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# On face vectors of polytopes: unimodality and generalized Euler's relations

(1, 10, 45, 120, **205**, 222, 140, 40, 1)

## Introduction

One can construct different triangle arrays from the face vectors of polytopes and observe the recurrences of them. These recurrences allow to conclude by induction on the number of rows about certain combinatorial properties of face vectors.

The unimodality conjecture is proved for cyclic and stacked polytopes. A general method, based on Stanley's construction of a Pascal's type triangle, is considered to deduce log-concavity of the face vectors of certain polytopes by taking advantage of the symmetry and the unimodality of their $h$-vectors.

A well-known necessary condition for an arbitrary $(d+2)$-tuple of integers to be the face vector of some $d$-polytope is Euler's relation. According to Euler's relation any polytope $P$ has as many faces of even dimension as it has faces of odd dimension. As a generalization of this fact one can compare the number of faces whose dimension is congruent to $i$ modulo $m$ with the number of all faces of $P$ for some positive integer $m$ and for some $0 < i < m+1$. The powers of certain doubly stochastic matrices are discussed to show that for pyramids, bipyramids, prisms and for simpplicial polytopes the above proportion is asymptotically equal to $1/m$.

## $h$-vector

**Definition: The $h$-vector of of a simplicial $d$-polytope is given by**

$$h_k := \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}, \quad \text{for} \quad 0 \le k \le d.$$

It is easy to invert the above equations to return to

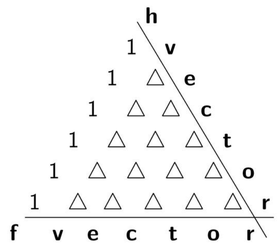
**the components of the $f$-vector:**

$$f_k = \sum_{i=0}^{d} \binom{d-i}{d-k-1} h_i, \quad \text{for} \quad -1 \le k \le d-1.$$

In other words, to know the $h$-vector is equivalent to know the $f$-vector. These relations can be easily made visible by using the following method based on Stanley's technique (see [5]).

## Pascal type triangle to express the above relations

We replace the first $d+1$ bordering 1's on the right-hand side of Pascal's triangle by the components of the $h$-vector, while the internal entries (indicated by $\triangle$) obey the usual "sum of two entries above" rule. The $f$-vector emerges in the $(d+1)^{th}$ row of this modified Pascal's triangle.




This version of Stanley's technique is used by Lee in [5].

## Modular $f$-vector

For any positive integer $m$ and for any $d$-polytope $P$, we define its *$m$-modular $f$-vector* $f^m(P)$ to be the vector with $i^{th}$ component $f_i^m(P)$ equal to the number of faces whose dimension is congruent to $i$ modulo $m$ $(i = -1, 0, \ldots, m-2)$. Consider the following example for the $f$-vector of the 9-dimensional simplex:

$$f(S_9) = (1,\ 10,\ 45,\ 120,\ 210,\ 252,\ 210,\ 120,\ 45,\ 10,\ 1)$$

$$f^4(S_9) = (1,\ 10,\ 45,\ 120) + (210,\ 252,\ 210,\ 120) + (45,\ 10,\ 1) = (256,\ 272,\ 256,\ 240)$$

Using this notation, Euler's relation can be given in a simple form:

**Euler-Poincaré formula**

$$f_{-1}^2(P) = f_0^2(P).$$

## Example of prisms

It is reasonable to investigate the classes of polytopes $\mathcal{P}$ for which one can construct some infinite triangle array from $f$-vectors (or just a part of it: infinite trapeze). For example, let $P$ be any polytope. Let us define a sequence $P_k$ of polytopes by the recursion

$(i)$ $P_0 = P$

$(ii)$ $P_{k+1}$ is a prism with base $P_k$.

The related $f$-vectors form a trapeze in which the $k^{th}$ row is $f(P_k)$. For example let $P$ is a triangle, then one obtains the following few $f$-vectors and 3-modular $f$-vectors:

| | |
|---|---|
| 1 3 3 1 | 2 3 3 |
| 1 6 9 5 1 | 6 7 9 |
| 1 12 24 18 7 1 | 19 19 25 |
| 1 24 60 43 26 9 1 | 45 50 69 |

It is easy to observe the role of the matrix $2I + Q$ in the above structure. The asymptotic Euler's relation is proved for some classes of polytopes (see [6]). Probably it holds for other natural classes of polytopes e.g. ordinary polytopes, cubical polytopes, possibly even for the class of all polytopes.

