken circuit complex with disjoint minimal broken circuits. It is easy to see from his description together with Theorem 2.8 that for these graphs $\chi_G(t)$ factors completely over $\mathbb{Z}$ exactly if they are chordal and at most 3-chromatic, including the class of all 2-trees.

Also as an application of Theorem 2.8 we get a simple proof for Stanley's characterization of supersolvable graphs, which was given in [27] without proof:

Corollary 2.10: [27, Proposition 2.8]

Let $G = (V, E)$ be a connected finite simple graph and $L(G)$ the geometric lattice of its graphic matroid. Then the following conditions are equivalent:

(1) $L(G)$ is supersolvable,

(2) G is a triangulated graph (i. e., every circuit of length at least four has a chord),

(3) There is a linear order $v_0, v_1, \ldots, v_n$ of the vertices such that for every i, $1 \leq i \leq n$, the neighbors of v_i contained in the set $\{v_0, \ldots, v_{i-1}\}$ form a clique.

Proof:

(1) $\Longrightarrow$ (2): By condition *(4)* of Theorem 2.8 there exists some order ω of E such that all minimal broken circuits are of size 2. Hence if C is any circuit of size at least 4, it contains a broken circuit $B - \min_\omega(B)$, where B is a 3-circuit. Now $\min_\omega(B)$ clearly is a chord in C.

(3) $\Longrightarrow$ (1): For $1 \leq i \leq n$, let $X_i = \{(v_j, v_i) \in E : 0 \leq j < i\}$. Then the number of non-empty classes X_i equals the rank of $L(G)$. Hence, the implication *(5) $\Longrightarrow$ (1)* of Theorem 2.8 applies.

(2) $\Longrightarrow$ (3): This follows by induction from the existence of a simplicial vertex (i. e., a vertex whose neighbors form a clique) in every connected triangulated graph, which is a well known property (see [18, Theorem 4.1 and Lemma 4.2]). $\square$

3. Rooted Complexes

We will now investigate a class of complexes associated with a geometric lattice that is more general than the class of broken circuit complexes. The main reason for this is to find more far-reaching structural explanation for factorizations of the characteristic polynomial and to throw new light on how the broken circuit construction works. Although these more general complexes can also be defined by a notion of "broken circuits", cf. Lemma 3.3, we have found more convenient another approach which does not emphasize circuits.

Definition 3.1:

Let L be a geometric lattice, E its set of atoms (points), $L_{>\hat{0}} = L - \{\hat{0}\}$.

(1) A rooting map for L is a function $\pi : L_{>\hat{0}} \longrightarrow E$ that assigns to every nonempty flat X a point $\pi(X) \in X$, such that $\pi(X) \in Y \leq X$ implies $\pi(Y) = \pi(X)$.
For simplicity we extend every rooting map π to a map $\pi : 2^E - \{\emptyset\} \longrightarrow E$ via

$$\pi(A) = \pi(\overline{A}).$$

(2) A subset $A \subseteq E$ is called unbroken (with respect to the rooting map π) if $\pi(A) \in A$ and broken otherwise. A is rooted if $\pi(B) \in B$ for all nonempty subsets $B \subseteq A$ (that is, if A does not contain a broken set).

(3) The collection of rooted sets for L and π is called the rooted complex of L with respect to π, and denoted by $\mathtt{RC}(L, \pi)$.

The following result collects the basic facts.

Theorem 3.2:

(1) $\mathtt{RC}(L, \pi)$ is a simplicial complex (that is, every subset of a rooted set is rooted.)
(2) $\mathtt{RC}(L, \pi)$ is a cone for $r > 0$, with apex $p_0 = \pi(E)$ (i. e., $p_0 \cup A$ is rooted for every rooted set $A \subseteq E$.)
(3) $\mathtt{RC}(L, \pi)$ is a subcomplex of the matroid complex (i. e., rooted sets are independent.)
(4) $\mathtt{RC}(L, \pi)$ is pure of dimension $r - 1$ (that is, every maximal rooted set has size r.)

Proof:

(1) This is clear by definition.
(2) If B is rooted, but $p_0 \cup B$ is not, then there is a $B' \subseteq p_0 \cup B$ such that $\pi(B') \notin B'$. But B is rooted, hence $p_0 \in B'$, which implies $\pi(B') = p_0$, with the definition of a rooting map.
(3) Let A be rooted and $C \subseteq A$ a circuit. Then C contains $p = \pi(C)$, and hence $B = C - p$ fails to satisfy $\pi(B) \in B$.
(4) If B is a maximal rooted set, then $p_0 \in B$ by *(2)*. Now by *(3)*, $B - p_0$ is contained in a hyperplane which can be chosen not to contain p_0. Since rooting maps restrict to flats so that the rooted sets of the flat are exactly those rooted sets of the whole matroid which are contained in the flat, we are done by induction. (The cases $r = 0$ and $r = 1$ are trivial.) $\square$

For every rooting map π on a geometric lattice L and for every flat $X \in L$, the restriction $\pi|_{[\hat{0},X]}$ defines a rooting map on the interval $[\hat{0}, X]$, and the corresponding rooted complex is the restriction of the rooted complex of L to X,

$$\mathtt{RC}([\hat{0}, X], \pi|_{[\hat{0},X]}) = \mathtt{RC}(L, \pi)|_X.$$

Thus all the observations of Theorem 3.2 about rooted complexes also apply to restrictions of rooted complexes to flats: they are again rooted complexes and thus pure cones of dimension $r(X) - 1$. In particular this implies that every flat has a rooted basis. (In fact every flat X has $|\mu(\hat{0}, X)|$ rooted bases, as we will see in Theorem 3.11.)

The precise relationship between the construction of rooted complexes and of broken circuit complexes is described in the following lemma and proposition:

Lemma 3.3:

Let $\pi : L_{>\hat{0}} \longrightarrow E$ be a rooting map. Define broken circuits as sets of the form $C - \pi(C)$ for circuits C such that $\pi(C) \in C$. Then

$$\{ \text{ minimal broken sets } \} = \{ \text{ minimal broken circuits } \}.$$

Thus $\mathtt{RC}(L, \pi)$ is the complex of all subsets of E that do not contain a broken circuit.

