ON THE POSET OF PARTITIONS

OF AN INTEGER

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Abstract: We study the posets (partially ordered sets) P_n of

partitions of an integer n, ordered by refinement, as

defined by BIRKHOFF. In particular we disprove the conjecture

that the posets P_n are COHEN-MACAULAY for all n, and show

that even the MÖBIUS function on the intervals does not alternate

in sign in general.

INTRODUCTION

Let P_n for $n \geqslant 1$ denote the poset of (unordered) partitions of the integer n, ordered by refinement, as introduced by BIRKHOFF [2, p. 16 and p. 104]:

We write partitions as $x = (a_1, a_2, \ldots, a_k)$, $y = (b_1, b_2, \ldots, b_\ell)$ etc., where we assume that $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_k > 0$, $a_1 + a_2 + \ldots + a_k = n$, and similarly for y. Then $x \leqslant y$ is defined to mean that there is a partition $\{1, \ldots, k\} = J_1 \cup J_2 \cup \ldots \cup J_\ell$ of the index set of x into ℓ disjoint, nonempty subsets, such that $b_i = \sum_{j \in J_i} a_j$ for all $1 \leqslant i \leqslant \ell$ (compare Fig. 1).

The posets P_n have been discussed by BJÖRNER [3, p. 176], who raised the question about their topological properties.

This discussion is organized as follows:
After some preliminary remarks in §1, we interpret in §2 the structure of intervals in P_n in terms of associated "puzzles". This technique is applied in §3 to disprove the shellability and COHEN-MACAULAY property for large n. In §4 we study the MÖBIUS function on P_n, and elaborate on the possible structure of intervals of P_n.

(1) GENERAL STRUCTURE

We first discuss the general structure of the posets P_n, numerical invariants, the natural embeddings and the connection to the YOUNG lattice of all partitions, ordered by containment of their YOUNG diagrams (see e.g. [1, p. 17]).

For general poset notations as well as for the notions of shellability, COHEN-MACAULAY poset and related concepts, the reader is referred to [3] or [5].

Fix $n \geqslant 1$, then P_n is a graded modular poset of rank $n-1$, with maximal element $\hat{1} = (n)$ and minimal element $\hat{0} = (1,\ldots,1)$. Its rank function is given by $r((a_1,\ldots,a_k)) = n - k$.

We note that for $m \leq n$, P_m has a natural order preserving embedding $i_{n,m}: P_m \to P_n$ given by $(a_1,\ldots,a_k) \to (a_1,\ldots,a_k,1\ldots1)$. These embeddings are full and faithful in the sense that for $x \leq y$ in P_m, we have $[i_{m,n}(x), i_{m,n}(y)] \cong i_{m,n}([x,y])$ isomorphic. As we obviously have $i_{k,n} \circ i_{n,m} = i_{k,m}$ for $m \leq n \leq k$, the direct limit of the sequence $(P_n)_{n \geqslant 1}$ of posets is an infinite poset (fig. 2):

$$P_\infty = \{a_1,a_2,\ldots) \mid a_1 \geqslant a_2 \geqslant \ldots ; a_i \in \mathbb{N}, a_n = 1 \text{ for all } n \gg 1\},$$

$$\cong \{a_1,a_2\ldots,a_N) \mid a_1 \geqslant a_2 \geqslant ,,, \geqslant a_N > 1 \text{ for some } N > 0\},$$

endowed with the obvious (induced) order-relation (see Fig. (2)). This poset does not seem to have been studied before. We disregard the infinite sequence of parts of size one in every element of P_∞. P_∞ inherits its rank function and its local properties (structure of intervals) from the posets P_n, has however now maximal element.

The WHITNEY numbers of the second kind (cardinalities of the rank levels) are

for P_n: $W_k = p(n,n-k) =$ number of partitions of n into n-k parts

for P_∞: $W_k = p(k)$ $\quad$ = number of partitions of k.

This suggests a relation between P_∞ and the YOUNG lattice Y of all partitions, ordered by containment of their YOUNG diagrams, which has the same WHITNEY numbers W_k. Indeed, there is the following order preserving, bijective map:

$\phi: Y \to P_\infty$

$(a_1,\ldots,a_k) \to (a_1+1,a_2+1,\ldots,a_k+1).$

Now Y is a distributive lattice, and as such even EL-shellable (see e.g. [4]), it is COHEN-MACAULAY and has all the related "nice" combinatorial properties. We suggest as a partial explanation of the "bad" behavior of P_n and P_∞ (as discussed in §§3,4) the fact that P_∞ can be thought of as an extension of the well-behaved lattice Y, where the additional order-relations (respectively the additional faces in the corresponding complexes) spoil the topological properties of Y. For example, it is easy to see that for $x \le y$ in Y with $r(x,y) > 2$, $[\phi(x),\phi(y)]$ has connected proper part in P_∞, contrary to the behavior observed in §3.

