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Multiarrangements of Hyperplanes and their Freeness

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Abstract:

We introduce *hyperplane arrangements with multiplicities* (*multiarrangements*) as natural objects arising in different geometric and combinatorial situations.

This leads to a generalization of the theory of *free* hyperplane arrangements as defined by Terao to *free multiarrangements.*

The restriction of a (simple) free arrangement to one of its hyperplanes is always a free multiarrangement in this sense. Multiarrangements arising from unitary reflection groups are discussed.

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1 Introduction

A *hyperplane arrangement* is a finite set X of hyperplanes (subspaces of codimension one) in a finite dimensional vector spaces. The theory of hyperplane arrangements has made considerable progress in recent years. We refer to P. Cartier's talk [2] in the Bourbaki seminar for a survey. In particular, the theory of *freeness* of hyperplane arrangements initiated by H. Terao [16] has produced interesting insight into the algebraic structure of hyperplane arrangements (see especially [16, 17, 5, 6, 14]).

In this note, we propose an extension of this theory to arrangements $\widetilde{X}$ in which every hyperplane $H \in \widetilde{X}$ has a positive integer *multiplicity* $k(H)$. We call them *multiarrangements of hyperplanes.*

Section 2 will give basic definitions and a combinatorial interpretation in terms of non-simple matroids. The interpretation of matroid loops is considered. We show that the restriction of a hyperplane arrangement to a hyperplane has a natural structure of a multi-arrangement of hyperplanes.

In Section 3 we interpret multiarrangements as possibly non-reduced linear divisors on $\mathbf{C}^n$. K. Saito's theory of logarithmic vector fields and differential forms [12] can be extended to the corresponding non-reduced hypersurfaces. This yields definitions of logarithmic vector fields (tangent of order $k(H)$ to every hyperplane H in $\widetilde{X}$). We call a multiarrangement *free* if the corresponding module of logarithmic vector fields is a free module. Multiarrangements in vector spaces of dimension $n \leq 2$ turn out to be always free. A basis criterion as in [12] is given.

In Section 4 we show that the restriction of a free simple arrangement to one of its hyperplanes is always a free multiarrangement. This relates to a conjecture by P. Orlik. As a second application, we discuss the multiarrangement defined by a finite unitary reflection group and its freeness.

The material of this note is part of the author's Ph.D. thesis [22] written under the direction of Anders Björner, which continues this discussion. The author wants to thank Anders Björner, Louis Billera, Peter Orlik and Gian-Carlo Rota for valuable comments and inspiring discussions.

2 Definitions and Combinatorics

Definition 1:

Let V be an n-dimensional vector space over an arbitrary field $\mathbf{k}$. A *multiarrangement of hyperplanes* is a finite multiset $\widetilde{X} = \{H_1^{k(H_1)}, ..., H_m^{k(H_m)}\}$ of linear hyperplanes in V that is, $(n-1)$-dimensional subspaces of V containing the origin). For a hyperplane $H_i \in \widetilde{X}$, $k(H_i)$ is a positive integer, called its *multiplicity.* The *order* of $\widetilde{X}$ is $\left|\widetilde{X}\right| = \widetilde{m} = k(H_1) + ... + k(H_m)$.

The *associated arrangement* to $\widetilde{X}$ is the set $X = \{H_1, ..., H_m\}$ of order $\left|X\right| = m$. A multiarrangement $\widetilde{X}$ is a *simple arrangement* if $\widetilde{X} = X$, that is if $k(H) = 1$ for all $H \in \widetilde{X}$.

Let $X = \{H_1, ..., H_m\}$ be a simple arrangement of hyperplanes in V. For every $H \in X$ we choose a non-zero linear functional $\ell_H \in V^*$ that defines H, i.e., such that $H = \{y \in V : \ell_H(y) = 0\}$. Now the set $\{\ell_H : H \in X\}$ of vectors in V^* forms a (represented) matroid. This matroid has received considerable attention. Especially the associated geometric lattice (which is isomorphic to the lattice of intersections of the hyperplanes in X, ordered by reverse inclusion) was extensively studied by Zaslavsky [21] for enumerative questions and by Orlik and Solomon [5] under topological aspects.