Proof:

Let $B = C - \pi(C)$ be a broken circuit. Then $\pi(B) = \pi(C) \notin B$, so B is a broken set. Conversely, let B be a minimal broken set, and let $p = \pi(B)$. All proper subsets of B are unbroken, hence rooted. Let $A = B - b$ be a maximal proper subset of B.
If B were dependent, then $\overline{B} = \overline{A}$ and $\pi(A) = p \notin B$. If $A \cup p$ were dependent, then $\overline{A \cup p} = \overline{A}$ and since $p \in \overline{A \cup p} \le \overline{B}$ we get $\pi(A) = \pi(A \cup p) = p \notin B$. Since A is rooted, we conclude that B and $A \cup p$ are independent. On the other hand $B \cup p$ is clearly dependent, and hence a circuit. Since also $\pi(B \cup p) = p$ it follows that B is a broken circuit. □

Proposition 3.4:

Let ω be a linear order on E, and for $X \in L_{>\hat{0}}$ define

$$\pi(X) = \min_\omega(X).$$

Then π is a rooting map, and the associated rooted complex is

$$\mathtt{RC}(L, \min_\omega) = \mathtt{BC}(L, \omega).$$

Proof:

With the definition of "broken circuits" as in Lemma 3.3 it is easy to see that

$$\{ \text{ broken circuits } \} \subseteq \{C - \min_\omega(C) : \ C \text{ is a circuit}\} \subseteq \{ \text{ broken sets } \}.$$

Hence, the result follows by Lemma 3.3. (Note that the first inclusion is usually strict, meaning that the concept of broken circuit used here is more restrictive than the usual one.) □

Examples in Section 4 will show that not every rooted complex arises in this way from a linear order on E. However, there turns out to be a canonical linear order on every rooted set. A rooted complex arises as a broken circuit complex exactly if these linear orders are compatible (i. e., can be extended to a global linear order on E.)

Lemma 3.5:

Let π be a rooting map on L, and $T \subseteq E$.

(1) T is rooted if and only if there is a (unique) linear order $t_1 \prec t_2 \prec \ldots \prec t_k$ on T such that for every non-empty subset S of T, $\pi(S) = \min_\prec(S)$.

(2) If T is not rooted, then T contains a unique maximal broken subset T^0; the elements of $T^1 = T - T^0$ can be ordered as $t_1 \prec t_2 \prec \ldots \prec t_l$ such that for $S \subseteq T$, $\pi(S) = \min_\prec(S \cap T^1)$ if $S \cap T^1 \neq \emptyset$.

Proof:

The required linear orders can be described inductively by $t_i = \pi(T - \{t_1, t_2, \ldots, t_{i-1}\})$. If this process stops, that is, if $t_i \notin T - \{t_1, t_2, \ldots, t_{i-1}\}$, then $T^0 = T - \{t_1, t_2, \ldots, t_{i-1}\}$ contains every broken subset of T and thus is the unique maximal broken subset of T. (Uniqueness also follows from the observation that the union of any two broken subsets is broken.) For the converse of *(1)*, simply observe that $\pi(S) = \min(S) \in S$ for all non-empty subsets $S \subseteq T$. □

The considerable size and complexity of geometric lattices suggest that we should look for a formulation of rooting maps in more manageable terms, e.g., just using the points and lines of a combinatorial geometry. We will later see, however, that such a formulation does not exist (cf. Example 4.3). Nevertheless, one has the following reformulation that turns out to be useful in some instances.

Construction 3.6:

If π is a rooting map on L, then define a simple digraph $D_\pi = (E, A)$ which has an arc from x to y $(x, y \in E)$ iff $\pi(\{x, y\}) = y$. Then D restricted to a flat X has the property that there is one vertex π_X in X such that for every $y \in X - \{\pi_X\}$, $y \longrightarrow \pi_X$ is an arc of $D|_X$.

Conversely, every simple digraph $D = (E, A)$ such that every restriction to a flat has a "complete sink" (in the described sense) defines a unique rooting map.

Here a digraph D corresponds to the construction of a broken circuit complex exactly if it is acyclic.

The following sequence of results studies factorization of rooted complexes. By Theorem 3.2*(2)* there is always the trivial factorization $\mathtt{RC}(L, \pi) = \{\pi(E)\} * \mathtt{RC}'(L, \pi)$, where $\mathtt{RC}'(L, \pi)$ is the *reduced rooted complex*.

Proposition 3.7:

Let π be a rooting map on L such that $\Delta = \mathtt{RC}(L,\pi)$ factors as $\Delta = \Delta_1 * \Delta_2$ on $X_1 = \bigcup \Delta_1$, $X_2 = \bigcup \Delta_2 = E - X_1$. If X_1 is a flat, then it is modular.

Proof:

Let Y be any complement of X_1 in L, and let B_1 and B_2 be bases of X_1 and Y, respectively. We can assume $B_1 \in \Delta_1$ and $B_2 \in \Delta$, which implies $B_2 \in \Delta_2$ since $Y \cap X_1 = \emptyset$. Now $B_1 \cup B_2 \in \Delta$, hence $r(X_1) + r(Y) = |B_1| + |B_2| = |B_1 \cup B_2| \leq r(L)$. This implies that X_1 is modular. □

From Theorem 2.5 and Proposition 3.4 we know that for suitable rooting maps, modular flats induce factorization of the rooted complex. The converse, as conjectured for the case of broken circuit complexes, is false for rooted complexes in general: as will be seen from examples in Section 4, factorization of rooted complexes may or may not be induced by modular elements.

However, rooted complexes do *always* factor at distributive elements, that is, whenever the geometric lattice is a product, all associated rooted complexes factor.

Lemma 3.8:

If L is an irreducible geometric lattice and π a rooting map on L, then the reduced rooted complex $\mathtt{RC}'(L,\pi)$ is not a cone.

Proof:

Assume that $\mathtt{RC}'(L,\pi)$ is a cone over x_1. Then $\mathtt{RC}(L,\pi)$ is a cone both over x_1 and $x_0 = \pi(E)$. Now if L is connected, then there is a circuit $C \subseteq E$ containing both x_0 and x_1. Then $C - \{x_0, x_1\}$ is independent of rank $r(C) - 1$. Let C' be a rooted basis of $\overline{C - \{x_0, x_1\}}$. But now $C' \cup \{x_0, x_1\}$ is rooted, hence independent, and has the same closure as C, which is dependent, but of the same size: contradiction.

(For finite L this lemma alternatively will follow from Theorem 3.12, since the reduced Euler characteristic of $\mathtt{RC}'(L,\pi)$ is $\beta(L)$, which vanishes only if the matroid $\mathcal{M}$ is not connected, that is, if L is reducible.) □

Theorem 3.9:

(1) If $L = L_1 \times L_2$ and π is a rooting map on L, then

$$\mathtt{RC}(L,\pi) = \mathtt{RC}(L_1,\pi) * \mathtt{RC}(L_2,\pi).$$

(2) For every L and π the singleton factors of $\mathtt{RC}(L,\pi)$ (that is, the cone points of $\mathtt{RC}(L,\pi)$) are exactly the roots of the irreducible factors of L.

Proof:

(1) $\Delta_1 = \mathtt{RC}(L_1,\pi)$ and $\Delta_2 = \mathtt{RC}(L_2,\pi)$ are pure subcomplexes of $\Delta = \mathtt{RC}(L,\pi)$, since $X_1 = \bigcup \Delta_1$ and $X_2 = \bigcup \Delta_2$ are flats. We have to show that $\pi(F_1 \cup F_2) \in F_1 \cup F_2$ for $F_1 \in \Delta_1$ and $F_2 \in \Delta_2$. Because of $\overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2}$ we can assume (without loss

of generality) that $\pi(F_1 \cup F_2) \in \overline{F_1}$. But this implies $\pi(F_1) = \pi(F_1 \cup F_2)$, and thus $\pi(F_1 \cup F_2) \in F_1 \subseteq F_1 \cup F_2$ since F_1 is rooted.

