

GL₂ Langlands correspondence.

- Function field case.

§ Introduction :

Notation

k : global field of $\text{char} > 0$. X smooth. proj curve over \mathbb{F}_q . s.t. $\text{Frac}(X) = k$.

~~we may assume X is a~~

A : adeln of k .

W_k : Weil group of k

k_v : completion of k at $v \in |X|$,

W_v : Weil grp of k_v .

$A_{\text{cusp}, r}$: cuspidal automorphic representations of rank r .

G_r : Galois rep. of $r k = r$.

Langland corresp. is a bijection $A_{\text{cusp}, r} \rightarrow G_r$, preserve L-function.

Thm (Marco's talk) to give Langlands correspondence we just have to give a (automatically surjective) mor.

$$\rho_r: A_{\text{cusp}, r} \rightarrow G_r$$

$\forall r$. ~~we~~ preserve L function. (loc. Hecke eigen value = \mathbb{F}_q -loc. Frobenius eigen)

Remark: $r=1$, ~~we~~ G_1 is given by class field theory.

Then we can do a little bit twist. Let $J \in \mathbb{A}^+$ of $\text{deg} = 1$.

$\rightarrow \rho$... construct $\rho^0: \mathbb{A}^+ \rightarrow G^+$

(Rank: right hand side) is the corresponding Galois character of J given by class field theory).

~~For~~

Today We mainly focus on GL_2 case.

§ Strategy of the proof

observation:

Lemma: G, H locally profinite grp. $V: G \times H$ ined. rep.

then $V \cong \bigoplus_G U \otimes_{\mathbb{C}} T$ where U is an ined. G -rep.

~~T~~ T is an ined. H -rep.

Rank: if we have $V \cong \bigoplus_{\substack{V_i \text{ ined} \\ V_i \subseteq V}} V_i = \bigoplus_i U_i \otimes_{\mathbb{C}} T_i$

Idea: Naive one: construct a natural (big) $GL_2(\mathbb{A}_K) \times W_K$ rep V .

has ~~the~~ decomposition $V = \bigoplus_{\pi \in \mathcal{A}_{\text{usp}, 2}} \pi \otimes P_{\pi}$

if Hecke eigenvalue of $\pi = \text{Frob. eigenvalue of } P_{\pi}$.

then $\varphi_{\pi}: \pi \mapsto P_{\pi}$.

(~~V~~ do not exist since P_{π} has obstruction.)

~~Final one:~~

~~Final one:~~ construct rep: V of $GL_2(\mathbb{A}_K) \times W_K \times W_K$.

~~has decomp. $V = \bigoplus_{\pi \in \mathcal{A}_{\text{usp}}} \pi \otimes P_{\pi} \otimes \check{P}_{\pi}$.~~

Final one:

Thm (Deligne, 1973 - 1987 He proved the theorem during these years).

\exists representation V of $GL_2(\mathbb{A}) \times W_K \times W_K$ st.

$$V = \bigoplus_{\pi \in \mathcal{A}_{\text{cusp}, 2}} \pi \otimes P_{\pi} \otimes P_{\pi}^{\vee}$$

Story: The representation is constructed as follows:

① • construct "moduli ~~space~~^{space} of sheaves" $M_D \rightarrow X \times X$, which is a relative surface. $K = \text{Frac}(X \times X)$. $M_{D,K}$ the generic fiber $\bar{M}_{D,K}$ the compactification. equipped an action of $GL_2(\mathbb{A}) \times W_K$

• $\varprojlim_D H^*(\bar{M}_{D,K}, \bar{\mathcal{O}}_e)$ inherit an action. factor through $GL_2(\mathbb{A}) \times W_K \times W_K$.

