Here is the outline for our research seminar on \(p\)-adic non-Abelian Hodge theory:

1. A mini-course given by a colleague of Michael on the classical non-Abelian Hodge theory (over complex numbers).
2. A brief introduction to [OV07].
3. For the main part we concentrate on [LSZ13].
4. Introduce the open problems or continue to study Faltings’ approach.

We aim to get an understanding of the \(p\)-adic Simpson correspondence and the work of Lan-Sheng-Zuo [LSZ13]. Suggestions and comments are welcome.

In below we take a brief review on the history of Non-Abelian Hodge Theory and a short introduction to the references.

### 1. History and background

The theory of non-abelian Hodge theory starts from Hitchin. Hitchin in [Hit87] studied the self-dual Yang-Mills equations and obtained the famous Hitchin equations as follows.

Let \(X/\mathbb{C}\) be a smooth projective curve over the complex numbers. \(E\) is a \(C^\infty\)-complex vector bundle of rank 2 on \(X\) together with a \(C^\infty\)-Hermitian metric, \(\nabla\) is a \(C^\infty\)-connection on \(E\), and \(\Phi \in \text{End}(E) \otimes \mathcal{A}^{(1,0)}\) is an \(\text{End}(E)\)-valued \((1,0)\) form satisfying

\[
\begin{align*}
F(\nabla) + [\Phi, \Phi^*] &= 0, \\
\nabla''(\Phi) &= 0
\end{align*}
\]

where \(F(\nabla)\) is the curvature of \(\nabla\), \(\Phi^*\) is the complex conjugation of \(\Phi\), and \(\nabla''\) is the \((0,1)\)-component of the connection \(\nabla\).

Theorem [Kob14, Chapter I, Proposition 1.3.7] tells us that on a Riemann surface, a \(C^\infty\)-complex vector bundle with a \(C^\infty\)-connection has a unique holomorphic structure determined by the \((0,1)\)-part of the connection. So if \((E, \nabla, \Phi)\) is a solution of the Hitchin equations, \(\nabla''\) will induce a holomorphic structure \(E^{\nabla''}\) on \(E\). From the second equation, we see that \(\Phi\) is holomorphic with respect \(E^{\nabla''}\). In particular, the pair \((E^{\nabla''}, \Phi)\) forms a Higgs bundle. By calculation, the connection \(\tilde{\nabla} := \nabla + \Phi + \Phi^*\) is flat and induces another holomorphic structure on \(E\) (and the Chern classes of \(E\) vanish).

Hitchin shows in [Hit87] that the solutions of the Hitchin equations modulo gauge equivalent are in one-to-one correspondence with poly-stable Higgs bundles with vanishing Chern classes via

\[
(E, \nabla, \Phi) \mapsto (E^{\nabla''}, \Phi).
\]
In [Don87] after [Hit87], Donaldson showed that every irreducible flat connection is gauge equivalent to a connection of the form $\nabla + \Phi + \Phi^*$ where $(E, \nabla, \Phi)$ is a solution of the Hitchin equations. Therefore we get a bijection between poly-stable Higgs bundles with vanishing Chern classes and semi-simple local systems on $X$ as follows:

$$\begin{align*}
\text{semi-simple } & \leftrightarrow & \text{ bundles with flat connections } & \leftrightarrow & \text{ solutions of Hitchin equations } & \leftrightarrow & \text{ polystable Higgs-bundles }
\end{align*}$$

For $\Phi = 0$, we recover the famous result by Narasimhan and Seshadri [NS65].

Simpson in [Sim88, Sim92, Sim94, Sim97] generalized this correspondence to the general case where $X$ is a smooth projective variety of arbitrary dimension, $E$ is a $C^\infty$-vector bundle of rank $r$, and he refined the correspondence with additional information given by variation of Hodge structures. He showed the corresponding moduli spaces $M_{\text{dr}}(X/\C), M_B(X/\C), M_{\text{Dol}}(X/\C)$, which we mean rank $r$ integrable connections, local systems, semi-stable Higgs bundles with vanishing Chern class respectively, are real analytically isomorphic. Furthermore, he gave a comparison of Dolbeault cohomology and de Rham cohomology and show that the correspondence between semi-stable Higgs bundles and flat connections are compatible with extensions. Simpson called this correspondence non-abelian Hodge theory, and some other mathematicians called this Simpson correspondence.

2. $p$-adic non-abelian Hodge Theory

The first step of $p$-adic analogue of Simpson correspondence is given by Faltings in [Fal05]. For curves or small affine space over $p$-adic field, Faltings constructed an equivalence between the category of Higgs bundles (the Chern classes may not vanish) and “generalised representations” using $p$-adic Hodge theory and almost étale coverings (the local computation looks similar to the calculation in [Ols05, Chapter 3,4]). This kind of representations include usual representations as a subcategory.

Faltings’ construction appears to be satisfactory only for curves and, even in this case, many fundamental questions remain open.

The equivalence depends on the choice of an exponential function for the multiplicative group.

If we restrict the inverse functor as a functor from usual representations of geometric $\pi_1(X)$-representations to Higgs bundles. It is difficult to characterise its image. All Higgs bundles in the image are semi-stable of slop zero, but people don’t know whether the image contains all those semi-stable Higgs bundles of slop zero. People also don’t know what kind of Higgs bundles comes from “genuine representations”.

After [Fal05], Abbes , Gros and Tsuji in [AGT16] explicitly explained the theory of Faltings.

Liu Ruochuan and Zhu Xinwen in [LZ17] formalized the Simpson correspondence functor on a rigid analytic space via pulling back to pro-étale site [Sch12, Sch13]. (One can show that this is implied by the results of Faltings et al., but this is difficult).
They used Scholze’s machinery to construct a Riemann-Hilbert correspondence for $p$-adic local systems on rigid analytic varieties using this Simpson correspondence functor. As a consequence, they obtained rigidity theorems for $p$-adic local systems on a connected rigid analytic variety. Finally, they gave an application of their results to Shimura varieties.

On the other hand, the $p$-adic Simpson correspondence can also be built from lifting a characteristic $p$ correspondence. Ogus and Vologodsky in [OV07] established the nonabelian Hodge theorem in positive characteristic. They constructed a functor, which they called “inverse Cartier transform” from a category of certain nilpotent Higgs modules to a category of certain nilpotent flat modules on a $W_2(k)$-liftable smooth variety. (Here $k$ is some perfect field of positive characteristic)

Lan Guitang, Sheng Mao and Zuo Kang in [LSZ13] lifted the inverse Cartier transform to the truncated Witt ring $W_n(F_q)$ and introduced the notion of Higgs-de Rham flows. They showed that there is a Higgs correspondence from crystalline representations to periodic Higgs-de Rham flows by passing form $W_n(F_q)$ to $W(F_q)$.

The theory developed in [LSZ13] also turns out to be useful in the study of Higgs bundles with nontrivial Chern classes. This has been demonstrated in the recent work [Lan15] of A. Langer on an algebraic proof of the Bogomolov’s inequality for Higgs sheaves on varieties in positive characteristic $p$ that can be lifted modulo $p^2$.

References


$p$-adic Non-abelian Hodge Theory