(2) The factors Δ_1 and Δ_2 of part (1) are cones over $\pi(X_1)$ and $\pi(X_2)$, respectively. Thus for every irreducible factor of L we get a cone point of Δ. The converse follows from Lemma 3.8. □

The following can be said about generalizations of Theorem 2.8 to rooted complexes.

Remark 3.10:

Let L be a geometric lattice of rank r, and π a rooting map on L. Consider the following statements:

(2) $\mathtt{RC}(L, \pi)$ factors completely,
(2') $\mathtt{RC}(L, \pi)$ is completely balanced,
(3) $\mathtt{RC}(L, \pi)^{[1]}$ is complete r-partite,
(3') $\mathtt{RC}(L, \pi)^{[1]}$ is r-partite.
(4) The minimal nonfaces of $\mathtt{RC}(L, \pi)$ (minimal broken circuits) all have size 2.

Then (2) $\Longrightarrow$ (2') $\Longrightarrow$ (3'), (2) $\Longrightarrow$ (3) $\Longrightarrow$ (3') and (2) $\Longrightarrow$ (4) are again trivial.
(3') $\Longrightarrow$ (2') follows from the fact that $\mathtt{RC}(L, \pi)$ is pure of dimension $r-1$, by Theorem 3.2(4).
However, we do not know whether the other converse implications hold.
As for the conditions *(1)* and *(5)* of Theorem 2.8, we do not know valid analogues for the case of rooted complexes. Example 4.3 shows some of the obstacles to finding analogues to Theorem 2.8 and Corollary 2.9 for rooted complexes.

Now let $f = (f_0, \ldots, f_r)$ be the *f-vector* of a rooted complex $\mathtt{RC}(L, \pi)$, where f_i is the number of faces of $\mathtt{RC}(L, \pi)$ of cardinality i ($f_0 = 1$). The maximal faces (all of size r, by Theorem 3.2(4)) are called *facets* of $\mathtt{RC}(L, \pi)$. The Whitney-Rota formula (1.1) has the following generalization.

Theorem 3.11:

The f-vector of $\mathtt{RC}(L, \pi)$ is given by

$$\sum_{i=0}^{r} f_i t^{r-i} = (-1)^r \chi_L(-t).$$

Proof:

For every flat $X \in L$, let $f(X)$ be the number of bases of X that are elements of $\mathtt{RC}(L, \pi)$. Since

$$(-1)^r \chi(-t) = \sum_{X \in L} (-1)^{r(X)} \mu(\hat{0}, X) t^{r-r(X)}$$

by definition, it suffices to show that $f(X) = (-1)^{r(X)} \mu(\hat{0}, X)$.

Now we note that $\sum_{r(X)=i} f(X) = f_i$, and hence

$$\sum_{X \in L} (-1)^{r(X)} f(X) = \sum_{i=0}^{r} (-1)^i f_i = -\tilde{\chi}(\mathtt{RC}(L,\pi))$$

is the reduced Euler characteristic of the rooted complex (up to sign.) Since rooting maps restrict correctly to intervals, we get that for $Y \leq X$, $f(Y)$ is also the number of bases of Y that belong to $\mathtt{RC}([\hat{0}; X]; \pi)$. Thus we get more generally that

$$\sum_{\substack{r(Y)=i \\ Y \leq X}} f(Y) = f_i(\mathtt{RC}([\hat{0}, X], \pi)),$$

and hence for $X > \hat{0}$,

$$\sum_{Y \in [\hat{0}, X]} (-1)^{r(Y)} f(Y) = \sum_{i=0}^{r(X)} (-1)^i f_i(\mathtt{RC}([\hat{0}, X], \pi)) = -\tilde{\chi}(\mathtt{RC}([\hat{0}, X], \pi)) = 0,$$

since for $X \in L_{>\hat{0}}$, $\mathtt{RC}([\hat{0}, X], \pi)$ is a cone by Theorem 3.2(2), and hence has vanishing reduced Euler characteristic. Thus $\mu(\hat{0}, X)$ and $(-1)^{r(X)} f(X)$ satisfy the same recursion on L. With this, $f(\hat{0}) = 1$ proves the claim. □

Observe that the f-vector of $\mathtt{RC}(L, \pi)$ does not depend on the rooting map π chosen, but can be computed from L alone, although different rooting maps can give rise to non-isomorphic rooted complexes. By Theorem 3.2(2), $\mathtt{RC}(L, \pi)$ is a cone and hence topologically trivial. However, the *reduced rooted complex* $\mathtt{RC}'(L, \pi)$ obtained by deleting the apex $\pi(E)$ from $\mathtt{RC}(L, \pi)$ has interesting topological structure. Clearly, $\mathtt{RC}'(L, \pi)$ is a pure $(r-2)$-dimensional complex on the set of atoms $E - \pi(E)$.

In the following theorem, $\beta(L)$ denotes Crapo's beta-invariant, defined, e.g., in [34] and [36].

Theorem 3.12:

$\mathtt{RC}'(L, \pi)$ *is homotopy equivalent to a wedge of* $(r-2)$*-dimensional spheres.*
If L *is finite, the number of the spheres in the wedge equals* $\beta(L)$.

Proof:

We refer to [6] for definitions and a survey of topological methods in the analysis of posets as used in the following proof.
Let L be finite, and $\overline{L} = L_{>\hat{0}} - [\pi(\hat{1}), \hat{1}]$. By a result of Wachs and Walker [32] the order complex $\Delta(\overline{L})$ is shellable, hence is homotopy equivalent to a wedge of $(r-2)$-spheres. In the finite case, the number of these spheres is equal to the absolute value of the reduced Euler characteristic of $\Delta(\overline{L})$, hence to

$$\Big| \sum_{\pi(\hat{1}) \not\leq x} \mu(\hat{0}, x) \Big|,$$

which by a result of Zaslavsky [39, p.76; 36] is equal to the beta-invariant $\beta(L)$.
Now one checks that the mapping $\pi : \overline{L} \longrightarrow E - \{\pi(\hat{1})\}$ induces a simplicial mapping $\Delta(\overline{L}) \longrightarrow \mathtt{RC}'(L, \pi)$, hence an order preserving mapping $\pi : P \longrightarrow Q$ of their face posets $P = P(\Delta(\overline{L}))$, $Q = P(\mathtt{RC}')$. For $F \in Q$, that is, F a face of the reduced rooted complex, look at the fiber $\pi^{-1}(Q_{\leq F})$. This fiber is contractible – in fact it is meet-contractible via $\vee F$. Explicitly, the fiber is the order complex of a subposet of $\overline{L}$ obtained as a certain ideal containing $\vee F$ inside the union of the principal filters above a, for all $a \in F$. Hence, by the Fiber Theorem of Quillen [24] (see also [33]), π induces a homotopy equivalence $\Delta(\overline{L}) \sim \mathtt{RC}'(L, \pi)$. □

Theorem 3.12 is probably not best possible. We see no reason to believe that $\mathtt{RC}(L, \pi)$ does not share all the nice topological properties that are known for the special case of broken circuit complexes. Specifically, we expect rooted complexes to be shellable, from which Theorem 3.12 would follow in a stronger form (also for all links, which would show that rooted complexes are homotopy Cohen-Macaulay). However, the known proofs for shellability of broken circuit complexes [4] [23] do not generalize straightforwardly.