• $\varprojlim_D H^*(\bar{M}_{D,K}, \bar{\mathcal{O}}_e) = H_{\text{cusp}}^*(\bar{M}_{D,K}, \bar{\mathcal{O}}_e) + \underbrace{\text{Sigma.}}_{\text{generated by coh. class of rational cur}} \mathbb{V}$

② compute ~~$\sum (-1)^i \text{Tr } H^i(\bar{M}_{D,K}, \bar{\mathcal{O}}_e) =$~~

$$\sum_n (-1)^n \text{Tr} (g \circ \text{Frob}_n^i \text{Frob}_n^j, H^n(\bar{M}_{D,K}, \bar{\mathcal{O}}_e)) = \# \text{Fixed points}$$

$R-K$: If we want to use Fixed point thm. the space must be compact
(but finite order iso or Frobenius)

• Arthur - Selberg trace formula ↙ cuspidal part.

$$\text{Tr}_{\mathcal{Q} \in \text{deck Alg}} (A_{\text{auto}, 2}^{\mathcal{Q}}) = \begin{matrix} \text{Sum of different} \\ \text{kinds of integrals} \\ \text{of function determined} \\ \dots \end{matrix} + \begin{matrix} \text{Error terms.} \\ \text{+ come from parabolic} \\ \text{induction of} \\ \text{characters} \end{matrix}$$

③: compare them: Ξ can be compute directly.

$$\begin{aligned} & \# \text{ Fixed points of } g \cdot F_u^i F_w^j \\ &= \sum \int_{\text{orbits}} \text{char. function} \end{aligned}$$

then: we have:

$$\text{Tr}_{g \cdot F_u^i F_w^j} H_{\text{Cuspidal}}^* (\bar{M}_K/J, \bar{\mathcal{O}}_e) = \text{Tr}_{\phi_g} \cdot \chi_u^i \chi_w^j (A_{\text{cusp},2}^J)$$

where χ_u gives the local Hecke eigenvalues.

$$\text{Then: } H_{\text{Cusp}}^* (\bar{M}_K/J, \bar{\mathcal{O}}_e) = \bigoplus_{\pi} \pi \otimes P_{\pi} \otimes \eta_{\pi}$$

$$\left(\begin{aligned} & \text{Left } F_u^i F_w^j = \text{right } \chi_u^i \chi_w^j \Rightarrow \\ & \textcircled{1} \text{ Hecke eigenvalue } (\pi) = \text{Frob. e.v. } (P_{\pi}). \\ & \textcircled{2} \eta_{\pi} = P_{\pi}^{\vee} \end{aligned} \right)$$

~~thus we have \mathbb{F}_2~~

$$\left(\begin{aligned} & \text{Trace equality + mult: 1 then} \\ & \Rightarrow \pi \text{ run over all cuspidal rep in } A_{\text{cusp},2}^J. \end{aligned} \right)$$

$$\text{we have } \varphi_2^{\circ}: A_{\text{cusp},2}^J \rightarrow \mathbb{F}_2^J \quad \pi \mapsto P_{\pi}.$$

challenges:

① truncation: Trace will not converge.

(In Selberg trace formula, Trace of infinite differential trace will not converge. so we should do truncation)

$$\text{Tr}(\psi, \Delta_{\text{cusp}}^J \text{ on } \mathcal{H}^g) = \int \text{cusp. par.} + \text{Error term}^{\text{sp.}}$$

long from parabolic induction.

② $M_{(1,1)}$ has easier geometry, but its too big. we do truncation and fix level structure. \rightarrow $M_{(1,1)}^{\text{CR}}$ D./S. is a sphere.

③ Bad things: $GL_2(\mathbb{A})$ action will change v. D. !!!

(• $GL_2(A)$ action on shuffles

| • geo. property of $M_{D(\mathbb{Z}^n)}^{(r)}$.

(• compute trace / baby case . show how to compute trace .)

\$ GL_2(A) \$ action and geo. prop. of stratas.

• ~~vector~~ interpret "bundles" as cosets in $GL_2(A)$.

Notation. we assume X is geo. loc. $k_\infty = k \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ $A_\infty = \text{adele of } k_\infty$.