For a multiarrangement $\widetilde{X}$, we consider the matroid $M(\widetilde{X})$ given by the multiset

$$\{\ell_{H_1}^{k(H_1)}, ..., \ell_{H_m}^{k(H_m)}\}$$

and its linear dependencies.

The matroid $M(\widetilde{X})$ has multiple points unless $\widetilde{X}$ is simple. The associated simple matroid is $M(X)$, the corresponding lattice of flats is again the intersection lattice of X.

A property of a multiarrangement $\widetilde{X}$ is *combinatorial* if it only depends on the associated abstract matroid $M(\widetilde{X})$. This generalizes a definition by Terao for the simple case. It was shown that f-vector [21] and "being simplicial" are combinatorial properties of simple arrangements. H. Terao's famous conjecture [16, 19] asks whether freeness of an arrangement is combinatorial. We extend this question in Section 3.

Example 2:

Let X be an arrangement in V, H a hyperplane in V. Then the restriction of X to H is the arrangement $X\big|_H = \{K \cap H : K \in X \backslash \{H\}\}$. This restriction in fact has the natural structure of a multiarrangement $\widetilde{X\big|_H}$ with

$$k(K') = \#\{K \in X : K \cap H = K'\}.$$

This construction generalizes naturally to restrictions of multiarrangements. In fact it suggests to even allow *loops* for the matroids $M(\widetilde{X})$, with V having a certain multiplicity in $\widetilde{X}$ – namely, H having multiplicity $k(H)$ in $\widetilde{X\big|_H}$, and $\widetilde{X\big|_H}$ being loop free iff $H \notin X$. However, we will *not* follow this suggestion here and exclude ("delete") loops.

Hence, $\widetilde{X\big|_H}$ will have order $\left|\widetilde{X\big|_H}\right| = \left|\widetilde{X}\right| - k(H)$, where we agree that $k(H) = 0$ if $H \notin \widetilde{X}$.

Example 3:

C. Greene [3] considers hyperplane arrangements associated to finite simple graphs. In this context, graphs with multiple edges naturally give rise to multiarrangements. The main result of [3], stating that there is a natural bijection between the regions of the arrangement and acyclic orientations of the graph, generalizes to the case of multigraphs and the associated multiarrangements. We omit details.

We note that topological considerations [22] (compare also to the projective situation in [20]) suggest to allow infinite multiplicity of hyperplanes in $\widetilde{X}$, too. For the problems considered in this note, this will not be necessary.

3 Freeness

K. Saito's theory of logarithmic differential forms and logarithmic vector fields [12] can be extended to the case of a non-reduced hypersurface in a complex manifold. To avoid unnecessary generality, we will describe the main points of this development only for the case where the hypersurface is given by a multiarrangement in $\mathbf{C}^n$ (linear hypersurface singularity), which allows us to work in the rational instead of the holomorphic category and also provides independence of the field and its characteristic.

For the following let $\widetilde{X}$ be a multiarrangement in $V = \mathbf{k}^n$, $S = \mathbf{k}[x_1, ..., x_n]$ be the ring of polynomial functions on V, and $S' = \mathbf{k}(x_1, ..., x_n)$ the quotient ring of rational functions. Let

$$Der(V) = \{\theta = \sum_{i=1}^{n} p_i \frac{\partial}{\partial x_i} : p_i \in S \text{ for } 1 \leq i \leq n\}$$

be the S-module of (algebraic) derivations on V, and

$$\Omega^s_{alg}(V) = \{\omega = \sum_{|I|=s} \omega^I dx_I : \omega^I \in S \text{ for } I \subseteq [n]\}$$

for $0 \leq s \leq n$ the S-module of algebraic s-forms on V. Similarly, let $\Omega^s_{rat}(V) = S' \otimes_S \Omega^s_{alg}(V)$ denote the S-module of rational s-forms.

