Forschungsseminar FS2016
Motivic Galois group and periods

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1 Introduction

A method for linearising the problem of studying topological spaces is to pass to (co)homology. The theory of motives is an attempt to linearise the study of algebraic varieties. Unlike in algebraic topology, we have many (co)homology theories for algebraic varieties together with canonical comparison isomorphisms: Betti, de Rham, ℓ-adic, crystalline, etc.. Moreover, these cohomology theories satisfy Künneth-type formulas and often come with extra structure which can be encoded as a representation of a pro-algebraic group: Galois representation, mixed Hodge structure, etc.

In the ideal world, for a field k, one wishes for a Tannakian category $\mathbf{MM}(k)$ of mixed motives over k and a functor $M : \mathsf{Sm}/k \to \mathbf{MM}(k)$ from the category of smooth schemes over k, such that cohomology theories of X with their additional structure and the higher Chow groups of X can be recovered from $M(X)$. There has been many attempts to construct such a category. In the pure case (restricting to smooth projective varieties), we have among others Grothendieck's original proposal of numerical motives (abelian semi-simple category, for which the existence of fiber functors depends on the standard conjectures), and André motives built out of motivated cycles (avoiding the standard conjectures and constructing a semi-simple tannakian category with a pure motivic Galois group). In the mixed case, Deligne and Beilinson observed that it might be easier to construct the derived category of $\mathbf{MM}(k)$ and then try to recover $\mathbf{MM}(k)$ as the heart of this category for the right t-structure. In the 90s a triangulated category $\mathbf{DM}(k)$ of motives were constructed by Voevodsky (along with similar constructions by Hanamura and Levine), based on his theory of $\mathbb{A}^1$-homotopy invariant sheaves with transfers. This category has natural realisation functors, and the higher Chow groups of X appear as extension groups. However, constructing the right t-structure on this category turned out to be at least as difficult as the standard conjectures [14]. Nori suggested another approch, constructing a tannakian category of mixed motives $\mathbf{MM}(k)$ and a mixed motivic Galois group based on his version of a weak tannakian formalism. The relation with algebraic cycles is unfortunately unclear.

In this seminar we will study yet another approach due to Ayoub. Like Nori's, it is unconditional and produces a certain pro-algebraic group as candidate for the motivic Galois group. Unlike Nori's, it builds on the work of Voevodsky and his successors.

Ayoub first constructs a category $\mathbf{DA}(S)$ of etale motivic sheaves on a scheme S with rational coefficients (which is equivalent with Voevodsky's triangulated category of motives with rational coefficients when S is the spectrum of a field). Then he develops a new variant of the tannakian formalism which works outside of the abelian category case, and applies it to the Betti realisation functor $\mathrm{Bti}^* : \mathbf{DA}(k) \to D(\mathrm{Vec}_{\mathbb{Q}})$ to obtain a Hopf algebra in $D(\mathrm{Vec}_{\mathbb{Q}})$, from which he constructs the motivic Galois group [9].

We will study the construction of $\mathbf{DA}(k)$ and the Betti realisation in details. A priori, these objects live in the world of monoidal triangulated categories; however, it is convenient to lift them to the more structured context of stable monoidal model categories. We will then present Ayoub's construction of the motivic Galois group.

We will then proceed to study applications. First, we have a non-trivial reformulation of the conjecture of Grothendieck and Kontsevich-Zagier on transcendance properties of periods. Then, we have a motivic version of the theorem of the fixed part [10], which states that if a "motivic local system" (i.e., a representation of Ayoub's motivic fundamental group!) over a variety X/k has the property that its underlying local system, after a base change $k \to \mathbb{C}$, on $X_{\mathbb{C},an}$ is trivial, then it comes from the base field. Finally, and most importantly, we have the geometric version of the Kontsevich-Zagier conjecture [13, Theorem 1.6] - this last item is a "true" application, not a motivic or conjectural one!

For a general view of these results and applications, André's Bourbaki talk "Groupes de Galois motiviques et périodes" and Ayoub's own survey [11] are recommended.

2 Talks

2.1 Overview

Give an introduction to the conjectural picture for the tannakian category of mixed motives, following [1, Chapter 21]. Sketch André's construction of a Tannakian category of pure motives and a pure motivic Galois group using motivated cycles [1, Chapter 9.2].