• $Bun_{2,X}(\overline{\mathbb{F}_q}) \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{Family of lattices } \mathcal{M} \\ \text{st. M.v. volat.} \\ \text{and } M_v \subseteq k_\infty^{\oplus 2} \\ \text{st. most v.} \\ M_v = \mathcal{O}_v \oplus \mathcal{O}_v \end{array} \right\} \xleftrightarrow{1:1} \frac{GL_2(A_\infty)}{GL_2(k_\infty) \backslash GL_2(\mathcal{O}_\infty)}$

$\Sigma \longmapsto \{ \Sigma_v \} \xrightarrow{\quad} \mathbb{X} \cdot GL_2(\mathcal{O}_\infty)$
 where $\Sigma_v = \Sigma_v = k_\infty e_1 \oplus k_\infty e_2$.
 $\{ \Sigma_v \} = \mathbb{X} \cdot GL_2(\mathcal{O}_\infty) \cdot \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$

$\mathbb{Q} \cup \{ \infty \} \xrightarrow{\Pi} \Pi(U, \Sigma) \longleftarrow \{ M_v \mid M_v = \mathbb{X} \cdot GL_2(\mathcal{O}_\infty) \cdot \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \subseteq k_\infty e_1 \oplus k_\infty e_2 \} \longleftarrow \mathbb{X} \cdot GL_2(\mathcal{O}_\infty)$

$\{ s \in k_\infty e_1 \oplus k_\infty e_2 \mid s_v \in \mathbb{X}_v \cdot GL_2(\mathcal{O}_v) \cdot \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \}$

Similarly $Bun_{2,X}(\overline{\mathbb{F}_q}) \xleftrightarrow{1:1} \frac{GL_2(A)}{GL_2(k) \backslash GL_2(\mathcal{O})}$

Bundle with level structure $\xleftrightarrow{1:1} \frac{GL_2(A)}{GL_2(k) \backslash GL_2(\mathcal{O})} / \underbrace{K_D}_{\{ \mathbb{X} \equiv 1 \pmod{\mathcal{O}_D} \}}$
 $\{ \Sigma, \Sigma_D \cong \mathcal{O}_D^{\oplus 2} \}$

Rank $D \uparrow \Rightarrow k_D \downarrow$.

(This helps us do all calculation by elements in $GL_2(A_\infty)$.)

Modification: $GL_2(A)$ acts on $Bun_{2, X}(\bar{\mathbb{F}}_q)$.

$$X \in GL_2(A), \quad \Sigma = \mathbb{P} \cdot G^{\oplus 2} \quad \mathbb{P} \in GL_2(A_\infty).$$

$$\text{the } \Sigma(X) \triangleq \mathbb{P} \cdot X \cdot G^{\oplus 2}.$$

Rule: ①. change level structure!

②. If we pick $\alpha_v \in GL_2(\mathcal{O}_v) \begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathcal{O}_v)$

$$\alpha_v \triangleq (1, \dots, \alpha_v, \dots, 1) \in GL_2(A_\infty).$$

then $\bullet \rightarrow \Sigma(\bullet \alpha_v)$.

$$0 \rightarrow \Sigma(\alpha_v) \rightarrow \Sigma \xrightarrow{\varphi_{\alpha_v}} \bar{\mathbb{F}}_q(v) \rightarrow 0.$$

when α_v vary \bullet , φ_{α_v} go over all $\left\{ \begin{array}{l} 1\text{-dim quotient} \\ \text{of } \Sigma \otimes \bar{\mathbb{F}}_q(v) \end{array} \right\} = \mathbb{P}^1(\Sigma \otimes \bar{\mathbb{F}}_q(v))$

Def: " (α_v) " is called an Hecke modification.

Recall: shukla: Σ/\bar{X} bundle. $u, w \in \bar{X}$

st. $\Sigma \hookrightarrow \Sigma' \hookrightarrow (I \times F)^* \Sigma$

$\Sigma'/\Sigma = U \times \bar{\mathbb{F}}_q$

$\Sigma' / (I \times F)^* \Sigma = W \times \bar{\mathbb{F}}_q$.

(In other words shukla is a bundle Σ st.

$$(I \times F)^* \Sigma = \Sigma(\alpha_v^{-1})(\alpha_w)$$

if we write $\Sigma = \mathbb{P} \cdot G^{\oplus 2}$

then Σ is shukla iff

$$\sigma_{\mathbb{P}} = \gamma \mathbb{P} \alpha_v^{-1} \alpha_w$$

$\sigma = \text{Frob. } \gamma \in GL_2(\mathbb{K}_v)$

Thm: $M_{D, (v, w)} = \{ \text{moduli sp. of shtuka over } (v, w) \}$
 (with level D).