For $H \in X$ let ℓ_H be a linear form in V^* defining H (as in Section 2). If we identify $\{X_1, X_2, ..., X_n\}$ with a basis of V^*, then every ℓ_H is a linear function in the variables X_i. Now let $\widetilde{Q} = \Pi_{H \in x} \ell_H^{k(H)}$ be a defining equation for the multiarrangement $\widetilde{X}$. $\widetilde{Q}$ is a polynomial in S of degree $deg\widetilde{Q} = |\widetilde{X}|$, well-determined up to a constant factor. Similarly, $Q = \Pi_{H \in X} \ell_H$ will denote a defining equation for the associated simple arrangement X, and $degQ = |X|$.

Definition 4:

Let $\widetilde{X}$ be a multiarrangement in V.

(a) The S-module

$$Der(\widetilde{X}) = \left\{\theta \in Der(V) : \theta(\ell_H) \in \ell_H^{k(H)} S \text{ for all } H \in X\right\}$$

is called the module of *logarithmic vector fields* along $\widetilde{X}$.

(b) The S-modules

$$\Omega^s(\widetilde{X}) = \left\{\frac{\omega}{\widetilde{Q}} : \omega \in \Omega^s_{alg}(V); \omega \wedge d\ell_H \in \ell_H^{k(H)} \Omega^{s+1}_{alg}(V) \ \text{ for all } H \in X\right\}$$

$(1 \leq s \leq n)$ are the modules of *logarithmic s-forms* (s-forms with at most logarithmic pole along $\widetilde{X}$).

Note that Definition 4 reduces to the definitions of [16] (in the form of [11]) in the case of simple arrangements, which were originally given by [12] for the general case of reduced complex hypersurface singularities, that is,

$$\begin{aligned}\Omega^s(X) &= \left\{\frac{\omega}{Q} : \ \omega \in \Omega^s_{alg}(V); \omega \wedge \mathrm{d}\ell_H \in \ell_H \Omega^{s+1}_{alg}(V) \ \text{ for all } H \in X\right\} \\ &= \left\{\omega \in \Omega^s_{rat}(V) : \ Q\omega \in \Omega^s_{alg}(V); \ Q\mathrm{d}\omega \in \Omega^{s+1}_{alg}(V)\right\}\end{aligned}$$

As in [15], we can see that $Der(\widetilde{X})$ is isomorphic to the module

$$P(\widetilde{X}) = \left\{\mathbf{p} \in S^n : \ell_H^{k(H)} \middle| \ell_H \circ \mathbf{p} \quad \text{for all} \ \ H \in X\right\}$$

of polynomial functions $V \to V$ that map every hyperplane into itself with sufficient multiplicity.

Theorem 5:

The pairing of S-modules

$$\begin{array}{ccc} \Omega^1(\widetilde{X}) \times Der(\widetilde{X}) & \longrightarrow & S \\ (\theta; \frac{\omega}{\widetilde{Q}}) & \mapsto & \frac{\omega(\theta)}{\widetilde{Q}} \end{array}$$

is well defined and non-singular. Hence $Der(\widetilde{X})$ and $\Omega^1(\widetilde{X})$ are dual S-modules, and $Der(\widetilde{X})$ is free if and only if $\Omega^1(\widetilde{X})$ is free.

Proof:

See [12]. Some arguments are simplified by noting $Der(\widetilde{X}) = \bigcap_{H \in X} Der(\{H^{k(H)}\})$, and assuming $H = \{x_1 = 0\}$ for a particular hyperplane after suitable transformation of coordinates. □

Definition 6:

A multiarrangement $\widetilde{X}$ is called *free* iff $Der(\widetilde{X})$ is a free S-module.

Corollary 7:

For dimension $n \leq 2$, every multiarrangement $\widetilde{X}$ is free.