Recall briefly the notion of triangulated category and the example of the derived category of an abelian category. Explain how to form the Verdier localisation of a triangulated category. See e.g. [20, Chapter 10,13]. Define (symmetric) monoidal triangulated categories, and explain how the monoidal structure descends to a Verdier localisation at a thick tensor ideal, i.e. a thick triangulated subcategory $\mathcal{I}$ of $\mathcal{T}$ such that $A \in \mathcal{I}, B \in \mathcal{T} \Rightarrow A \otimes B \in \mathcal{I}$.

Next use Verdier localisation to give a first construction of the categories of motives $\mathbf{DA}^{\text{eff}}(S, \Lambda)$ and $\mathbf{DA}(S, \Lambda)$ as monoidal tensor triangulated categories, following [3, §2.1-3] (with S a base scheme and Λ a commutative ring). As we will only consider motives for the étale topology, we drop the ét from the notation in loc.cit.

Let $\sigma : k \to \mathbb{C}$. Explain the construction of the Betti realisation functor

$$\text{Bti}^* = \text{Bti}^*_\sigma : \mathbf{DA}(k, \Lambda) \to D(\Lambda - Mod)$$

as follows (for more details, see [7], where everything is stated in the model category language). The "naive" construction of the category of motives can be repeated using the site of complex analytic varieties over a fixed analytic variety X with its usual topology instead of the site of smooth k-varieties with the étale topology. One also replaces the affine line $\mathbb{A}^1$ with the open disk $\mathbb{D}^1$. This produces monoidal triangulated categories $\mathbf{AnDA}^{\text{eff}}(X, \Lambda)$ and $\mathbf{AnDA}(X, \Lambda)$. Given a smooth algebraic variety S over k, we have an associated analytic variety S^{an} and there is an induced analytification functor

$$\text{An}^* : \mathbf{DA}(S, \Lambda) \to \mathbf{AnDA}(S^{\text{an}}, \Lambda)$$

Moreover, there is a functor

$$\iota_S^* : D(S^{\text{an}}, \Lambda) \to \mathbf{AnDA}(S^{\text{an}}, \Lambda)$$

from the derived category of sheaves of Λ-modules on S^{an} to $\mathbf{AnDA}(S^{\text{an}}, \Lambda)$. Because complex analytic varieties are, locally for their topology, covered by polydisks, and because the analytic Tate object is already invertible so that the T^{an}-spectrum construction does not change the category, one can show that the functor ι_s^* is an equivalence of categories [7, Theorem 1.6]. There is a natural quasi-inverse ι_{S*}, and one then defines

$$\text{Bti}^* = \iota_{S*}\text{An}^*$$

Let $S = \mathbf{Spec}(k)$ and $X \in \mathsf{Sm}/k$: one can show that $\text{Bti}^* M(X)$ is quasi-isomorphic to the singular chain complex of X^{an} with Λ-coefficients, i.e. it is an object in $D(\Lambda-\text{Mod})$ whose homology groups are canonically isomorphic to singular homology groups of X^{an}.

Now, the functor Bti^* turns out to have a right adjoint, which we denote by Bti_*; its existence can be obtained by triangulated category theory, but will also be proven later via model category theory. By adjunction, the object $\text{Bti}_*\Lambda$ represents singular homology in $\mathbf{DA}$. We then define (with $\Lambda = \mathbb{Q}$):

$$\mathcal{H}_{\text{mot}}(k, \sigma) = \text{Bti}^*\text{Bti}_*\mathbb{Q}[0]$$

This is Ayoub's motivic Hopf algebra. One of the main results of the later talks is that $H_n(\mathcal{H}_{\mathrm{mot}}(k,\sigma)) = 0$ for $n < 0$. This shows that $H_0(\mathcal{H}_{\mathrm{mot}}(k,\sigma))$ is an Hopf algebra over $\mathbb{Q}$. We can finally put

$$G_{\mathrm{mot}}(k,\sigma) := \mathbf{Spec}(H_0(\mathcal{H}_{\mathrm{mot}}(k,\sigma))).$$

This is Ayoub's motivic Galois group, a pro-algebraic group defined over $\mathbb{Q}$.

