CHARACTERIZING GRAPHS WITH GRAM DIMENSION AT MOST FOUR

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Problem description

Given a graph $G=(V,E)$ on n vertices its *Gram dimension* is the minimum $k \in \mathbb{N}$ such that: for every family of vectors $\{p_i\}_{i=1}^n \subseteq \mathbb{R}^d$ there exist $\{q_i\}_{i=1}^n \subseteq \mathbb{R}^k$ satisfying

$$ij \in V \cup E \Rightarrow p_i^T p_j = q_i^T q_j.$$

This graph parameter is denoted by $\mathrm{gd}(G)$. Since $\mathrm{gd}(G)$ is minor-monotone, the class of graphs for which $\mathrm{gd}(G) \le k$ can be characterized by a finite set of forbidden minors.

The main goal of this project is to identify the forbidden minors for $k=4$.

Applications

The graph parameter $\mathrm{gd}(G)$ can be used to guarantee the existence of bounded rank solutions to semidefinite programs (SDP). Specifically, consider the SDP:

$$\max_{X \succeq 0} \langle A_0, X\rangle \text{ s.t. } \langle A_l, X\rangle = b_l \quad \forall l \in [m],$$

and let $G = \cup_{l=0}^m G(A_l)$. Here, $G(A)$ denotes the *support graph* of matrix A, where i,j form an edge iff $A_{ij} \neq 0$.

Then, whenever the SDP is feasible, it has an optimal solution of rank at most $\mathrm{gd}(G)$.

Examples

1. $\mathrm{gd}(K_n) = n$.
2. For G a partial k-tree, $\mathrm{gd}(G) \le k+1$.
3. $\mathrm{gd}(K_{2,2,2}) = 5$.
4. $\mathrm{gd}(G) \le 2 \iff K_3 \not\preceq G$.
5. $\mathrm{gd}(G) \le 3 \iff K_4 \not\preceq G$.

Related parameters

Let $\mathrm{ed}(G) = \min k$ s.t. $\forall\, p_1,\dots,p_n \in \mathbb{R}^d$ there exist $q_1,\dots,q_n \in \mathbb{R}^k$ satisfying:

$$ij \in E \Rightarrow \|p_i - p_j\| = \|q_i - q_j\|.$$

The graph parameter $\mathrm{ed}(G)$ is minor-monotone. It is shown in [1, 2] that

$$(*) \qquad \mathrm{ed}(G) \le 3 \iff K_5, K_{2,2,2} \not\preceq G.$$

The parameters $\mathrm{ed}(G)$ and $\mathrm{gd}(G)$ are related as follows:

1. $\mathrm{gd}(G) = \mathrm{ed}(\nabla G)$.
2. $\mathrm{gd}(\nabla G) = \mathrm{gd}(G) + 1$.
3. $\mathrm{ed}(\nabla G) \ge \mathrm{ed}(G) + 1$.

Remarks

The crux of the proof of theorem $(*)$ is the inequality $\mathrm{ed}(C_2 \times C_5) \le 3$. Our main theorem combined with (1) and (3) above, imply this. If moreover $\mathrm{ed}(\nabla G) \le \mathrm{ed}(G)+1$ holds, we have that $\mathrm{gd}(G) \le 4 \iff \mathrm{ed}(G) \le 3$.

Main Theorem: $\mathrm{gd}(G) \le 4 \iff K_5, K_{2,2,2} \not\preceq G$

The proof consists of three main steps:

1. $\mathrm{gd}(K_5) = \mathrm{gd}(K_{2,2,2}) = 5$.
2. If G is 2-connected and is $\{K_5, K_{2,2,2}\}$ minor-free then it is contained in the clique sum of copies of $K_4, V_8, C_2 \times C_5$.
3. $\mathrm{gd}(V_8), \mathrm{gd}(C_2 \times C_5) \le 4$.