Proof:

Over a ring of homological dimension at most two, reflexive modules are free. □

The following theorem collects basic facts about bases of $Der(\widetilde{X})$ and $\Omega^1(\widetilde{X})$. Both the results and proofs are straightforward generalizations from the case of simple arrangements, as in [12] and [16].

Theorem and Definition 8:

Let $\widetilde{X}$ be a multiarrangement in $V = \mathbf{k}^n$.

(a) $Der(\widetilde{X})$ is an S-module of rank n. Let $\theta_i = \sum_{j=1}^{n} p_{ij} \frac{\partial}{\partial x_j} \in Der(\widetilde{X})$ for $1 \leq i \leq n$. Then $\{\theta_1, ..., \theta_n\}$ is a basis of $Der(\widetilde{X})$ if and only if $det(p_{ij}) = c \cdot \widetilde{Q}$ for some $c \in \mathbf{k}^*$. If $\widetilde{X}$ is free, then $Der(\widetilde{X})$ has a homogeneous basis, that is, we can choose $\{\theta_1, ..., \theta_n\}$ such that for all i and j, p_{ij} is either 0 or a homogeneous polynomial of degree m_i.

The multiset $\{m_1, ..., m_n\}$ of *multiexponents* for a homogeneous basis of $Der(\widetilde{X})$ defined this way does not depend on the particular homogeneous basis chosen. The multiexponents satisfy $m_1 + ... + m_n = \left|\widetilde{X}\right|$.

(b) $\Omega^1(\widetilde{X})$ is an S-module of rank n. Let $\omega_i = \sum_{j=1}^{n} q_{ij} \mathrm{d}x_j \in \Omega^1(\widetilde{X})$ for $1 \leq i \leq n$. Then $\{\omega_1, ..., \omega_n\}$ is a basis of $\Omega^1(\widetilde{X})$ if and only if $det(q_{ij}) = c \cdot \widetilde{Q}^{-1}$ for some $c \in \mathbf{k}^*$.

$\{\omega_1, ..., \omega_n\}$ is a dual basis to $\{\theta_1, ..., \theta_n\}$ if (q_{ij}) is the transposed inverse matrix of (p_{ij}). Thus, if $\widetilde{X}$ is free, then $\Omega^1(\widetilde{X})$ has a homogeneous basis: $\{\omega_1, ..., \omega_n\}$ can be chosen such that for all i and j, q_{ij} is either 0 or a homogeneous rational function of degree $-m_i$.

Corollary 7 in fact implies that every multiarrangement in dimension $n = 2$ is "inductively free" in the sense of [16]. Using Theorem 8, this can be verified constructively.

However, it is nontrivial to write down bases for large 2-dimensional multiarrangements as in the simple case [16, p.296]. This has several reasons:

One difficulty stems from the fact that 2-dimensional *simple* arrangements are supersolvable [15, 4] and hence in suitable coordinates have a triangular basis for $Der(\widetilde{X})$ (that is, $p_{ij} = 0$ for $i > j$, see [22]) and $\Omega^1(\widetilde{X})$ – but this is no longer true for multiarrangements, as the following computations show.

Example 9:

The multiarrangement $\widetilde{X}_1$ defined by $\tilde{Q}_1 = X_1^2X_2^2(X_1 - X_2)$ is free with multiexponents $\{2,3\}$ and a "triangular basis" of $Der(\widetilde{X})$ given by

$$\begin{aligned} \theta_1 &= X_1^2\frac{\partial}{\partial X_1} + X_2^2\frac{\partial}{\partial X_2} \\ \theta_2 &= (X_1 - X_2)X_2^2\frac{\partial}{\partial X_2} \end{aligned}$$