(2) PUZZLE INTERPRETATION

Let n be fixed, $x, y \in P_n, x \leq y$. To study the structure of the interval $[x,y]$, we observe that it can be visualized as a puzzle, where the "board" is given as the multiset Y of parts of y, the "pieces" as the multiset X of parts of x. (Depicting y as its YOUNG diagram, X as a collection of rectangles, the connection to the notion of a puzzle as described by ROTA and JONI [7] becomes obvious.)

Now a "solution" of the puzzle $[x,y]$ corresponds to a subposet of the interval $[x,y]$ with connected proper part (for $r(x,y) > 2$).

Examples: (a) The puzzle corresponding to $[(5,4,3,2,1),(8,6,1)]$ in P_{15} has a unique solution (in the obvious sense) given by $8 = 5 + 3, 6 = 4 + 2$, $1 = 1$. It is easy to see that this interval is isomorphic to the Boolean algebra B_2.

(b) The puzzle $[(5,5,5),(15)]$ in P_{15} has the unique solution $15 = 5 + 5 + 5$. The interval is a chain, because all possible ways to "split 15" are here essentially equivalent.

(3) Shellability of P_n

The posets P_n can be viewed as quotients of the (geometric) partition lattices π_n under the natural action of the symmetric group S_n. The posets P_n are semimodular, however not locally semimodular for $n \geqslant 8$, as first pointed out by A. BJÖRNER [3] in view of the not-semimodular interval $[(3,2,1,1,1);(5,3)]$ in P_8. Similarly P_n^* is semi-modular, but not locally semimodular for $n \geqslant 8$. Local semimodularity would imply that the posets are even CL-shellable [4]. Björner remarks that P_8 is nevertheless shellable. Indeed, a shelling of P_n for $n \leq 9$ is given by the reverse lexicographic order of the maximal chains of P_n, as induced by the lexicographic order of the partitions in P_n themselves. This method however breaks down in the interval $[(3,2,2,2,1),(6,4)]$ in P_{10}. But P_{10} can still be checked to be shellable.

We use now the technique developed in §2 to show that P_n does not have these nice topological properties for larger n. In particular we give a negative answer to the question raised by A. BJÖRNER in [3, p. 177]:

Theorem: For $n \geqslant 19$ the posets P_n contain an interval of rank 3 with disconnected proper part. The P_n are therefore not COHEN-MACAULAY and (a fortiori) not shellable for $n \geqslant 19$.

Proof: Consider the interval $J_1 = [(6,5,3,2,2,1),(8,7,4)]$ in P_{19}. The corresponding puzzle has two distinct solutions, given by

	8	7	4
1st Solution:	6 + 2	5 + 2	3 + 1
2nd Solution:	5 + 3	6 + 1	2 + 2

which are totally disjoint in the sense that they do not allow any "common split".

Thus the maximal chains in $\bar{J}_1$ are split into two disjoint classes, which do not have any point in common, i.e. J_1 is an interval of rank 3 with disconnected proper part, which contradicts COHEN-MACAULAYness of J_1, P_{19}

and (via the embedding in §1) of P_n for all $n \geqslant 19$. (The interval J_1 has actually the structure of two boolean algebras of rank 3, identified at their maximal and minimal elements: $\bar{J}_1 = \bar{B}_3 + \bar{B}_3$, $|J_1| = 14$, $\tilde{H}_0(\bar{J}_1) \cong \mathbb{Z}$, $H_1(\bar{J}_1) = \mathbb{Z}^2$.)

We remark that the interval J_1 in P_{19} is not a singular "bad" incident, as can be seen from the intervals $[(6,4,4,3,2,1);(8,7,5)]$ in P_{20} or $[(5,4,4,3,3,2);(8,7,6)]$ in P_{21}, which have the same structure as the interval J_1 just discussed. In fact there are infinitely many intervals isomorphic to J_1 in P_∞, even if intervals obtained by scalar multiplication or addition of constants are not counted as different. This can be seen from the study of the four-parameter set of intervals $[(a,b,c,d,e,f);(a+b,c+d,e+f)]$ where $a,b,c,d,e = a + b - d$ and $f = c + d - a$ are positive rational coordinates. The intervals failing to have the proper structure will lie on a finite number of hyperplanes in four-space. Clearing denominators will yield integral solutions.