Time permitting: state the motivic t-structure conjecture, in the form of [12, 3.26]. To put it in our context, replace **DM** with **DA**, specialize to $\Lambda = \mathbb{Q}$, and use the fact that $\mathbf{DM}_{gm}(k,\mathbb{Q})$ is equivalent to the category $\mathbf{DA}_{\mathrm{ct}}(k)$ which we just introduced. This conjecture would imply that $\mathbf{MM}_{\mathrm{ct}}(k)$ is a Tannakian category, with dual group isomorphic to $G_{\mathrm{mot}}(k,\sigma)$.

2.2 Model categories

We have seen in the overview talk that categories of motives can be thought of as monoidal triangulated categories, obtained as Verdier localisation of derived categories of certain abelian categories. For many constructions and results, a more structured approach is required. A way to present monoidal triangulated categories is through stable monoidal model categories. In this talk, we explain the basic of (monoidal) model categories. The main reference is the book [21, Chapters 14-18], which is available at the FU library.

Because of time constraints, we are going to sweep under the rug some technical aspects. In particular, any mention of "left/right proper", "cell complexes", "cofibrantly generated" or "compactly generated" can be safely ignored for this introduction.

First, give the definition of model categories:

- 14.1: Present everything up to 14.1.11, except material related to transfinite compositions. In the definition of a weak factorisation system, assume factorisations to be functorial.
- 14.2: Present everything, except the proof of 14.2.5 and remark 14.2.6.

Give the basic examples of model categories:

- 18.1: Present everything, but only mention the q-model structure (and call it the "projective model structure", to keep closer to Ayoub's terminology in later talks). A look at [18, 2.3] could be useful.
- 18.4: give 18.4.1 and state 18.4.2-4.
- 18.5: State 18.5.2-3, and give the remark just after that about projective resolutions. State 18.5.4 (and call the model structure there the "injective model structure").
- 17.2: State Thm 17.2.2. (optional, to connect model category theory with algebraic topology).

In the rest of the talk, the notions can be illustrated on the example $\mathbf{Cpl}(\Lambda)$ of complexes of Λ-modules with the projective and injective model structures.

Explain how homotopical algebra works in model categories. This section is not essential for the rest of the seminar, since we can usually compute morphisms groups using the universal property of Bousfield localisations explained in the next talk.

- 14.3: Give definition 14.3.1 and 14.3.4. State 14.3.2-3. State 14.3.9.(iii)-(iv), 14.3.10-11, 14.3.14-15.
- 14.4: Give definition 14.4.5, state 14.4.6 and 14.4.7.

Explain the functoriality of model categories. Those are the tools we will use most in the rest of the seminar.

- 16.1: Present everything up to 16.1.7
- 16.2: Present everything up to 16.2.3

Explain the interaction of model categories with monoidal structures.

- 16.3: everything without too much details, focussing on the "homological algebra" examples1 6.3.1.(v)-(vi)
- 16.4: Give definition 16.4.7. State Lemma 16.4.8. Discuss 16.4.10-12. For an alternative treatment, one can also consult [18, Chapter 4]

2.3 Bousfield localisation and local homotopy theory

Given a model category, we can construct new ones by a general process called Bousfield localisation. Bousfield localisation is used several times during the construction of the categories of motives. A particular important example for us is the homotopy theory of presheaves of complexes of abelian groups over a Grothendieck site; Bousfield localisation is used to incorporate the topology into the picture.

For Bousfield localisations:

- Give definition [21, 19.4.1], its enriched variant [21, 19.5.1], and specialize the latter to the case of model categories enriched in complexes of Λ-modules.
- Assert that, in practice, Bousfield localisations at a class of maps exist. It is not necessary to get into the precise hypotheses, which are satisfied in all the examples we encounter.
- Present some of the properties of weak equivalences and fibrations in the localised model structure, for instance [17, 3.3.3.(1),3.3.4.(1),3.4.1.(1),3.4.4].

For local homotopy theory, we use [6, Chapitre 4] as the main reference. You should specialize to the case $\mathcal{M} = \Lambda - \mathrm{Mod}$, which is the most important for us. We also cover at the same time the notion of stable model categories, which provides us with triangulated category structures.