Adding an extra hyperplane, we get $\widetilde{X}_2$ defined by $\tilde{Q}_2 = X_1^2X_2^2(X_1 - X_2)^2$, which is free with multiexponents $\{3,3\}$. $Der(\widetilde{X}_2)$ does not have a "triangular basis". A basis is given by

$$\begin{aligned} \theta_1 &= (X_1 - X_2)X_1^2\frac{\partial}{\partial_1} + (X_1 - X_2)X_2^2\frac{\partial}{\partial X_2} \\ \theta_2 &= X_1^2X_2\frac{\partial}{\partial X_1} + X_2^2(2X_1 - X_2)\frac{\partial}{\partial X_2}. \end{aligned}$$

(We note that for general multiarrangements, the radial Euler vector field $\theta_E = \Sigma X_i\frac{\partial}{\partial X_i}$ is not in $Der(\widetilde{X})$. In fact $\theta_E \in Der(\widetilde{X})$ iff $\widetilde{X}$ is simple, and 1 is not a multiexponent of $\widetilde{X}$ unless $\widetilde{X}$ has a nonempty simple arrangement as a summand.)

Even bigger difficulties in the construction of bases have to do with the observation that multiexponents *cannot* be computed combinatorially. The following example at the same time shows that multiarrangements with triangular basis cannot be characterized combinatorially, either.

Proposition 10:

Multiexponents are not combinatorial in general.

Proof/Example:

Let $\widetilde{X}$ be a multiarrangement in $\mathbf{R}^2$, given by $\tilde{Q} = X_1^3X_2^3(X_1 - X_2)(X_1 - \alpha X_2)$, with $m = 4$ and $\tilde{m} = 8$ ($\alpha \in \mathbf{R}\backslash\{0,1\}$).

Case 1: For $\alpha = -1, \widetilde{X}$ is free with basis of $\Omega^1(\widetilde{X})$ given by

$$\begin{aligned} \omega_1 &= \frac{1}{(X_1^2 - X_2^2)}\left(\frac{dX_1}{X_1^3} - \frac{dX_2}{X_2^3}\right) \\ \omega_2 &= \frac{dX_1}{X_1^3} \end{aligned}$$

Hence in this special case $\widetilde{X}$ is free with multiexponents $\{3, 5\}$; it has a triangular basis.

Case 2: For $\alpha \in \mathbf{R}\backslash\{0, \pm 1\}$, $\widetilde{X}$ is free with a basis for $\Omega^1(\widetilde{X})$ given by

$$\begin{aligned} \omega_1 &= \frac{1}{X_1 - X_2}\left(\frac{dX_1}{X_1^3} - \frac{dX_2}{X_2^3}\right) \\ \omega_2 &= \frac{1}{X_1 - \alpha X_2}\left(\frac{dX_1}{X_1^3} - \frac{dX_2}{\alpha^2 X_2^3}\right), \end{aligned}$$

hence $\widetilde{X}$ is free with multiexponents $\{4, 4\}$ in this *generic* case. $\Omega^1(\widetilde{X})$ does *not* have a triangular basis. □

4 Two Examples and a Conjecture

In the following we will briefly discuss two examples of free multiarrangements occuring "in nature".

Theorem 11:

Let X be a simple hyperplane arrangement that is free with exponents $\{1, e_2, ..., e_n\}$. Let H be a hyperplane in X and let $\widetilde{X\big|_H}$ be the multiarrangement in H given by the restriction of X to H with the canonical multiplicities. Then $\widetilde{X\big|_H}$ is free with multiexponents $\{e_2, ..., e_n\}$.

Proof:

After change of coordinates we can assume that $\ell_H = X_1$, that is, $H = \{X_1 = 0\}$. Let $\theta_1, ..., \theta_n$ be a homogeneous basis of $Der(X)$, with $\theta_1 = \sum_{j=1}^{n} X_j \frac{\partial}{\partial X_j} = \theta_E$ and $\theta_i = \sum_{j=1}^{n} p_{ij} \frac{\partial}{\partial X_j}$. Now $H \in X$ implies $X_1 \big| p_{i1}$ for $1 \leq i \leq n$, and we can assume $p_{i1} = 0$ for $2 \leq i \leq n$, by transforming the basis $\theta_i \longrightarrow \theta_i - \frac{p_{i1}}{X_1}\theta_1$ for $i = 2, ..., n$. With this we get $\{\theta_2\big|_{X_1=0}, ..., \theta_n\big|_{X_1=0}\}$ as a basis of $Der(\widetilde{X})$: an easy computation shows $\theta_i\big|_{X_1=0} \in Der(\widetilde{X})$ for $2 \leq i \leq n$, and Theorem 8 together with $\widetilde{Q}(\widetilde{X\big|_H}) = \frac{Q(X)}{X_1}\Big|_{X_1=0}$ imply the basis property. □

Theorem 9 appears to be the "first half" of an argument that would verify P. Orlik's conjecture [19, Problem 2; 9, (1.10)]:

Conjecture 12:

For every (simple) free arrangement X and every $H \in X$, $X\big|_H$ is free.

In fact the "second half" of the argument would be supplied by a proof of the following, stronger conjecture:

Conjecture 13:

If $\widetilde{X}$ is a free multiarrangement, is X always free?

There is some evidence for Conjecture 12. First, it is trivial for $n \leq 2$, and the case $|\widetilde{X}| = |X|+1$ is also easy to check. If $\widetilde{X}$ is a restriction of a simple supersolvable arrangement to one of its hyperplanes, then $\widetilde{X}$ is free and X is also free, because it is supersolvable [15, 4]. If X is generic, then $\widetilde{X}$ is free if and only if X is free [22].

We note that P. Orlik's Conjecture 12 was motivated by the observation that for arrangements X defined by Coxeter groups, restrictions to hyperplanes in X are free [10] and restrictions to intersections of hyperplanes in X seem to be free, too [6].

This implies that Conjecture 13 seems to hold in the case where $\widetilde{X}$ is the restriction of a Coxeter arrangement to one of its hyperplanes.

A natural test case for a converse of Conjecture 13 (see below) is given by the arrangement defined by finite unitary reflection groups [13]. Let G be such a group, acting on $\mathbf{C}^n$, generated by unitary reflections, and let X_G denote the associated arrangement $X_G = \{Fix(g) : g \in G;\ \dim Fix(g) = n-1\}$. There are canonical multiplicities for the hyperplanes in X given by

$$k(H) = \#\{g \in G : Fix(g) = H\}.$$

This defines a multiarrangement $\widetilde{X}_G = \{H^{k(H)} : H \in X\}$ whose order is the number of unitary reflections in G. A defining equation of $\widetilde{X}$ is given by the determinant of the Jacobian matrix of a system of basic invariants for G [13, Thm. 5.2; 1, p.113]. See [8, 9] for detailed discussions.

The finite irreducible groups generated by unitary reflections in $\mathbf{C}^n$ have been classified by Shephard and Todd [13]. Their list includes many groups (Coxeter groups and others) for which $\widetilde{X}$ is simple and hence free by Terao's theorem [18]. Also, many of the groups live in $\mathbf{C}^2$, where every multiarrangement is free by Corollary 7. Of the remaining groups, the only infinite class will be discussed in the following Example 15. Only three more exceptional cases remain to check, if we wish to conclude that all multiarrangements defined by unitary reflection groups are free. They correspond to the groups denoted by (25), (26) and (31) in [13]. However, these arrangements are so large (with $\widetilde{m} = 24, 33, 81$ and $m = 12, 21, 19$, respectively) that we feel unable to do these verifications without massive support by symbolic computation, as successfully used in [9]. Of course a classification free proof would be desirable. One approach would be to use [18] together with a special explanation, why in this case $\widetilde{X}$ is free when X is free. We observe, however:

Example 14:

In general, a multiarrangement $\widetilde{X}$ need not be free if the underlying simple arrangement X is free:

The smallest example is come from the supersolvable arrangement X_1 defined by $Q_1 = xy(x-y)(x-z)(y-z)$. The simple arrangement X_1 is free with exponents $\{1, 2, 2\}$ whereas the multiarrangement $\widetilde{X_1}$ defined by Q_1^2 is not free. The degrees of a minimal set of homogeneous generators of $Der(\widetilde{X_1})$ is $\{3, 4, 4, 4\}$, with a relation of degree 5.