As in [1, 2] the most tedious part is to show that $\mathrm{gd}(C_2 \times C_5) \le 4$.

Rough sketch of the proof

To prove that $\mathrm{gd}(C_2 \times C_5) \le 4$ we follow a line of attack similar to [1]. We "stretch" the initial vector configuration $\mathbf{p} = \{p_1,\dots,p_n\} \subseteq \mathbb{R}^d$ along a non-edge (i_0, j_0), while preserving the norms and the inner products on the edges of the graph. As suggested in [3] (in the context of Euclidean distance realizations) this can be achieved by solving an SDP.

The stretched vector configuration $\mathbf{q} = \{q_1,\dots,q_n\}$ in general will not lie in $\mathbb{R}^4$. However, using an optimal solution of the dual SDP (a stress matrix) and the corresponding optimality condition we can infer the existence of linear dependencies among the q_i. Using these we can show that the configuration can be "folded" in $\mathbb{R}^4$.

Stress via SDP duality

Consider the pair of primal-dual SDP's:

$$\max_{X \succeq 0} X_{i_0 j_0} \text{ s.t. } X_{ij} = a_{ij} \quad \forall ij \in V \cup E,$$

$$\min \sum_{ij \in V \cup E} a_{ij} w_{ij} \text{ s.t. } \sum_{ij \in V \cup E} w_{ij} E_{ij} - E_{i_0 j_0} \succeq 0.$$

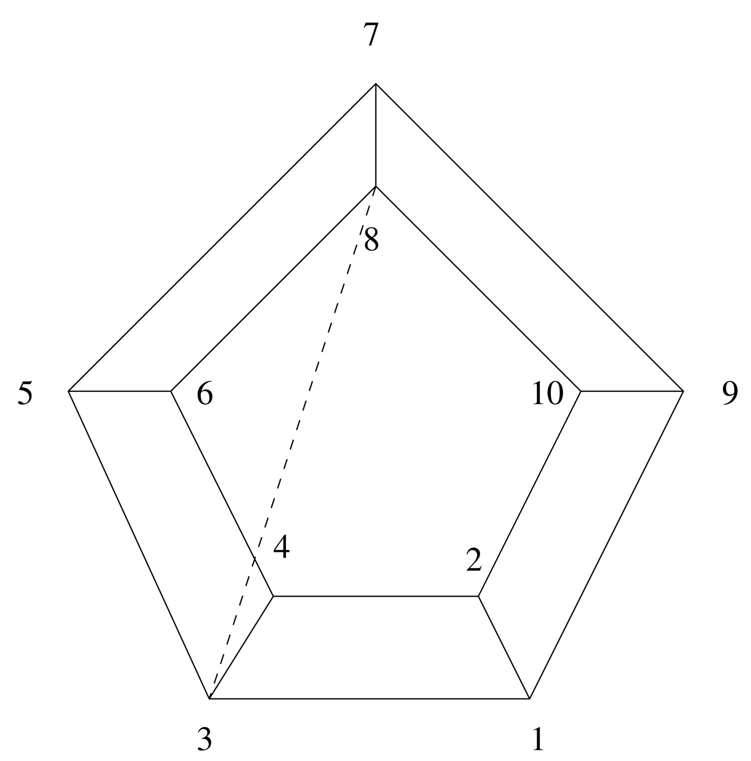
Here, $a_{ij} = p_i^T p_j$ and $E_{ij} = (e_i e_j^T + e_j e_i^T)/2$.

Since the primal program is bounded and the dual is strictly feasible, there is no duality gap and the primal attains its maximum. Assuming that the primal is also strictly feasible, the dual value is also attained. Then, the optimality condition translates into:

$$w_{ii} p_i + \sum_{j \in N(i)} w_{ij} p_j = 0 \quad \forall i \in V.$$

Sketch of $\mathrm{gd}(C_2 \times C_5) \le 4$

We stretch along the pair $(3, 8)$.