Furthermore it is easy to construct product intervals $J_1 \times B_k$: the puzzle $[(2^{2k},2^{2k-1},\ldots,2^1),(2^k+2^{2k},2^{k-1}+2^{2k-1},\ldots,2^1+2^{k+1})]$ is uniquely solvable because binary representation is unique. Thus this interval as well as any scalar multiple corresponds to B_k. To get an interval $J \times B_k$, we multiply "board-parts" and "pieces" of this puzzle by $\ell \geqslant 5$ and adjoin them to those of the puzzle J_1. Similarly we can construct intervals of the form $J_1^k = J_1 \times \ldots \times J_1$ by duplicating J_1 with parts and pieces of larger size, e.g. $[(8+6,8+5,8+3,8+2,8+2,8+1,6,5,3,2\ 2,1),$ $(16+8,16+7,16+4,8,7,4)] = [(14,13,11,10,10,9,6,5,3,2,2,1),(24,23,20,8,7,5)] \cong$ $\cong J_1 \times J_1$. Now by [6 thm. 4.3] and [8, thm. 62.5] we compute the homology of $\overline{J_1 \times B_k}$ to be $H_p(\overline{J_1 \times B_k}) = \tilde{H}_p(S(\bar{J}_1 * \bar{B}_k)) = \tilde{H}_{p-1}(\bar{J}_1 * S^{k-2}) = \tilde{H}_{p-k}(\bar{J}_1)$, which shows that the COHEN-MACAULAY property is violated in arbitrarily

high homology groups.

On the other hand standard arguments in homology theory (EILENBERG-ZILBER-theorem, KÜNNETH formula and MAYER-VIETORIS-sequence, see [8]) allow to compute that the BETTI numbers of J_1^k satisfy the recursion

$$\tilde{\beta}_{p+1}(\overline{J_1^{k+1}}) = \tilde{\beta}_{p-1}(\overline{J_1^k}) + 2\tilde{\beta}_{p-2}(\overline{J_1^k}), \quad \text{hence:}$$

$$\tilde{\beta}_p(\overline{J_1^k}) = \binom{k}{p-2k+2} 2^{p-2k+2}, k \geq 1.$$

This shows that the BETTI numbers below the top-dimension become arbitrarily large, and the number of nonvanishing homology groups is not limited either.

Thus, in a certain sense, the COHEN-MACAULAY-property fails "to unbounded extent" on the intervals of P_∞.

(4) MÖBIUSFUNCTION

As P_n is COHEN-MACAULAY for $n \leq 10$, its MÖBIUSfunction will alternate in sign, i.e.

$$\mu(x;y) \cdot (-1)^{r(x,y)} \geq 0 \tag{1}$$

for $x \leq y$. The counterexample in §3 has $\mu(J) = -1$, which does not violate this condition (as $r(J_1) = 3$). However we construct:

Theorem: The MÖBIUSfunction does not alternate in sign on P_n for $n \geq 111$. For sufficiently large n, the property (1) fails on intervals of arbitrary rank $r \geq 7$.

Proof: We study the following interval of rank 7 in P_{111}:

$$J = [(21,20,11,11,8,8,6,6,6,5,5,5);(27,26,25,18,15)].$$

The corresponding puzzle has only the two following solutions.

	27	26	25	18	15
Solution 1:	21 + 6	20 + 6	11 + 8 + 6	11 + 7	5 + 5 + 5
Solution 2:	11 + 11 + 5	21 + 5	20 + 5	6 + 6 + 6	8 + 7