- Define pointed categories (i.e. the map from the initial object to the final object is an isomorphism).
- State, without details, the existence of the suspension and loop adjunction (Σ^1, Ω^1) in a pointed model category (part of Thm 4.1.38). The terminology comes from pointed spaces and it may be useful to present that example.
- Give definition 4.1.44-45. Explain that the category $\mathbf{Cpl}(\Lambda)$ (with both the projective and injective structures) is stable, while the standard model structure on topological spaces is not.
- State Thm 4.1.49, and how this recovers the triangulated structure on derived categories.
- Recall the material of 4.4.1 on presheaves and sheaves with values in a category, without giving details for the proof of the existence of sheaffification.
- Give definition 4.4.15 and state Proposition 4.4.16 (as usual, ignore the hypothesis "présentable par cofibrations").
- Prove Lemma 4.4.19: deduce in particular that, if $\mathcal{M} = \mathbf{Cpl}(\Lambda)$ (with its projective model structure) and $U \in \mathcal{S}$, then the object $U \otimes \Lambda$ is cofibrant in the projective model structure.
- In the following, assume that $\mathcal{M} = \mathbf{Cpl}(\Lambda)$ with the projective model structure. It is a category of coefficients in the sense of 4.4.23, with $\mathcal{E}$ consisting of one element, the complex $R[0]$, so the results apply.
- Give definition 4.4.27. Note that for presheaves of complexes, this takes a more concrete form: a map is a top-local equivalence iff it induces isomorphism of cohomology sheaves.
- Give definition 4.4.34, and note without proof that the weak equivalences in the localised structure are exactly the top-local equivalences. State and prove 4.4.35.
- Explain the elementary functoriality of presheaves and sheaves: 4.4.44,4.4.47-49,4.4.50

- Explain the corresponding results for model categories: 4.4.46 and 4.4.51.
- State 4.4.63. (without insisting on the condition that the site has enough points, this is a technicality).
- Conclude that we have constructed a (top-local) stable monoidal model categories of presheaves of complexes of Λ-modules over a site, whose homotopy category is the derived category of sheaves of Λ-modules on that same site.

2.4 Spectra and motives

In the original definitions of categories of pure motives by Grothendieck, an important step is to pass from effective motives to non-effective motives by inverting the Lefschetz motive. We have to perform a similar step in our context, using the technique of spectra borrowed from classical homotopy theory. We then have all the ingredients necessary to define the categories of motives; we then present without proofs some of their main properties.

For spectra, we follow [19]. Note that in later talks we do not need to use spectra so often, so you don't need to give details.

- Present §1 up to 1.6. In 1.2, use Ayoub's notation Sus^n instead of F_n.
- Give definition 1.8 and state Theorem 1.14 (ignoring the "cofibrantly generated" and "left proper" part, just the existence of the model structure) and Proposition 1.16.
- Present 3.1, 3.3, 3.4, 3.6, 3.7, 3.8. Explain that 1.16 is still valid for the stable model structure.
- State 5.1.
- Discuss Lemma 5.10 together with the paragraphs above and below it. To obtain a monoidal category structure on a model category of spectra, it is necessary to use a more complicated notion of "symmetric spectra". We will not need to know the details in this seminar, and we will pretend that plain spectra are good enough!

Using all the ingredients introduced so far, give the definition of the categories of motives $\mathbf{DA}^{\mathrm{eff}}(S,\Lambda)$ and $\mathbf{DA}(S,\Lambda)$, following [9, 2.1.1] and [8, §3]. We always work with the étale topology (and not with the Nisnevich topology) so we put $\tau = \text{ét}$ in loc.cit. and omit it from the notation. The construction makes sense for S any (noetherian) scheme and Λ any commutative ring, but we are most interested in the case $S = \mathbf{Spec}(k)$ and $\Lambda = \mathbb{Q}$, for which we use the notation $\mathbf{DA}^{(\mathrm{eff})}(k)$.

