Helly Numbers of Disconnected Families

Éric Colin de Verdière¹, Grégory Ginot², and Xavier Goaoc³

1École normale supérieure and CNRS, Paris, France
2École normale supérieure and Université Pierre et Marie Curie, Paris, France
3INRIA, Loria, Nancy, France

Helly-Type Theorems

The Helly number $h(F)$ of a finite family of sets $F$ is the smallest integer $k$ such that

(for every $G \subseteq F$ of size at most $k$, $\bigcap G \neq \emptyset$) $\implies$ (every $F$ has non-empty intersection).

Example. Every family $F$ of convex sets in $\mathbb{R}^d$ has Helly number at most $d + 1$: If every $(d + 1)$-tuple of sets $F$ has non-empty intersection, then $F$ has non-empty intersection [Helly, 1923].

Main Theorem

Let $F$ be a finite family of open sets in $\mathbb{R}^d$ such that for each $G \subseteq F$, $\bigcap G$ is the disjoint union of at most $r$ contractible sets. Then $F$ has Helly number at most $r(d + 1)$.

Extensions:
- “Contractible” can be replaced with “$Q$-homology cells”;
- extends to arbitrary (paracompact) topological spaces;
- constraints can be relaxed for small subfamilies.

Remarks

- Strengthens a result by [Kalai and Meshulam, 2008] on $r$-families of good covers (also [Amenta, 1996]);
- related to [Matoušek, 1997] on “unions of sets” (no explicit bound on Helly number);
- the bound on the Helly number is tight.

Application: Geometric Transversal Theory

Let $C_1, \ldots, C_n$ be disjoint convex sets in $\mathbb{R}^d$ and let $F_r$ be the set of lines meeting $C_r$. What conditions on $C_r$ ensure that $h'(\{F_1, \ldots, F_n\})$ is bounded?

- Central question in geometric transversal theory.
- Our Main Theorem gives unified proofs and sharpenings of existing results.

Nerves and Good Covers

The nerve of a family $F$ of sets is the simplicial complex $N(F)$ with vertex set $F$ such that $G \subseteq F$ is a simplex of $N(F)$ if $\bigcap G \neq \emptyset$.

If $r = 1$, $F$ is a good cover: the intersection of every subfamily is either empty or contractible.

Nerve Theorem: If $F$ is a good cover, then $N(F)$ is homotopy equivalent to (the geometric realization of) $N(F)$ [Borsuk, 1948].

Warm-Up: Case $r = 1$ (Topological Helly Theorem)

Let $F \subseteq G$ of size $h(F)$ such that $\bigcap F = \emptyset$ and $\forall H \subseteq F$, $\bigcap H \neq \emptyset$.

- $N(F)$ is a simplicial hole of size $|G| = h(F)$; All possible simplices on vertex set $G$, except the full simplex $G$ itself.
- $N(G)$ has non-trivial homology in dimension $|G| - 2 = h(F) - 2$.
- However, $N(G) \cong \bigcup_r N_r$ (Nerve Theorem), so $N(G)$ has trivial homology in dimension $\geq d$.
- So $h(F) < 2 < d$, qed [Helly, 1930].

Main New Object: the Multinerve

The multinerve $M(F)$ of a family $F$ of sets is the set of all connected components of the intersections of any subfamily of $F$, ordered by reverse inclusion:
- not a simplicial complex, but a simplicial poset;
- a blown-up version of the nerve $N(F)$;
- in our setting, $M(F) \cong \bigcup_r N_r$ (see below).

Proof Overview

As above, we bound the size of a simplicial hole in the nerve by a function of $d$. However, in general $N(G) \neq \bigcup_{r}$. Instead:
- Homological Multinerve Lemma: for every family $G$ satisfying the hypotheses of the theorem, $M(G) \cong \bigcup_r$. In particular, $M(G)$ has trivial homology in dimension $\geq d$.
- Projection Lemma: a simplicial hole in $N(F)$ of size $h(F)$ implies that $M(G)$ has non-trivial homology in some dimension $\geq h(F)/r - 2$, for some family $G$.

$M(F)$ (solid) extends to arbitrary (paracompact) topological spaces; $N_r$ (dashed) extends to arbitrary (paracompact) topological spaces; $N(F)$ (dotted).

Open Problems

- Explore relations with [Matoušek, 1997], which allows non-trivial homotopy in high dimensions;

Bibliography


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