In [5], Björner gave an inductive definition of *neat base families* for matroids as follows: for a matroid of rank 0 or 1 a neat base family just contains the one basis of the matroid. For a matroid of higher rank, one chooses a distinguished point $p \in E$, and a neat base family for every hyperplane not containing p. The neat base family is then the set of all bases of the form $p \cup B$, where B belongs to one of the base families chosen for the hyperplanes. From the inductive description of the maximal faces of a rooted set complex in the proof of Theorem 3.2(4) we get:

Corollary 3.13:
The facets of a rooted complex form a neat base family.

Since in the same way the maximal faces of the restriction of a rooted complex to each flat X gives a neat base family in X, Theorem 3.11 also follows from the results of [5].

4. Some Examples

It is natural to ask to what extent factorization of a rooted complex can be used to "explain" factorizations of the characteristic polynomial for more general matroids than the supersolvable ones. The most general class of geometric lattices for which $\chi(t)$ is known to factor over $\mathbb{Z}$ is the class of intersection lattices of "free" arrangements of hyperplanes as defined by Terao [29,30] (see also [41, Chapter 3]). This includes the case of representable supersolvable matroids, and that of the intersection lattices of Coxeter arrangements. The following examples will show that

(1) $\mathtt{RC}(L, \pi)$ can factor completely for non-supersolvable matroids.
(2) $\chi(t)$ may factor over $\mathbb{Z}$ without $\mathtt{RC}(L, \pi)$ factoring for any rooting map π.
(3) Not every intersection lattice of a (free) Coxeter arrangement admits a rooting map such that $\mathtt{RC}(L, \pi)$ factors completely.
(4) There are non-representable matroids for which $\mathtt{RC}(L, \pi)$ factors (but the theory of free representable matroids (i.e., free arrangements) does not apply).

Example 4.1:

(1) **The non-Fano matroid.**
The non-Fano plane F^- is the matroid in Figure 4.1. This matroid is not supersolvable, but its characteristic polynomial factors as

$$\chi(t) = (t-1)(t-3)(t-3).$$

This factorization, which was discussed in [11], is not "accidental". For example, the corresponding hyperplane arrangement is free in the sense of Terao [29], and hence the factorization has to hold (for rather an algebraic than combinatorial reason).
Furthermore, it turns out that for a suitable rooting map, the rooted complex factors completely: if π is the rooting map defined by

$$\begin{aligned} \pi(E) &= \mathbf{1}, \\ \pi(\mathbf{245}) &= \mathbf{5}, \\ \pi(\mathbf{267}) &= \mathbf{2}, \\ \pi(\mathbf{346}) &= \mathbf{6}, \\ \pi(\mathbf{357}) &= \mathbf{3}, \text{ and} \\ \pi(\mathbf{47}) &= \mathbf{4} \text{ or } \mathbf{7}, \end{aligned}$$

then $\mathtt{RC}(F^-, \pi)$ factors completely as the join of the zero-dimensional (discrete) complexes on $\{\mathbf{1}\}$, $\{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$ and $\{\mathbf{5}, \mathbf{6}, \mathbf{7}\}$. Here it is easy to see that this π does not come from any linear order: the second to fifth condition on π would in turn imply $\mathbf{5} \prec \mathbf{2}$, $\mathbf{2} \prec \mathbf{6}$, $\mathbf{6} \prec \mathbf{3}$ and $\mathbf{3} \prec \mathbf{5}$. In fact, if $\mathtt{RC}(F^-, \pi)$ was a broken circuit complex for some ω, then L had to be supersolvable by Theorem 2.8.

Figure 4.1:

Figure 4.2:

(2) **Stanley's Example [27].**
Let St_7 be the matroid of Figure 4.2. Its characteristic polynomial again factors as $\chi(t) = (t-1)(t-3)(t-3)$, but this factorization seems to be really accidental (i.e., not supported by any available theory.) In particular, no coordinatization of St_7 is free in the sense of Terao [41, Example 3.10.2].
To see that $\mathtt{RC}(St_7, \pi)$ does not factor completely for any π, we can assume without loss of generality that $\pi(\mathbf{12345}) = \mathbf{1}$. This implies that **23**, **24**, **25**, **34**, **35** and **45** are not rooted, and thus for every π, the complement of the 1-skeleton of $\mathtt{RC}(St_7, \pi)$ is isomorphic to the union of K_4 with three isolated vertices.

(3) **Matroid of the regular icosahedron.**
The Coxeter arrangement H_3 is the set of symmetry planes of a regular icosahedron. This arrangement is free in the sense of Terao [29], with

$$\chi(t) = (t-1)(t-5)(t-9).$$

It is, however, not hard to check that the corresponding matroid (whose geometric lattice is the set of intersections of the hyperplanes, ordered by reverse inclusion) does not have a rooting map π such that $\mathtt{RC}(H_3, \pi)$ factors completely. The same applies to the Coxeter arrangements of type D_n for $n \geq 4$, as can easily be checked using the combinatorial desciption of their matroids in [40]. Thus we still lack a combinatorial explanation for the fact that $\chi_L(t)$ factors completely over $\mathbb{Z}$ when L is the intersection lattice of a Coxeter arrangement.

(4) **The affine plane of order** 3
Let $\mathtt{AG}(3,3)$ denote the matroid of the affine plane of order 3, as given by Figure 4.4. This matroid is not supersolvable, but very symmetric. Its symmetry in fact suggests the rooting map given by

$$\begin{aligned}
\pi(E) &= \mathbf{5},\\
\pi(\mathbf{123}) &= \mathbf{2},\\
\pi(\mathbf{369}) &= \mathbf{6},\\
\pi(\mathbf{987}) &= \mathbf{8},\\
\pi(\mathbf{741}) &= \mathbf{4},\\
\pi(\mathbf{267}) &= \mathbf{7},\\
\pi(\mathbf{681}) &= \mathbf{1},\\
\pi(\mathbf{843}) &= \mathbf{3},\\
\pi(\mathbf{429}) &= \mathbf{9}.
\end{aligned}$$

For this rooting map π the complex $\mathtt{RC}(\mathtt{AG}(3,3), \pi)$ factors completely into discrete complexes on $\{\mathbf{5}\}$, $\{\mathbf{1}, \mathbf{3}, \mathbf{7}, \mathbf{9}\}$ and $\{\mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{8}\}$, with

$$\chi(t) = (t-1)(t-4)^2.$$

Now by [42] the corresponding hyperplane arrangement in $\mathbf{k}^3$ is free if and only if $\mathbf{k}$ is a field of characteristic different from 3. Thus complete factorization of a rooted complex does not directly imply freeness for the corresponding hyperplane arrangements.