To verify these data, we used the help of the computer algebra system MACAULAY.

To decide freeness for arrangements like this one without computer assistance, the theory of multiarrangements and their freeness is not yet in a satisfactory state. In particular, we do not know any addition/deletion theorem in the spirit of [16] – such a result would suffice to decide Conjecture 13 and give criteria for freeness in the situation of Example 14.

More comprehensively, we ask for a combinatorial characterization of free multiarrangements, from which Conjecture 13 and Question 14 can be derived.

Example 15:

We consider the multiarrangement $\widetilde{X} = \widetilde{X}_{p,m,n}$ defined by the unitary reflection group $G(p;m;n)$ [13, p.277] acting $\mathbf{C}^n$, where $p \geq 1, n \geq 2, m\big|p$. A defining equation is given by

$$\tilde{Q} = (X_1 X_2 \cdot \ldots \cdot X_n)^{\frac{p}{m}-1} \Pi_{1 \leq i < j \leq n} (X_i^p - X_j^p),$$

hence $\widetilde{X}$ has order

$$n(\frac{p}{m} - 1) + \frac{n(n-1)}{2} p.$$

Note that $\widetilde{X}$ is a simple arrangement iff $m = p$ or $m = \frac{p}{2}$. This includes the cases of all infinite families of Coxeter arrangements, with $G(1;1;n) = A_{n-1} \oplus \phi_1$ (where ϕ_1 is the empty arrangement in $\mathbf{C}^1$ and $\oplus$ denotes the obvious direct sum of arrangements), $G(2;1;n) = B_n$, $G(2;2;n) = D_n$ and $G(p;p;2) = I_2(p)$. For $p \geq 3$ and $n \geq 3$, the arrangements do not have a real representation, i.e., the corresponding groups are not orthogonal reflection groups.

Now for $i \geq -1$, let $\theta_i = \sum X_i^{k_i} \frac{\partial}{\partial X_i}$, where $k_i = pi + 1$. Then $\theta_i \in Der(\widetilde{X})$ for $i > 0$, and $\theta_0 \in Der(\widetilde{X})$ if $m = p$ or $m = \frac{p}{2}$, that is, if $\widetilde{X} = X$. In fact homogeneous bases for $Der(\widetilde{X})$ can explicitly be given as

$$\{(X_1 X_2 \cdot \ldots \cdot X_n)^{p-1} \theta_{-1}, \theta_0, \theta_1, \ldots, \theta_{n-2}\} \quad \text{for } m = p$$

and

$$\{(X_1 X_2 \cdot \ldots \cdot X_n)^{\frac{p}{m}-2} \theta_0, \theta_1, \ldots, \theta_{n-1}\} \quad \text{for } m < p.$$

The verification that these vector fields are in $Der(\widetilde{X})$ is easy because of the symmetry of $\tilde{Q}$. The determinant of the basis criterion Theorem 8 reduces to a Vandermonde. From the bases of $Der(\widetilde{X})$, we read off the multiexponents as

$$\{(n-1)(p-1), 1, p+1, 2p+1, \ldots, (n-2)p+1\} \quad \text{for } m = p.$$

and

$$\{n(\frac{p}{m}-2)+1, p+1, 2p+1, ..., (n-1)p+1\} \text{ for } m < p.$$

We compare the results for $\widetilde{X}$ with the associated simple arrangement X: for $m = p$, we have $\widetilde{X} = X$ and nothing new, for $m < p$, X is free with basis $\{\theta_0, ..., \theta_{n-1}\}$ and exponents $\{1, p+1, ..., (n-1)p+1\}$.

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