Explain why, as homotopy categories of stable monoidal model categories, the categories $\mathbf{DA}^{\mathrm{eff}}(S,\Lambda)$ and $\mathbf{DA}(S,\Lambda)$ are monoidal triangulated categories. Using the Quillen adjunction $(\mathrm{Sus}^0, \mathrm{Ev}_0)$, define the derived adjunction (with the left adjoint being monoidal):

$$L\,\mathrm{Sus}^0 : \mathbf{DA}^{\mathrm{eff}}(S,\Lambda) \leftrightarrows \mathbf{DA}(S,\Lambda) : R\,\mathrm{Ev}_0$$

Recall from the first talk that $\mathbf{DA}^{\mathrm{eff}}(S,\Lambda)$ (resp. $\mathbf{DA}(S,\Lambda)$) are actually Verdier localisations of the derived category of the abelian category of presheaves (resp. T-spectra of presheaves) of complexes of Λ-modules on Sm/S. Time permitting, present the results [3, §2.4], which lay out explicitely what the Bousfield localisation looks like in the $\mathbb{A}^1$-local model structures.

Define the effective motive $M^{\mathrm{eff}}(X) \in \mathbf{DA}^{\mathrm{eff}}(S,\Lambda)$ of a smooth S-scheme Sm/S as the complex of presheaves $X \otimes \Lambda[0]$ with X the representable presheaf of sets and $\Lambda[0]$ the constant presheaf with value the complex $\Lambda[0]$. Explain why this is a cofibrant object in the model category defining $\mathbf{DA}^{\mathrm{eff}}(S,\Lambda)$ (recall that Bousfield localisation does not change cofibrations). Similarly, define $M(X) \in \mathbf{DA}(S,\Lambda)$ as $\mathrm{Sus}^0_T(X \otimes \Lambda[0])$. Since $X \otimes \Lambda[0]$ is cofibrant, we have $M(X) \simeq \mathrm{LSus}^0 M^{\mathrm{eff}}(X)$. Explain that we have

$$M^{(\mathrm{eff})}(X) \otimes M^{(\mathrm{eff})}(Y) \simeq M^{(\mathrm{eff})}(X \times Y)$$

Define the Tate twists $\Lambda(n)$ in $\mathbf{DA}^{\mathrm{eff}}(S,\Lambda)$ for $n \geq 0$ (resp. in $\mathbf{DA}(S,\Lambda)$ for $n \in \mathbb{Z}$) following [8, Notation 3.1.(ii)].

Recall the notion of compactly generated triangulated categories. State that if S is noetherian and finite dimensional, the triangulated category $\mathbf{DA}^{(\mathrm{eff})}(S,$

Recall the structure of compact objects in a compactly generated triangulated category [5, Proposition 2.1.24]. Deduce from the previous paragraph that, for S noetherian finite dimensional, the subcategory $\mathbf{DA}_c^{\mathrm{eff}}(S)$ of compact objects in $DA^{\mathrm{eff}}(S)$ (resp. $\mathbf{DA}_c(S)$ in $\mathbf{DA}(S)$) is the smallest triangulated subcategory with direct factors containing $M^{\mathrm{eff}}(X)$ for $X \in \mathsf{Sm}/S$ (resp. $M(X)(-n)$ for $X \in \mathsf{Sm}/S$, $n \in \mathbb{N}$).

We need the existence of Gysin triangles in $\mathbf{DA}^{(\mathrm{eff})}(k, \Lambda)$ for k a field. Let X be a smooth variety over k and Z be a smooth closed subvariety of codimension c. Then there are distinguished triangles

$$M^{(\mathrm{eff})}(X \setminus Z) \to M^{(\mathrm{eff})}(X) \to M^{(\mathrm{eff})}(Z)(c)[2c]$$

The proof uses a deformation to the normal cone argument and a projective bundle theorem for motives, and could be sketched. See for instance [22, Chapter 15] (this is in **DM**, but a similar argument work in **DA**). Deduce from resolution of singularities (existence of smooth compactifications) and the results of the previous paragraph that, if k is a field of characteristic 0, $\mathbf{DA}^{(\mathrm{eff})}(k, \Lambda)$ is compactly generated by motives of smooth projective varieties (with negative Tate twists in the non-effective case).