(5) **A non-representable matroid.**
Let L be the geometry of rank 4, on 11 points, depicted in Figure 4.3. It is not representable since it contains both the non-Fano matroid and the Fano matroid as subgeometries. The non-Fano matroid in L in fact forms a modular coatom H_0. Hence, if we extend the rooting map π on $[\hat{0}, H_0]$ given by part (1) above (cf. Figure 4.1) to L via

$$\pi^*(X) = \begin{cases} \pi(X \wedge H_0) & \text{for } X \wedge H_0 \neq \hat{0}, \\ X & \text{otherwise,} \end{cases}$$

then $\mathtt{RC}(\mathrm{L}, \pi^*)$ factors completely, with

$$\chi_L(t) = \chi_{H_0}(t)(t-4) = (t-1)(t-3)^2(t-4).$$

The preceding examples raise the problem of characterizing combinatorially those geometric lattices which admit a rooting map such that the associated rooted complex factors completely.

We next observe that the converse of Corollary 3.13 is false: not every neat base family is the set of facets of a rooted complex. In fact, Example 4.2 will show that the Whitney-Rota formula of Theorem 3.11 does not in general hold for the hereditary closures of neat base families. Only an inequality stays valid for this case.

Example 4.2:
Consider the combinatorial geometry of Figure 4.5 of rank 4. An affine coordinatization in $\mathbb{R}^3$ is given by $\mathbf{4} = (0,0,0)$, $\mathbf{5} = (0,1,0)$, $\mathbf{6} = (0,-1,0)$, $\mathbf{1} = (1,0,0)$, $\mathbf{2} = (0,0,1)$ and $\mathbf{3} = (1,0,1)$. We will now construct a neat base family in the sense of [5] for this geometry that does not come from a rooting map and in fact does not satisfy the equation of Theorem 3.11 for its hereditary closure.
Let $\mathbf{1}$ be the distinguished point for a neat base family B. The planes not through $\mathbf{1}$ are **2456, 3456, 235** and **236**. Let $\mathbf{2}$ be the distinguished point in **2456** and then $\mathbf{5}$ the one in **456**, so that $B_{\mathbf{2456}} = \{\mathbf{245}, \mathbf{256}\}$. Let $\mathbf{3}$ be the distinguished point in **3456** and then $\mathbf{6}$ the one in **456**, so that $B_{\mathbf{3456}} = \{\mathbf{346}, \mathbf{356}\}$. Then already **45, 46** and **56** belong to the hereditary closure of the neat base family. So the line $\ell = \mathbf{456}$ has three "representatives", making $f(\ell) = 3$ in the notation of the proof of Theorem 3.11, whereas $\mu(\hat{0}, \ell) = 2$. This makes f_2 larger than the Whitney number w_2, (i. e., the coefficient of t^{r-2} in $\chi(t)$). In general, the f-vector of the complex generated by a neat base family is componentwise larger than or equal to the vector of Whitney numbers of the second kind.

We will now describe a geometry L of rank 4 such that the 3-truncation $L^{[3]}$ has a rooting map π' for which $\mathtt{RC}(L^{[3]}, \pi')$ has complete 4-partite 1-skeleton, but π' does not

Figure 4.3:

Figure 4.4:

Figure 4.5:

Figure 4.6:

extend to a rooting map on L (as it would trivially if it was of the form $\pi' = \min_\omega$) and $\texttt{RC}(L, \pi)$ does not factor completely for any rooting map π.

This explains why it is *not* true that whenever there is a rooting map π' on $L^{[3]}$ such that $\texttt{RC}(L^{[3]}, \pi')^{[1]}$ is r-partite, then there is a rooting map π on L such that $\texttt{RC}(L, \pi)$ factors completely. (Compare this to Remark 3.10.) In particular, we do not know whether some analogue of Corollary 2.9 holds for rooted complexes.

Example 4.3: (cf. [5, p. 117])

Let L be the geometric lattice of the geometry given by Figure 4.6, that is, the sum of a point with the geometry of six points on the vertices and edges of a triangle. Let H_0 be the coatom (plane) determined by this triangle.

Then a rooting map π' on $L^{[3]}$ such that

$$\begin{aligned} \pi'(E) &= \mathbf{1} \\ \pi'(\mathbf{234}) &= \mathbf{2} \\ \pi'(\mathbf{456}) &= \mathbf{4} \\ \pi'(\mathbf{672}) &= \mathbf{6} \end{aligned}$$

is easily constructed. Since the only non-rooted 2-sets are **34**, **56** and **27**, the corresponding rooted complex has a 1-skeleton isomorphic to $K_{1,2,2,2}$, that is, complete 4-partite. This $\texttt{RC}(L^{[3]}, \pi')$ cannot be a broken circuit complex of $L^{[3]}$: this would require $\mathbf{2} \prec \mathbf{4} \prec \mathbf{6} \prec \mathbf{2}$.

Clearly, π' cannot be extended to a rooting map π of L: there is no consistent choice for $\pi(H_0)$. Also, $\texttt{RC}(L, \pi)$ cannot factor completely for *any* π since

$$\chi_L(t) = (t-1)^2(t^2 - 5t + 7)$$

does not factor completely over $\mathbb{Z}$.

Remark 4.4:

The Definition 3.1 of "rooting map" and "rooted complex" generalizes straightforwardly to other classes of atomic lattices. For example, let L be the face lattice of a convex polytope P. Order the vertices of P arbitrarily and for every non-empty face F of P define $\pi(F)$ to be the minimal vertex of F. Then π is a rooting map for L. In this case the rooted complex $\texttt{RC}(L, \pi)$ gives a simplicial subdivision of P with no new vertices.

In general the face numbers of rooted complexes $\texttt{RC}(L, \pi)$ are dependent on the particular rooting map π. For example, if P is a simplicial polytope with face lattice L and rooting map π as before, then $\texttt{RC}(L, \pi)$ is a cone with apex $\pi(P)$, hence

$$f_1(\texttt{RC}(L, \pi)) = f_1(P) + f_0(P) - f_0(\mathrm{star}(\pi(P))),$$

which depends on the choice of the rooting map π.

5. The Orlik-Solomon Algebra

With each finite geometric lattice L is associated a certain anticommutative algebra $\mathcal{A}(L)$ over $\mathbb{Z}$, as defined by Orlik and Solomon [21]. For the case when L is the intersection lattice of a finite set of hyperplanes in $\mathbf{C}^d$ it was shown in [21] (with complex coefficients; see also [19, Theorem (4.5)]) that $\mathcal{A}(L)$ is isomorphic to the singular cohomology algebra of the complement of the union of these hyperplanes. This algebra has been studied in several papers by Orlik, Solomon and Terao [21], [22], [19], [31].