## Unimodality conjecture

In the late 1950's, Motzkin (according to Björner [2]) conjectured that $f$-vectors are *unimodal*, that is, for every $d$-polytope $P$ there exists some $k$ such that

$$f_{-1} \le f_0 \le \ldots \le f_k \ge \ldots \ge f_{d-1} \ge f_d.$$

This conjecture was apparently first disproved by Danzer (presented in a lecture in Graz in 1964, see [13]), consequently the question raised: Which polytopes have unimodal $f$-vectors? For further historical background related to the unimodality of $f$-vectors, see e.g. [2, 3, 12, 13, 14].
In [13, 14] Ziegler asks, whether the unimodality conjecture holds for cyclic polytopes. This question is answered in the affirmative by proving log-concavity. A sequence $f_0, \ldots, f_d$ is called *log-concave* if $f_{i-1} f_{i+1} \le f_i^2$ for all $0 < i < d$. Clearly, every log-concave $f$-vector is also unimodal.
Let $C_d(n)$ denote the $d$-dimensional cyclic polytope with $n$ vertices (the combinatorial type is determined by $d$ and $n$)

## Dehn-Sommerville equations

For the $h$-vector of $C_d(n)$ and for any $0 \le k \le d$, we have $h_k = h_{d-k}$, that is, the $h$-vector is symmetric. Consequently, it is enough to know the first half of the $f$-vector to compute the rest of its components

**like this:**

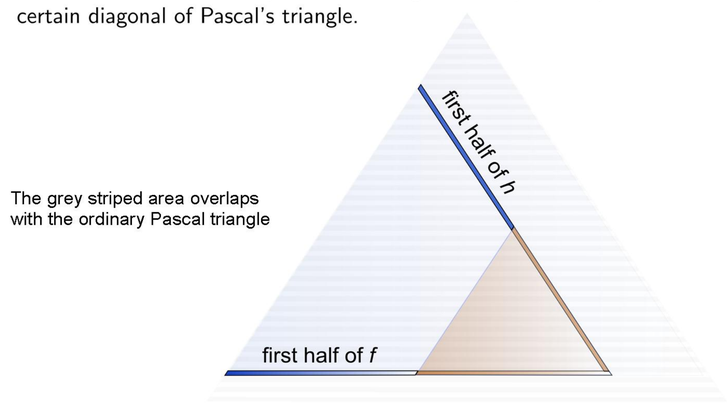
first half of $f$ $\rightarrow$ firts half of $h$ (Stanley's technique)
firts half of $h$ $\rightarrow$ entire $h$-vector (Dehn-Sommerville equations)
entire $h$-vector $\rightarrow$ entire $f$-vector (Lee's version of Stanley's technique)

That is, from equations (1) we have for

**the first half of the $h$-vector of $C_d(n)$**

$$h_k(C_d(n)) = \binom{n-d-1+k}{k}, \quad \text{for} \quad 0 \le k \le \left\lfloor \frac{d}{2} \right\rfloor \qquad (2)$$

From the above formula, it is clear, that the first half of the $h$-vector is embedded into the ordinary Pascal's triangle. Namely, it is a prefix of a certain diagonal of Pascal's triangle.




**Example (modified Pascal triangle related to $C_7(10)$)**

The underlined row in the middle is a prefix of a row of Pascal's triangle.

**1**
1 **3**
1 4 **6**
1 5 10 **10**
1 6 15 20 **30**
1 7 21 35 30 **6**
1 8 28 56 65 36 **3**
1 9 36 84 121 101 39 **1**

$f(C_7(10)) = (1,\ 10, 45, 120, 205, 222, 140, 40, 1)$

## Powers of stochastic matrices

In order to be able to deal more efficiently with the $m$-modular $f$-vectors of polytopes we use the following special *doubly stochastic matrix* and some useful properties of it (see e.g. Horn [4]).