Recall the notion of strongly dualizable object in a symmetric monoidal category . State [3, Theorem 3.11]. This duality statement is the reason we work in **DA** instead of $\mathbf{DA}^{\mathrm{eff}}$. It is part of the six operation formalism for **DA**; again, this could be summarized if there is interest. From the previous paragraph, deduce that if k is a field of characteristic 0, any compact object in $\mathbf{DA}(k)$ is strongly dualizable. This is a fundamental result for the construcion of the motivic Galois group.

2.5 Betti realisation and the motivic Hopf algebra

We first develop Ayoub's weak tannakian formalism, and look for a fiber functor to apply it to. In characteristic 0, this is provided by the Betti realisation, the motivic incarnation of the singular homology of complex algebraic varieties. It is defined by copying the construction of DA in the complex analytic setting: the resulting homotopy category is then just $D(\mathrm{Vec}_{\mathbb{Q}})$. Once we have the Betti realisation, the weak Tannakian formalism provides us with a motivic Hopf algebra. We follow [9] supplemented by [7]

For the weak tannakian formalism, we follow [9, §1]:

- Recall the definitions of bialgebras and Hopf algebras, following 1.1.1-3, and of comodules in 1.3.1
- State Hypothesis 1.20 and Theorem 1.21, giving an idea of the proof if possible.
- State Proposition 1.28.(a)-(b), and remark 1.29.
- State Hypothesis 1.40 and Theorem 1.45 (no need to give details of the definition of the antipode).
- Explain the basic functoriality of the construction as in Proposition 1.48.
- State the universality property 1.55.

The comparison with usual Tannakian theory goes as follows. Let $\mathcal{A}$ be a (small) Tannakian category and $\omega : \mathcal{A} \to \mathrm{Vec}_K^{\mathrm{fd}}$ be a fiber functor. Consider the Ind-category $\mathrm{Ind}(\mathcal{A})$. This is a Grothendieck abelian category (see e.g. [20, Chap. 8.6.5]). The monoidal exact functor ω extends to a monoidal exact functor $f : \mathrm{Ind}(\mathcal{A}) \to \mathrm{Vec}_K$. This extension is an exact functor which commutes with colimits, hence by [20, 8.3.27.(iii)], it admits a right adjoint $g : \mathrm{Vec}_K \to \mathrm{Ind}(\mathcal{A})$. There is also a section $e : \mathrm{Vec}_K \to \mathrm{Ind}(\mathcal{A})$ obtained from the unit of $\mathcal{A}$. These functors satisfy Hypothesis 1.40, and the resulting Hopf algebra in Vec_K is the algebra of functions on the Tannakian dual group of $\mathcal{A}$.

For the Betti realisation, we have already explained some ingredients in the first talk. Follow the sketch in [9, 2.1.2], without proofs.

We can now put everything together and define the motivic bialgebra and Hopf algebra. For simplicity, in the rest of the seminar, we restrict to rational coefficients ($\Lambda = \mathbb{Q}$). Recall also that we ignore everything related to the Nisnevich topology. Present the material in [9, 2.1.3] in details.

The motivic Hopf algebra $\mathcal{H}_{\mathrm{mot}}(k,\sigma)$ is obtained from the effective motivic bialgebra $\mathcal{H}^{\mathrm{eff}}_{\mathrm{mot}}(k,\sigma)$ by a simple localisation procedure: this is the content of Theorem 2.14. Define the relevant maps and state the theorem. Sketch the proof. This is about the only place where we need to explicitely manipulate spectra, precisely because Theorem 2.14 reduces the study of the motivic Hopf algebra to the study of the effective motivic bialgebra.

2.6 The $\mathbb{D}^1$-localisation functor

To understand the structure of $\mathcal{H}_{\mathrm{mot}}(k,\sigma)$, we need to understand the Betti realisation. By definition, this requires getting our hands on fibrant replacement functor for the $\mathbb{D}^1$-local model structures in the analytic setting. Because of Thm 2.14, we only need to do this effectively. All the references are to [9].

Discuss (co)cubical objects in a pseudo-abelian category and the associated simple complex following App. A.1 (“karoubienne” is a synonym for ”pseudo-abelian”). The rest of App. A is devoted to quasi-isomorphic variants of the associated complex construction when the cubical object has some extra structure. Present the definition of enriched (co)cubical object and normalised complex from A.2.