It was shown by Gel'fand and Zelevinskii [17, Theorem II.1] and by Jambu and Terao [19] that any broken circuit complex induces a basis in $\mathcal{A}(L)$. Also, as remarked by Gel'fand and Zelevinskii, this follows from Theorem 5.4 of [5] together with Section 3 of [21].

In this section we want to show that more generally every rooted complex of L induces a basis of $\mathcal{A}(L)$. Also, the effect of factorizations of $\mathtt{RC}(L, \pi)$ on $\mathcal{A}(L)$ will be discussed. Our results here generalize work by Terao [31]. We start with a quick review of some definitions and notation.

Let A be the free abelian group over a given ground set E (i. e., $A \simeq \mathbb{Z}^{(E)}$ in the notation of [8]; E can canonically be identified with a basis of A), and let ΛA be the *exterior algebra* over A.

Thus $\Lambda A = \bigoplus_{p \geq 0} \Lambda_p A$ is a free and graded abelian group endowed with an anticommutative multiplication (so, if $u \in \Lambda_p A$ and $v \in \Lambda_q A$ then $u \wedge v = (-1)^{pq} v \wedge u$.)

If E is linearly ordered, we get a basis of ΛA of the form $\{e_S : S \subseteq E \text{ finite}\}$ by putting $e_S = e_{i_1} \wedge \ldots \wedge e_{i_p}$, where $i_1, \ldots, i_p$ are the elements of S arranged in increasing order, denoted by $S = \{i_1, i_2, \ldots, i_p\}_<$. In the same way, $\Lambda_p A$ is free with basis $\{e_S : S \subseteq E \text{ finite}, |S| = p\}$.

In particular, if E is finite with $|E| = n$, then ΛA is free of rank 2^n, and $\Lambda_p A$ is free of rank $\binom{n}{p}$.

See texts on multilinear algebra, e. g. Bourbaki [8, chapitre 3], for further discussion.

Define a mapping

$$\partial : \Lambda_p A \longrightarrow \Lambda_{p-1} A$$

by linear extension of

$$\partial(e_S) = \sum_{j=1}^{p} (-1)^{j-1} e_{S - \{i_j\}},$$

for $S = \{i_1, i_2, \ldots, i_p\}_<$, and $\partial(e_\emptyset) = 0$. (The mapping $\partial : \Lambda A \longrightarrow \Lambda A$ is actually left interior multiplication with the element d of the dual exterior algebra defined by $d(e_i) = 1$ for all $i \in E$, see [8].)

Now let L be a geometric lattice and E the set of its atoms. Identify E with a basis of $A = \mathbb{Z}^{(E)}$ as before. The *Orlik-Solomon algebra* $\mathcal{A}(L)$ is defined by

$$\mathcal{A}(L) = \Lambda A / I,$$

where I is the ideal of ΛA generated by all elements $\partial(e_C)$ where C is a circuit in L. For subsets $S \subseteq E$ we denote by $\overline{e_S}$ the class of e_S in $\mathcal{A}(L)$.

Since I is a homogeneous ideal the algebra $\mathcal{A}(L)$ inherits its grading from ΛA:

$$\mathcal{A}(L) = \bigoplus_{p \geq 0} \mathcal{A}_p(L),$$

where $\mathcal{A}_p(L) = \Lambda_p A / (I \cap \Lambda_p A)$.

For an even finer grading, we write

$$\Lambda A = \bigoplus_{X \in L} \Lambda_X A, \quad \text{where} \quad \Lambda_X A = \sum_{\vee S = X} \mathbb{Z} e_S = \mathrm{span}_{\mathbb{Z}}\{e_S : \vee S = X\}.$$

This defines a grading of ΛA because $e_S \in \Lambda_X A$ and $e_T \in \Lambda_Y A$ imply $e_S \wedge e_T \in \mathbb{Z} e_{S \cup T} \subseteq \Lambda_{X \vee Y} A$. Now the ideal I is homogeneous for this grading, since for its generators e_C (C a circuit) we have $\partial(e_C) \in \Lambda_{\vee C} A$. Thus we get a direct sum decomposition

$$\mathcal{A}(L) = \bigoplus_{X \in L} \mathcal{A}_X,$$

with $\mathcal{A}_X = \Lambda_X A / (I \cap \Lambda_X A)$. Actually we have

$$\mathcal{A}_p(L) = \bigoplus_{\substack{X \in L \\ r(X) = p}} \mathcal{A}_X,$$

and in particular $\mathcal{A}_p(L) = 0$ for $p > r(L)$, due to the following lemma (cf. [21, p.173]).

Lemma 5.1

A subset $T \subseteq E$ is dependent if and only if $\overline{e_T} = 0$.

Proof:

Suppose that T is dependent. Let C be a circuit and e_i a point such that $e_i \in C \subseteq T$. Then

$$e_T = \pm (e_i \wedge \partial(e_C)) \wedge e_{T-C} \in I.$$

For the converse, which will not be needed in subsequent proofs, let ω be an ordering of E in which a given independent set T comes first (i. e., $x \in T$ and $y \in E - T$ implies $x \prec y$.) Then $T \in \mathtt{BC}(L, \omega)$. Now, by Theorem 5.2 the set $\{\overline{e_S} : S \in \mathtt{BC}(L, \omega)\}$ is a basis of $\mathcal{A}(L)$, hence in particular $\overline{e_T} \neq 0$. □

The operator ∂ satisfies $\partial^2 v = 0$ and $\partial(u \wedge v) = \partial u \wedge v + (-1)^p u \wedge \partial v$ for $u \in \Lambda_p A$ and $v \in \Lambda A$, (which is easy to verify on basis elements.) This implies that ∂ preserves I, and thus induces a map

$$\overline{\partial} : \mathcal{A}_p(L) \longrightarrow \mathcal{A}_{p-1}(L)$$

that is given by $\overline{\partial}(\overline{e_S}) = \overline{\partial e_S}$. Clearly, $\overline{\partial}$ satisfies the two fomulas for ∂ as well.

We are now going to prove that any rooted complex $\mathtt{RC}(L, \pi)$ induces a basis of the Orlik-Solomon algebra $\mathcal{A}(L)$. The algorithmic nature of our proof is very similar in spirit to the concept of "algebra with straightening law" or "Hodge algebra" developed for the commutative case by Bacławski and Garsia [2], [3] and by De Concini, Eisenbud and Procesi [13], [14], [15]. This suggests that one might formalize a notion of "anticommutative algebras with straightening law", of which the Orlik-Solomon algebras and also the "face algebras" of Kalai [20], [7] are examples.

Theorem 5.2:

Let π be a rooting map on a geometric lattice L. Then $\{\overline{e_T} : T \in \mathtt{RC}(L,\pi)\}$ is a basis of $\mathcal{A}(L)$.