$M_{D, (v, w)}$ is a disjoint union of unirational surface.

Obs: $M_{D, (v, w)} \approx \text{Ban}_{2, X}(\mathbb{F}_q) \times \mathbb{P}'_v \times \mathbb{P}'_w$.

\uparrow
~~disc.~~ disc.
 \uparrow

\uparrow modification by v, w .

$$\Sigma|_{X(v, w)} \cong \Sigma|_{X(w, v)}$$

Prk: If $D' \geq D$, then \exists étale

$$\exists \text{ finite étale } \pi_D^{D'} : M_{D', (v, w)} \rightarrow M_{D, (v, w)}$$

Thm: $GL_2(\mathbb{A}_{v, w}^\times) \curvearrowright M_{(v, w)}$

Rf: $\forall \alpha \in GL_2(\mathbb{A}_{v, w}^\times) \cdot \mathbb{H}_{\text{ét}} \ni \alpha \cdot X = X$, and X com. with $\mathcal{O}_v, \mathcal{O}_w$.

$$\alpha \cdot X = \alpha(X) = \gamma \otimes \mathcal{O}_v \otimes \mathcal{O}_w \cdot X = \gamma(\alpha X) \otimes \mathcal{O}_v \otimes \mathcal{O}_w.$$

~~Truncation: $M_{D, (v, w)}^{(r)} = \{ \text{shtuka } \Sigma \text{ st. } 2 \cdot \text{deg}(\text{max. line bundle}) - \text{deg}(\Sigma) \leq r \}$
 (with level str. D).~~

~~Thm: $M_{D, (v, w)}^{(r)} \subseteq M_{D, (v, w)}^{(r+1)}$.~~

~~Def: $M_{D, (v, w)}^{m, n} = \{ \text{shtuka } \Sigma \text{ st. } \text{deg.}(\text{max. line bun.}) \leq m, \text{deg}(\Sigma) = n \}$.~~

Truncation: Def: $M_{D(u,w)}^{m,n} = \left\{ \begin{array}{l} \text{shtuka } \Sigma \text{ s.t. } \deg(\max(\text{line bundle})) \leq m \\ \deg(\Sigma) = n. \text{ level str. on } D \end{array} \right\}$

$$M_{D(u,w)}^{(r)} = \bigcup_{2m-n \leq r} M_{D(u,w)}^{m,n}$$

$$= \bigsqcup M_{D(u,w)}^{\lfloor \frac{n+r}{2} \rfloor, n}$$

$$M_{D(u,w)}^{(r)} = M_{D(u,w)}^{\lfloor \frac{n+r}{2} \rfloor, 0} \sqcup M_{D(u,w)}^{\lfloor \frac{n+r}{2} \rfloor, n}$$

Prop: $M_{D(u,w)}^{m-1, n} \subseteq M_{D(u,w)}^{m, n}$ (open)

Prop: $M_{D(u,w)}^{m,n} - M_{D(u,w)}^{m-1, n} = \text{Union of two family of affine lines}$
 horospherical curves
 (intersect at one point.)

not a: Pf: $\text{Left} = \left\{ \begin{array}{l} \text{shtuka } \Sigma \text{ has maximal line bundle } L \text{ of deg } m. \\ \deg \Sigma = n. \text{ Level str. on } D \end{array} \right\}$

$$L \otimes \mathcal{O} \rightarrow L \rightarrow \Sigma \rightarrow H \rightarrow 0.$$

Fact: either: $L \in \text{Shtuka}$. $H \in \text{Pic}_X(\mathbb{F}_q)$.

or: $L \in \text{Pic}_X(\mathbb{F}_q)$. $H \in \text{Shtuka}$. (shtuka of rank 1 has dim = 1).

then $\text{Left}_{L,H} \in \text{Ext}^1(H, L(-D))$

and $\xi \in \text{Left}$. $\xi \in \text{Ext}^1$ satisfying (ξ should be a shtuka)

$$r(\xi) - \phi_L^*(\xi^0) = 0.$$

• this eq. defines 1-dim group sch G_a .