By considering all possible stressed subgraphs we can conclude in every case.

Similarly to [1] there is one exceptional case, namely, when nodes 1,2,9 and 10 are not stressed and every other edge is stressed. To deal with this case we have to stretch *for a second time* along the pair $(4, 9)$. This can be done again by an SDP, but the formulation is more involved since we have to accommodate the presence of pinned nodes, as in [3].

Two basic lemmas

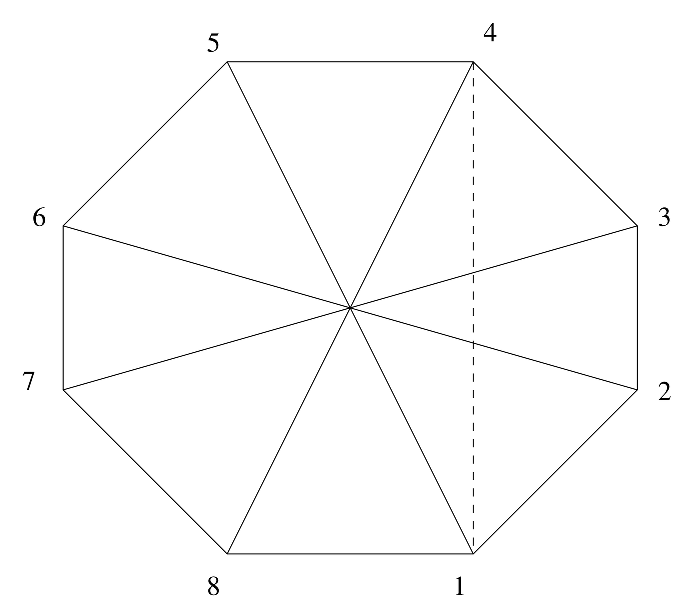
Let $G=(V,E)$ denote V_8 or $C_2 \times C_5$. Additionally, let $\mathbf{p} = \{p_i\}_{i=1}^n$ and for $T \subseteq V$ set $\mathbf{p}_T = \{p_i \mid i \in T\}$. For $i \in V$ with $N(i) = \{i_1, i_2\}$, let G/i be the graph obtained by contracting the edge ii_1.

Lemma 1: Let $T \subseteq V$ s.t. $V \setminus T$ is stable in G. If $\dim\langle \mathbf{p}_T\rangle \le 4$, then $\mathbf{p}$ can be folded in $\mathbb{R}^4$.

Lemma 2: Let i be a vertex of degree 2 in the stressed subgraph. Then, $\dim\langle \mathbf{p}\rangle = \dim\langle \mathbf{p}_{V\setminus i}\rangle$. Moreover, if Ω is a psd stress matrix for $(G, \mathbf{p})$, its Schur complement (w.r.t. w_{ii}) is a psd stress matrix for $(G/i, \mathbf{p}_{V\setminus i})$.

Proof sketch of $\mathrm{gd}(V_8) \le 4$

We stretch along the pair $(1, 4)$.



In any of the following cases we are done:

- $\exists\, T \subseteq V$ with $|T| = 4$ and $\dim\langle \mathbf{p}_T\rangle \le 2$.
- $\exists\, T \subseteq V$ with $|T| = 3$ and not consecutive on the outer circuit such that $\dim\langle \mathbf{p}_T\rangle \le 2$.

References

[1] M. Belk. Realizability of Graphs in Three Dimensions. In *Discrete Comput. Geom.* 37:139-162, 2007.

[2] M. Belk and R. Connelly. Realizability of Graphs. In *Discrete Comput. Geom.* 37:125-137, 2007.

[3] A. M.-C. So and Y. Ye. A Semidefinite Programming Approach to Tensegrity Theory and Realizability of Graphs. In *Proceedings of the 17th ACM-SIAM Symposium on Discrete Algorithms (SODA)*. 766-775, 2006.