Proof:

(1) We first verify that $\{\overline{e_T} : T \in \mathtt{RC}(L,\pi)\}$ is independent in $\mathcal{A}(L)$. Let $\sum \lambda_F \overline{e_F} = 0$ be a vanishing linear combination in $\mathcal{A}(L)$, where by induction on $r(L)$ and using the direct sum decomposition $\mathcal{A}(L) = \bigoplus_{X\in L} \mathcal{A}_X$ we can assume that the sum is over bases of the rooted complex $\mathtt{RC}(L,\pi)$. Every such basis contains $x_0 = \pi(E)$, so that we can rewrite the linear combination (possibly changing signs of some of the λ_F) as

$$\overline{e_{x_0}} \wedge \{\textstyle\sum \lambda_F \overline{e_{F'}}\} = 0,$$

where every $F' = F - \{x_0\}$ is a basis of a hyperplane (coatom) of L that does not contain x_0.
Applying $\overline{\partial}$, we now obtain

$$\{\textstyle\sum \lambda_F \overline{e_{F'}}\} - \overline{e_{x_0}} \wedge \{\textstyle\sum \lambda_F \overline{\partial} \overline{e_{F'}}\} = 0.$$

Since the decomposition $\mathcal{A}_{r-1}(L) = \bigoplus\limits_{H\in L_{r-1}} \mathcal{A}_H$ is direct, this implies

$$\textstyle\sum \lambda_F \overline{e_{F'}} = 0$$

(because $e_{x_0} \wedge \{\sum \lambda_F \overline{\partial} \overline{e_{F'}}\} \in \bigoplus_{x_0\in H\in L_{r-1}} \mathcal{A}_H$), and thus

$$\sum_{\vee F' = H} \lambda_F \overline{e_{F'}} = 0$$

for every hyperplane H not containing x_0. By induction on the rank we conclude from this that $\lambda_F = 0$ for all F.

(2) Now we will show that there is a function

$$\rho : \{\text{independent subsets of } E\} \longrightarrow \{0, 1, \ldots, r\},$$

such that $\rho(T) = 0$ if and only if $T \in \mathtt{RC}(L,\pi)$, and every element $\overline{e_T} \neq 0$ with $\rho(T) > 0$ can be written as a linear combination of the form

$$\overline{e_T} = \sum_{\rho(S)<\rho(T)} \lambda_S \overline{e_S}$$

("straightening relation"). Iterated application of this type of expansion shows (after at most r steps) that every $\overline{e_T}$ is a linear combination of $\{\overline{e_S} : S \in \mathtt{RC}(L,\pi)\}$.
For an independent set $T \subseteq E$, let $\rho(T)$ be the size of the maximal broken subset T^0 of T if T is not rooted (cf. Lemma 3.5(2)), and $\rho(T) = 0$ otherwise. Assume that T

is not rooted, and let $t^* = \pi(T^0)$. Clearly $t^* \notin T$, since T is independent. From this and the proof of Lemma 3.5 one sees that

$$\rho(T \cup t^* - t) = \rho(T^0 \cup t^* - t) < \rho(T^0) = |T^0| = \rho(T),$$

for any $t \in C - t^*$, where C is the unique circuit in $T^0 \cup t^*$.
Multiplying the relation $\overline{\partial e_C} = 0$ by $\overline{e_{T^0 \cup t^* - C}}$ we get a relation of the form

$$\overline{e_{T^0}} = \sum_{t \in C - t^*} \pm \overline{e_{T^0 \cup t^* - t}},$$

and thus a straightening relation

$$\overline{e_T} = \sum_{t \in C - t^*} \pm \overline{e_{T \cup t^* - t}},$$

as claimed. □

Theorem 5.2 does *not* require L to be finite. We note, however, that in this case of finite L Theorems 3.11 and 5.2 together imply the following basic result due to Orlik and Solomon.

Corollary 5.3: [21, Theorem 2.6]
If L is finite, then the Poincaré polynomial of $\mathcal{A}(L)$ is $(-t)^{r(L)}\chi_L(-\frac{1}{t})$.

Using Theorem 5.2, we can easily show how every factorization of a rooted complex induces a factorization of the associated Orlik-Solomon algebra as a tensor product of graded subspaces.

Theorem 5.4:
Let π be a rooting map on a geometric lattice L such that the rooted complex $\Delta = \mathtt{RC}(L, \pi)$ has a join decomposition $\Delta = \Delta_1 * \Delta_2$. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the subgroups of $\mathcal{A}(L)$ generated by $\{\overline{e_F} : F \in \Delta_1\}$ and $\{\overline{e_G} : G \in \Delta_2\}$, respectively. Then the $\mathbb{Z}$-linear map

$$\kappa : \mathcal{A}_1 \otimes \mathcal{A}_2 \longrightarrow \mathcal{A}(L),$$

defined by multiplication in $\mathcal{A}(L)$, is an isomorphism (of graded abelian groups).

Proof:

Every member of the rooted complex Δ is of the form $F \cup G$, where $F \in \Delta_1$ and $G \in \Delta_2$, and

$$\overline{e_{F\cup G}} = \overline{e_F} \wedge \overline{e_G} = \kappa(\overline{e_F} \otimes \overline{e_G}).$$

But $\{\overline{e_F} : F \in \Delta_1\}$ and $\{\overline{e_G} : G \in \Delta_2\}$ are linearly independent by Theorem 5.2, hence are bases of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively.
So, by Theorem 5.2, κ maps a basis of $\mathcal{A}_1 \otimes \mathcal{A}_2$ bijectively to a basis of $\mathcal{A}(L)$ and thus is an isomorphism. □

Of course, the isomorphism κ implies a factorization $P(\mathcal{A}, t) = P(\mathcal{A}_1, t) \cdot P(\mathcal{A}_2, t)$ of the Poincaré polynomial of $\mathcal{A}$, corresponding (via Corollary 5.3) to the factorization of $\chi_L(t)$ induced by the factorization of $\mathtt{RC}(L, \pi)$.

From Theorem 5.4 we deduce the following factorization result for $\mathcal{A}(L)$ in the case of a modular element, due to Terao [31, Theorem (3.8)].

Corollary 5.5:

If M is a modular flat in L, then the $\mathbb{Z}$-linear map

$$\kappa : \mathcal{A}([\hat{0}, M]) \otimes \{ \bigoplus_{X \wedge M = \hat{0}} \mathcal{A}_X \} \longrightarrow \mathcal{A}(L),$$

defined by multiplication in $\mathcal{A}(L)$, is an isomorphism. Conversely, if κ is an isomorphism for some flat $M \in L$, then M is modular.

Proof:

For this we choose a well-ordering ω such that $\mathtt{BC}(L, \omega)$ factors according to Theorem 2.5, and then apply Theorem 5.4. For the converse, observe that if M is not modular, then M has a complement M' such that $r(M) + r(M') > r(L)$. This means that κ kills $\mathcal{A}_M \otimes \mathcal{A}_{M'}$. □

6. Further Generalization

As was remarked after Theorem 3.2, the rooted complexes $\Delta = \mathtt{RC}(L,\pi)$ have the property that

$$\Delta|_X \text{ is a cone of dimension } r(X)-1 \text{ for every } X \in L_{>\hat{0}}. \tag{6.1}$$

For every complex with this property, we get a *covering* of $L_{>\hat{0}}$ by ideals in principal filters above atoms of L (i. e., subsets F of L such that all flats X in F contain a fixed atom x and $Y \le X$ implies $Y \in F$ for $X \in F$), by defining

$$\Pi(x) = \{X \in L_{>\hat{0}} : \Delta|_X \text{ is a cone with apex } x\}.$$

Here observe that rooting maps correspond to the special case of a *partition* of $L_{>\hat{0}}$ into ideals in principal filters above atoms, via $\Pi(x) = \{X \in L_{>\hat{0}} : \pi(X) = x\}$.