Present all of 2.2.1, up to Lemme 2.28. In 2.21, there is a typo: N_i should be C_i. Ignore remark 2.22. In the proof of 2.24, you can admit the result referenced as “[5. Lem 4.2.69]”.

2.7 Approximation of singular chains

Singular homology of a complex algebraic variety does not come solely from cycle classes of algebraic subvarieties (there are many obstructions to this, especially Hodge-theoretic ones). A surprising result of Ayoub is that it can be computed from a complex of ”algebraic chains” (in a suitable sense). Combined with the results of the previous talk, this provides a very nice model for the Betti realisation functor. All the references are to [9].

Present 2.2.4 up to Remarque 2.64. There is a lot of commutative algebra going on in Proposition 2.50, how much you present is up to taste and time constraints. On the other hand, Popescu’s theorem is a key ingredient which is of independent interest. The introduction of the paper of Spivakovsky referenced as [42] in [9] gives the basic idea and present the refinement in terms of subalgebras which is also used by Ayoub. The introduction of Swan’s survey ”Néron-Popescu desingulrization” is also a good read. The proof of 2.61 is quite involved: only discuss the surjectivity part.

2.8 Motivic Galois group and period torsor

The motivic Hopf algebra is an object in the derived category of $\mathbb{Q}$-vector spaces. We would like to extract an honest Hopf algebra over $\mathbb{Q}$ from this situation. The key step is to show that the motivic Hopf algebra is concentrated in positive homological degrees. The proof uses de Rham cohomology, and introduces at the same time a key object for the sequel, the motivic period torsor.

Introduce the de Rham complexes and the de Rham spectrum, and prove Proposition 2.88, replacing the Nisnevich topology with the étale topology both in statement and proof.

Discuss the corollaries of 2.88 up to Thm 2.93. The proof of Proposition 2.91 requires a digression into Proposition 2.83 (and paragraph above) on the multiplicative structure of the singular complex. Skip everything between 2.93 and 2.101 included, and prove 2.102-3. State Thm 2.104 and explain that it is obtained from Thm 2.90 by an explicit quasi-isomorphism. Discuss the rest of the section, without details in the proof of 2.108.(b). State conjecture A from section 2.4.

State without details the fact that Ayoub’s motivic Galois group is known to be isomorphic to Nori’s [15], and that consequently its maximal reductive quotient is known to be isomorphic to André’s pure motivic Galois group (this is a result of Arapura in [2]).

State the Kontsevich-Zagier conjecture on periods in the form of [11, §2.1] and state [11, Proposition 11, Remark 13].

2.9 Motivic fundamental group and the theorem of the fixed part

As in Tannakian theory, the weak Tannakian theorem of Ayoub can be applied to define versions of fundamental groups for local systems of various kinds. In particular, it can be applied to a version of motivic local systems, and yields a motivic fundamental Hopf algebra. It turns out that this object is easier to understand than the motivic Hopf algebra itself. In particular, it is controlled in a sense by the topological fundamental group of the associated complex analytic variety.

Recall briefly how the construction of **DA** for general base schemes work. Let $f : X \to Y$ be a morphism of schemes. There is an adjunction

$$f^* : \mathbf{DA}(Y) \leftrightarrows \mathbf{DA}(X) : f_*$$

obtained by deriving a Quillen adjunction at the level of the model categories of T-spectra. This Quillen adjunction is obtained from the elementary functoriality of categories of presheaves of complexes discussed in Talk 2.3. A result from [7] which is needed below and which should be stated without proof, is that these operations "commute" with Betti realisation in the sense that there are natural transformations

$$\mathrm{Bti}^* f^* \to (f^{\mathrm{an}})^* \mathrm{Bti}^*$$

and

$$\mathrm{Bti}^* f_* \to \mathrm{Bti}^* (f^{\mathrm{an}})_*$$

with the first one always an isomorphism and the second one an isomorphism on compact objects.