$$A = I_m + Q = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

For example, the asymptotic analog of Euler's relation for simplices can be shown by calculating the $m$-modular $f$-vectors. In fact, the $m$-modular $f$-vector of the $(n+1)^{th}$ row of Pascal's triangle is equal to the $m$-modular $f$-vector of the $n^{th}$ row multiplied by $A$. It is shown that the reduced spread of the matrix $A^n$ tends to zero as $n$ tends to infinity (see [6]),

## Modular $M$-matrix

The concept of $m$-modular $f$-vector can be extended for $M$-matrices involved in the $g$-theorem (see Billera and Lee [1], McMullen[9] and Stanley [10]). For any positive integer $m$ and $d$, we construct the $m$-modular $M_d$ matrix, whose rows are the $m$-modular vectors of the corresponding rows of $M_d$.

**Example ($M_4$ and 3-modular $M_4$)**

$$M_4 = \begin{pmatrix} 1 & 5 & 10 & 10 & 5 \\ 0 & 1 & 4 & 6 & 3 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}, \quad M_4^3 = \begin{pmatrix} 11 & 10 & 10 \\ 6 & 4 & 4 \\ 2 & 1 & 1 \end{pmatrix}.$$

The $m$-modular $M_d$ matrix can be given by using certain powers of matrix $A$. The convergence of the reduced spread of the powers of $A$ implies that the above described asymptotic Euler's relation holds for simplicial polytopes as well. In fact, we don't even take advantage of the special properties of the $g$-vectors to prove it.

## $f$-vector

**Definition**

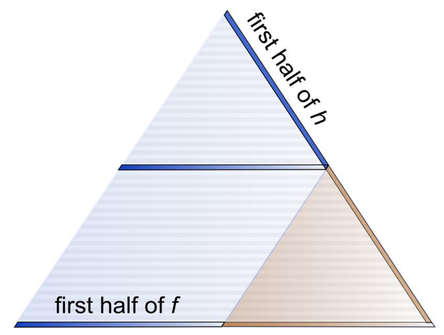
The *$f$-vector* of a $d$-polytope $P$ is the vector $f(P) = (f_{-1}, f_0, f_1, \ldots, f_d)$, where $f_k$ is the number of the $k$-dimensional faces of $P$.

The cyclic polytope $C_d(n)$ is a simplicial $\lfloor \frac{d}{2} \rfloor$-neighborly polytope, therefore the first half of its $f$-vector can be given obviously as follows

**The first half of the $f$-vector of $C_d(n)$**

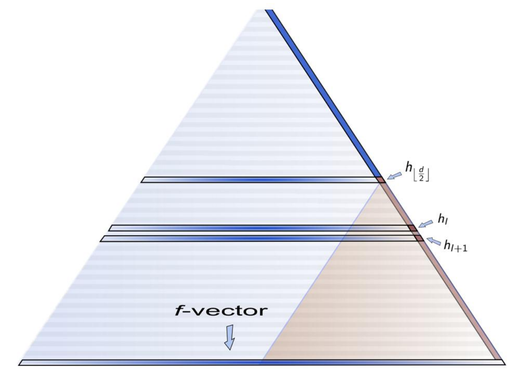
$$f_k(C_d(n)) = \binom{n}{k+1}, \quad \text{for} \quad -1 \le k < \left\lfloor \frac{d}{2} \right\rfloor \qquad (1)$$

That is, the first half of the $f$-vector of $C_d(n)$ is a prefix of the $n^{th}$ row of Pascal's triangle. For the second half of $f$, we don't need an explicit formula! Although it is determined by the first half.




## Induction on the rows of the modified Pascal triangle

It is easy to see that the log-concavity of the $l^{th}$ row in the modified Pascal's triangle implies the log-concavity of the $(l+1)^{th}$ row if $h_l \ge h_{l+1}$. The second half of the $h$-vector is decreasing, therefore the log-concavity of the f-vector relies on the log-concavity of the $\lfloor \frac{d}{2} \rfloor^{th}$ row in the middle, which is clearly log-concave as a prefix of a row of the ordinary Pascal triangle.




**Example (odd dimensional case)**

**1**
1 **3**
1 4 **6**
1 5 10 **10**
1 6 15 20 **6**
1 7 21 35 26 **3**
1 8 28 56 61 29 **1**

$f(C_6(9)) = (1,\ 9,\ 36,\ 84,\ 117, 90,\ 30,\ 1)$

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