To check that these solutions are actually disjoint, first observe that the corresponding two parts of J_2 have no atom in common, as no pair of numbers that occur in the same column in solution 1 also occur in the same column in solution 2. Second the two parts of J_2 have no coatom in common, as no column can be split into two parts in the same way in both solutions. For example, solution 1 allows 25 to be written as 11+14, or 8+17, or 6+19, whereas solution 2 splits 25 as 20+5. Now if the two solutions had any proper element in common or any relation, then the interval J_2 would contain a maximal chain that contains an atom of one and a coatom of the other. This maximal chain determines a third solution of the puzzle, which does not exist. Thus J_2 has disconnected proper part, is especially not shellable. Let C_1 and C_2 be the connected components of $\bar{J}_2$. Then from the equivalence of the "cuts" in $15 = 5 + 5 + 5$ and $18 = 6 + 6 + 6$ we see that $\underset{\sim}{3}$ is a factor of both $\hat{C}_1$ and $\hat{C}_2$. Hence $\mu(\hat{C}_1) = \mu(\hat{C}_2) = 0$, and $\mu(J_2) = + 1$, violating (1). (The structure of J_2 can be seen to be $\bar{J}_2 = \bar{C}_1 + \bar{C}_2$, where $C_1 \stackrel{\sim}{=} \underset{\sim}{3} \times B_5$, $C_2 \stackrel{\sim}{=} \underset{\sim}{3} \times B_3 \times M_5$, where M_5 is the lattice of rank 2 and five elements corresponding to "$25 = 11 + 8 + 6$". We have $\tilde{H}_0(\bar{J}_2) = \mathbb{Z}$, $H_p(\bar{J}_2) = 0$ for $p > 0, p \neq 5$ as $\hat{C}_1$ and $\hat{C}_2$ are COHEN-MACAULAY, and $H_5(\bar{J}_2) = 0$ can be read off from the structure of C_1 and C_2, as well as $|J_2| = 3 \cdot 32 + 3 \cdot 8 \cdot 5 - 2 = 214$).

Adding different pieces and boards as in §3, all of sizes larger than 27 and yielding a uniquely solvable puzzle, it is easy to construct intervals isomorphic to $J_2 \times B_k$ of rank $7 + k$ in P_∞, which still violate (1), as $\mu(J_2 \times B_k) = \mu(J_2)\mu(B_k) = (-1)^k$.

In fact the complicated structure of μ on P_n (or: P_∞) reflects the variety of patterns that can arise in puzzles as described in §2. On the other hand we can note that

- the number $i(r)$ of nonisomorphic intervals of given rank r in P_∞ is finite, e.g. $i(1) = 1, i(2) = 6$,

- the MÖBIUSfunction on intervals of rank 3 is indeed never positive.

The first assertion follows by induction on r, observing that each coatom in $[x,y]$ corresponds to splitting a part of y into two. Now the multisets of parts X,Y (as in §2) satisfy $|Y \backslash X| \leq r, |X \backslash Y| \leq 2r$, and the part in Y split to get a coatom has to be a sum of elements in $X \backslash Y$, i.e. there are less than 2^{2r} coatoms in $[x,y]$, and the number of nonisomorphic intervals of rank $r-1$ is finite by induction hypothesis. (The maximum value of six elements in the proper part of an interval of rank 2 is e.g. reached in $[(6,5,4,3,2,1);(7,6,5,3)]$ of P_{21}.) The second assertion is readily established by case-by-case analysis of the possible situations that can yield a poset of rank 3 with disconnected proper part. In the other case, the interval is COHEN-MACAULAY and has therefore never positive MÖBIUSfunction.

Finally we note the following extension (and correction) of the result in [2, p. 104]:

Theorem: In P_n let $x_1(r) := (r+1,1,\ldots,1), x_2(r) = (r,2,1,\ldots 1)$, and $S_1 := \{x_1(r) | 1 \leq r \leq n-1\}, S_2 := \{x_2(r) | 2 \leq r \leq n-2\}$. Then for $x \leq y$, $x \in S_1$:

$$\mu(x,y) = \begin{cases} (-1)^{r(x,y)} & \text{for } y \in S_1 \cup S_2 \\ 0 & \text{otherwise} \end{cases}$$

$\mu(\hat{0},y) = 0$ for all $y \in P_n$ with $r(y) \geqslant 2$.

Proof: We use induction over $r(x,y), x = y$ being trivial. The MÖBIUS-function satisfies $\mu(x,y) = - \sum_{x \leqslant z < y} \mu(x,z)$ for $x < y$, where $x \leq z < y$ implies $r(x,2) < r(x,y)$.

Now the theorem follows from the observation that $(S_1 \cup S_2) \cap [x,y]$ is an interval in $S_1 \cup S_2$, with minimal element x, and maximal element y_0, where $y = y_0$ if $y \in S_1 \cup S_2, y_0 < y$ otherwise (in this case $y_0 \in S_2$, as $y_0 \geqslant x_1(r)$ implies $y_0 \geqslant x_2(r+1) > x_1(r)$).

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