Introduce the category $\mathbf{SmDA}(X)$ as in 2.4.1. Recall the relative duality theorem for smooth projective morphisms and explain why motives of smooth projective morphisms are in **SmDA**. Recall the fact that $\mathbf{DA}(k)$ is generated by motives of smooth projective varieties and deduce that $\mathbf{DA}(k) = \mathbf{SmDA}(k)$.

State Neeman's adjoint functor theorem for compactly generated triangulated categories [5, Corollaire 2.1.22]. Introduce the adjunction ϕ_f^*, ϕ_{f*}, prove 2.42 and introduce definition 2.43 (with $S = \mathbf{Spec}(k)$ for simplicity).

Present the rest of 2.4.1, maybe only sketching the proof of 2.45, which is quite intuitive but lengthy.

Finally, present 2.4.2 till 2.54. In the proof of Step A of Theorem 2.49, instead of the diagram of strongly dualizable motives M provided by Lemma 2.51, you can proceed as if $\phi_{c*}\mathbb{Q}$ is strongly dualizable itself (this is not true, but the argument shows that for our purpose it behaves as such), and replace the mentions of i^*M with $\phi_{c*}\mathbb{Q}$. Similar arguments later on, again using Lemma 2.51, can also be glossed over. Still in Step A, for the reduction to $\mathbb{Q} \to f_*\phi_{c*}\mathbb{Q}$ being invertible, gloss over the arguments involving $p_\sharp$ and explain that the reduction essentially follows from the commutation of the Betti realisation with f_*. The "théorème classique de Chevalley" mentioned in Step D is treated, for instance, in Milne's book project "Algebraic Groups", Theorem 4.19.

Time permitting, recall the theorem of the fixed part for geometric variations of Hodge structures ([16, Corollaire 4.1.2]): a geometric VHS with constant underlying local system is constant. Reformulate it in Tannakian terms using the Tannakian category of geometric VHS and draw the analogy with Theorem 2.49. In fact, such a theorem in the situation of Theorem 2.49 (and its extension to geometric variations of mixed Hodge structures, which is known by Steenbrink and Zucker) would follow from Theorem 2.49 together with an appropriate Hodge realisation functor (which has not yet been constructed).

2.10 Relative motivic Galois groups

Building on the previous talk on motivic fundamental groups, we study relative motivic Galois groups attached to a field extension. References are to [10].

Discuss briefly the notion of semi-direct product of Hopf algebras, following 2.1.1. Explain the criterion for semi-direct product decomposition in the weak tannakian setting (Theorem 2.7).

Introduce nearby cycles in the topological setting (giving the definition, see e.g [7] before Proposition 4.8) and motivic nearby cycles (as a black box, following e.g. [4]). State the commutation with the Betti realisation [7, 4.8,4.9]. State Proposition 2.20, or rather (because we do not want to discuss rigid analytic geometry) its main ingredient, which is a canonical isomorphism

$$\mathrm{Bti}^* \simeq \mathrm{Bti}^* \Psi_\pi$$

stated right at the beginning of proof.

Discuss the notion of relative motivic Galois group for a field extension; state Thm 2.34 and the more precise Proposition 2.35. The argument for 2.35 is not so difficult, in particular the "rigid analytic" aspects occur mostly to show that Ψ_π has a right adjoint and to check some of the hypotheses of the weak tannakian formalism, which we can admit.

State and sketch the proof of Thm 2.55, combining what has just been done with the result of the previous talk. Finally, state and sketch the proof of Thm 2.57.

2.11 Geometric Kontsevich-Zagier conjecture

Using all the previous material, we can now explain the main ideas of the proof of the geometric Kontsevich-Zagier period conjecture from [13].

- State the main theorem, contrast with the period algebra computed in a previous talk.
- Define the tangential Betti realisation, the tangential motivic Hopf algebras, the groups, the relative versions... and state theorem 3.10 (a variant of the main result of the previous talk).
- Introduce the tangential period torsors, and state corollary 3.22 (sweeping under the rug the "prime" version, and trying to give an idea why the log appears). Present all of section 3.7.
- Recall the basics on the Riemann-Hilbert correspondence for integrable connexions with regular singularities on a curve.
- Explain how Riemann-Hilbert lifts to smooth motives on a curve (Theorem 4.8).
- Explain how everything ties together to prove the main theorem.

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