~~$\text{Left} = \bigsqcup_{L \in \text{Pic}_X(\mathbb{F}_q)} \text{Left}_{L,H} \sqcup \bigsqcup_{H \in \text{Shtuka}} \text{Left}_{L,H}$~~

• Left = $(\bigcup_{L,H \in \mathbb{D}} \text{left}_{L,H}) \cup (\bigcup_{L,H \in \mathbb{D}} \text{left}_{L,H})$. \square .

Baby case of trace formula.

Let $u = (v, w)$

Fact: $M_{D,(v,w)}^{(r)}$ is acted by finite group \odot

$G = GL_2(\mathbb{D}) \cdot \mathbb{A}^x / k^x \cdot J \cdot K_D$ where K_D is the congruence sub group of level D .

Thm: \exists equality in Gro. grp of reps. of $G \times W_u \times W_D$.

(1.1) $\sum_i (-1)^i H_c^i(M_{D,(v,w)}^{(r)} / J, \bar{\mathbb{Q}}_e) = H^0(M_{D,(v,w)}^{(r)} / J, \bar{\mathbb{Q}}_e) \otimes (\bar{\mathbb{Q}}_e + \bar{\mathbb{Q}}_e(-2))$

+ $\sum_{\substack{\pi \in \mathcal{A} \\ \pi \in \mathcal{A}_{\text{cusp}}}} \pi^D \otimes P_v(\pi) \otimes P_w^v(\pi) - 2 \sum_i (-1)^i H_c^i(Z_{D,(v,w)}, \bar{\mathbb{Q}}_e) + \underbrace{(2r+1)P + Q}_{\text{weight 2 part}}, r \geq 0.$

\uparrow
 curve.
 { compactification }
 - horospherical curves }
 - horosph. curves }

from parabolic induction
{ horospherical }
curves.

Gro. trace formula

Remark: In general Frobenius trace does not hold for general Automorphisms, but ok for these finite order ones. (and Frobenius).

Lemma.

~~Thm~~: X/\mathbb{F}_q var. $g \in \text{Aut}(X)$, finite order. then $\exists X^{(g)}$ st. $X^{(g)}/\mathbb{F}_q = X/\mathbb{F}_q$

a-1. $\text{Frob}_{X^g} = g \cdot \text{Frob}_X$.

Pf: $(g) \in H^1(\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q), \text{Aut}(X)) = \text{Hom}(\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q), \text{Aut}(X))$ finite order.

so $g \cdot \text{Frob}_X$ gives a G -torsor over \mathbb{F}_q where G is a finite group in $\text{Aut}(X)$. then $\text{Torsor}_{\text{Aut}(X)}^X$ is a variety over \mathbb{F}_q which is $X^{(g)}$. \square

Remark: Since G is finite group, to get this eq. we just consider trace.

if we show $\text{Tr}(g \cdot F_u^i F_w^j, \text{Left}) = \text{Tr}(\phi_g \chi_u^i \hat{\chi}_w^j, \text{Right})$ then we are done.

Pf: $\text{Tr}(g \cdot F_u^i F_w^j, \text{Left}) \stackrel{\text{Fix pt formula}}{=} \# \text{Fixed points of } g \cdot F_u^i F_w^j$

$$= \# \text{Fixed pts } g \cdot d_u^i d_w^j \text{ (Hecke modification)}$$

$$= \{x \mid \exists g a_u^i a_w^j = x\}$$

$$= \{ \text{orbits of gups like } G_{\mathbb{Z}}(A_{u,w}) \times \bar{D}_u \times \bar{D}_w \}$$

$$= \sum \int \text{characteristic functions on orbits}$$

A-S trace formula:

$$\text{Tr}^{SV}(\phi_g \chi_u^i \chi_w^j, A_{\text{quoto. 2. KD}}^J) = \sum \int \text{functions} \quad + \quad \sum \int$$

\swarrow cuspidal part
 \nwarrow parabolic induction part.

Compare these two equations we get. (2.2)

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