We will show that most properties proved in the last sections for rooted complexes generalize to complexes satisfying (6.1). This property (6.1) characterizes rooted complexes for matroids of rank $r \le 3$ (this is easy to verify), but *not* for higher rank, as the following example shows.

Example 6.1:

Let $E = \{\mathbf{1},\mathbf{2},\ldots,\mathbf{7}\}$ and let $\mathcal{M}$ be the matroid of rank 4 on E whose circuits are $\mathbf{1234}$, $\mathbf{1256}$ and all the sets of size 5 not containing any of these two. Let $\Delta \subseteq 2^E$ be the simplicial complex whose minimal nonfaces are $\mathbf{134}$, $\mathbf{256}$ and all the 4-subsets of $\{\mathbf{1},\ldots,\mathbf{6}\}$ that do not contain $\mathbf{134}$ or $\mathbf{256}$. Then Δ satisfies (6.1), but is not generated by a rooting map: Theorem 3.2(2) would require $\pi(\mathbf{1234}) = \mathbf{2}$ and $\pi(\mathbf{1256}) = \mathbf{1}$, which does not allow a consistent choice for $\pi(\mathbf{12})$.

In the following sequence of claims we sketch some of the properties that hold for an arbitrary simplicial complex Δ on the vertex set E which satisfies (6.1).

Claim 6.2:

Δ is a subcomplex of the independence complex.

Proof:

Let $C \in \Delta$ be a circuit, then $\Delta|_{\overline{C}}$ has dimension at least $|C|-1 = r(\overline{C})$. ∎

Claim 6.3:

For every flat X, $\Delta|_X$ is pure.

Proof:

Suppose not, then by induction we can assume that Δ has a maximal face F of size $|F| < r$. Now if Δ is a cone over x_0, then $F - \{x_0\}$ is contained in a hyperplane H of L that does not contain x_0. But $\Delta|_H$ is pure by induction, hence $F - \{x_0\} \subset F'$, with $F' \in \Delta|_H$, and $F' \cup \{x_0\} \in \Delta$ properly contains F. ∎

Claim 6.4:

If L is finite, then the f-vector of Δ is given by the Whitney-Rota formula (1.1).

Proof:

See the proof of Theorem 3.11. ∎

Lemma 6.5:

(i) Let Δ be a complex satisfying (6.1). Define Δ-broken sets as those sets $B \subseteq E$ that do not contain a cone point of $\Delta|_{\overline{B}}$.
Then Δ is the complex of all subsets of E that do not contain a Δ-broken set.
The union of any two Δ-broken sets is Δ-broken, such that every $B \notin \Delta$ contains a unique maximal Δ-broken subset.

(ii) If $\Delta = \mathtt{RC}(L, \pi)$ is a rooted complex, then every Δ-broken set is broken (but not conversely). In this case

$$\{ \text{ minimal broken sets } \} = \{ \text{ minimal } \Delta\text{-broken sets } \},$$

such that $\mathtt{RC}(L, \pi)$ is the complex of all subsets of E that do not contain a Δ-broken set.

Proof:

(i) A minimal non-face B of Δ cannot contain a cone point b of $\Delta|_{\overline{B}}$, because this would mean that $B - b$ is a non-face, too.
Conversely, every $B \in \Delta$ is a facet of $\Delta|_{\overline{B}}$, and thus contains a cone point of this complex.
If B_1 and B_2 are both Δ-broken and $b \in B_1$, say, is a cone point of $\Delta|_{\overline{B_1 \cup B_2}}$, then b is also a cone point of $\Delta|_{\overline{B_1}}$, contradiction.

(ii) The first statement is clear. For the second one, let B be a minimal broken set that is not Δ-broken. Then B contains a cone point b of $\Delta|_{\overline{B}}$. But $B - b \in \Delta$ because B is a minimal non-face of Δ, and thus $B \in \Delta$, because b is a cone point: contradiction. ∎

Claim 6.6:

$\{\overline{e_S} : S \in \Delta\}$ *is a basis of* $\mathcal{A}(L)$.

Proof:

Analogous to the proof of Theorem 5.2. For part (2), we consider Δ-broken sets instead of broken sets, define $\rho(T)$ to be the size of the maximal Δ-broken subset of T, let t^* be any cone point of $\Delta|_{\overline{T^0}}$, and use Lemma 6.5(i) instead of Lemma 3.5(2). ∎

Claim 6.7:

If $L = L_1 \times L_2$, then $\Delta = \Delta|_{X_1} * \Delta|_{X_2}$.

Proof:

Analogous to the proof of Theorem 3.9(1), using Lemma 6.5(i) and considering an arbitrary cone point b of $\Delta|_{\overline{F_1 \cup F_2}}$ instead of $\pi(F_1 \cup F_2)$. ∎

Claim 6.8:

(i) Let x_0 be a cone point of Δ, and let $\Delta = x_0 * \Delta'$. Δ' is a cone iff L is reducible.

(ii) If L is finite, then Δ' has reduced Euler characteristic $\pm\beta(L)$.

Proof:

(i) If L is reducible, then Δ' is a cone by Claim 6.7. If L is irreducible, then Δ' is not a cone by the argument of Lemma 3.8.

(ii) For each flat $X \in L$, let $f(X)$ be the number of bases of X that belong to Δ. In the proof of Claim 6.4 we saw that $f(X) = (-1)^{r(X)}\mu(\hat{0}, X)$. The result now follows from Zaslavsky's formula for $\beta(L)$ quoted in the proof of Theorem 3.12. □

We do not know whether Δ' necessarily has to be homotopy equivalent to a wedge of spheres, as in Theorem 3.12.

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Late Notes: (added in March 1989)

The following references have come to our attention since this paper was submitted.

(1) Matroids determined by their 3-truncation have been studied by M. D. Halsey (*Line closed combinatorial geometries,* Discrete Math. **65** (1987), 245-248). Corollary 2.9 also follows from the Theorem 3.2 of that paper.

(2) In addition to [17] and [19], the broken circuit complex version of Theorem 5.2 was also proved by M. Jambu and D. Leborgne (*Fonction de Möbius et arrangements d'hyperplans,* C. R. Acad. Sc. Paris **303** (1986), 311-314).

(3) M. Jambu (*Fiber-type arrangements and factorization properties*, preprint, 1987) applies the implication *(1)*$\Longrightarrow$*(4)* of Theorem 2.8 to the rational homotopy theory of hyperplane arrangements. He also discusses bases for the Orlik-Solomon algebra and their relation to free arrangements.