

Towards the Langlands Correspondence

- 1) On the Automorphic and Galois side
- 2) Statement of the correspondence (for GL_n)
- 3) Converse Theorem

X/\mathbb{F}_q smooth projective geometrically integral curve

$S \subseteq X$ reduced finite subscheme

F field of functions on X

\mathbb{A} adèle ring of F , \mathbb{A}^\times idèle ring

$\forall x \in |X|$, F_x is the completion at x

\mathcal{O}_x ring of integers

G connected split reductive group / F

Automorphic representations

We have a space of automorphic forms of G .

$$A_G^{\text{aut}} := \left\{ f: \frac{G(\mathbb{A})}{G(F)} \longrightarrow \mathbb{C} \mid \begin{array}{l} \exists K \text{ open in } G(\mathbb{A}) \\ \text{s.t. } f \text{ is } K \text{ invariant} \\ \text{on the right} \\ + \text{ admissibility} \\ \text{condition.} \end{array} \right.$$

$G(\mathbb{A})$ acts on A_G^{Aut} on the right.

The irr. sub-quotients of the repr. of $G(\mathbb{A})$ on A_G^{Aut} are the so-called automorphic representations of G .

Inside we have $A_G^{\text{cusp}} \subseteq A_G^{\text{Aut}}$, the subspace of cuspidal forms, namely $f \in A_G^{\text{Aut}}$ s.t.

$$\int_{N(\mathbb{A}_F)} f(gm) dm = 0$$

where $N = R_u(P)$.

The irr. sub-representations of $G(\mathbb{A})$ in A_G^{cusp} are the cuspidal automorphic representations. You can think them as the "building blocks" of the automorphic representations.

Flath's Theorem

Any irreducible admissible representation π of $G(\mathbb{A})$ can be written uniquely as $\bigotimes_{x \in |X|} \pi_x$ where π_x are admissible irr. representations of $G(F_x)$, unramified outside a certain $S \subseteq X$.

We recall that an irr. adm. repr of $G(F_x)$ is unramified iff $\pi_x^{G(\mathcal{O}_x)} \neq \{0\}$

$$\left\{ \begin{array}{l} \text{unramified repr} \\ \text{of } G(F_x) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{simple} \\ \text{Hom}(G(F_x), G(O_x)) - \\ \text{modules} \end{array} \right\}$$

Satake isomorphism: $S: \text{Hom}(G(F_x), G(O_x)) \xrightarrow{\sim} \mathbb{C}[X_i]^W$

Ex: If $G = GL_2$, the maximal torus is G_m^2 and the Weyl group is S_2 .

$$\xrightarrow{\text{S. iso}} \text{Hom}(GL_2(F_x), GL_2(O_x)) \xrightarrow{\sim} \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]^{S_2}$$

thus:

$$\left\{ \begin{array}{l} \text{unr. repr} \\ \text{of } GL_2(F_x) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \{z_1, \dots, z_n\} \in (\mathbb{C}^*)^2 / S_2 \\ \{z_1(\pi_1), \dots, z_2(\pi_2)\} \\ \text{Hecke eigenvalues} \end{array} \right\}$$

You can think this bijection as an unramified local Langlands correspondence

Langlands dual

$$G \rightsquigarrow G^\vee \text{ Langlands dual}$$

$$(X^*, \Phi, X_*, \check{\Phi})$$

$$(X_*, \check{\Phi}, X^*, \Phi)$$

G^\vee is related to the representation theory of G .

Ex:

G	G^\vee
GL_n	GL_n
SL_n	PGL_n
Sp_{2n}	SO_{2n+1}

Looking again at the Satake isomorphism -

$$\{ \text{unramified reps of } G(F_x) \} \xleftrightarrow{\sim} \text{Hom}(X_*(T), \mathbb{C}^*) / \mathcal{W}$$

$$\text{RHS} = \text{Hom}(X^*(\check{T}), \mathbb{C}^*) / \mathcal{W} = \check{T}(\mathbb{C}) / \mathcal{W} = \check{G}(\mathbb{C})^{ss} / \text{conj}$$

(Set of Satake parameters)

$$\pi_x \mapsto s(\pi_x) \in \check{G}(\mathbb{C})^{ss} / \text{conj}$$

Local L-factors: If we fix $\rho: \check{G} \rightarrow GL_n$,

$$\text{we can associate to } \pi_x \mapsto L_x^\rho(\pi, t) := \det_\rho (1 - s(\pi_x) t)^{-1}$$

⚠ The choice of a "good" ρ is in general a difficult $\mathbb{C}[[t]]$ problem. We would like to choose it in such a way to guarantee the functoriality of the Langlands Correspond. when G varies.

Ex: $G = GL_n$, $\rho = \text{id}$, $L_x(\pi, t) = \prod_{i=1}^n (1 - z_i(\pi_x) t)^{-1}$

"Strong multiplicity one" Theorem (Piatetski-Shapiro)

The isomorphism class of a cuspidal aut. repr. π of GL_n is determined by the datum of the Hecke eigenvalues for almost every point $x \in |X|$.

Galois l -adic representations

\bar{x} geom. pt., $S \subseteq X$ reduced finite, $\bar{X} := X \otimes \bar{\mathbb{F}}_q$, $\overline{X-S} := (X-S) \otimes \bar{\mathbb{F}}_q$
 We have an exact sequence

$$\begin{array}{ccccccc}
 1 & \rightarrow & \pi_1^{\text{ét}}(\overline{X-S}, \bar{x}) & \rightarrow & \pi_1^{\text{ét}}(X-S, \bar{x}) & \rightarrow & \text{Gal}(\bar{\mathbb{F}}_q / \mathbb{F}_q) \rightarrow 1 \\
 & & \parallel & & \uparrow & & \uparrow \\
 1 & \rightarrow & \pi_1^{\text{ét}}(\overline{X-S}, \bar{x}) & \rightarrow & W(X-S, \bar{x}) & \rightarrow & W(\bar{\mathbb{F}}_q / \mathbb{F}_q) \rightarrow 1 \\
 & & & & \text{Weil group} & & \text{Cyclic group generated} \\
 & & & & \text{of } X-S & & \text{by the } q\text{-Frobenius.}
 \end{array}$$

The topology on $W(X-S, \bar{x})$ is s.t. $\pi_1^{\text{ét}}(\overline{X-S}, \bar{x}) \hookrightarrow W(X-S, \bar{x})$ is an open embedding

Frobenii: Let $x \in |X-S|$ a closed point, $\kappa = \text{Spec } \mathbb{F}_{q^s}$. By the functoriality of the Weil group it induces a family of morphisms $W(\bar{\mathbb{F}}_q / \mathbb{F}_{q^s}) \rightarrow W(X-S, \bar{x})$. The set of images of the inverse of the q^s -Frobenius (geometric Frobenius) gives a conjugacy class $\text{Frob}_x \subseteq W(X-S, \bar{x})$.

Čebotarev Theorem The set $\bigcup_{x \in |X-S|} \text{Frob}_x$ is dense in $W(X-S, \bar{x})$.

Galois homomorphisms:

For every $l \neq p$, we will be interested in the continuous Galois homomorphisms unramified over S :

$$W(X-S, \bar{x}) \rightarrow G(\bar{\mathbb{Q}}_l)$$

If $G = G L_r$ we will call these homomorphisms $(l$ -adic) Galois representations of rank r .

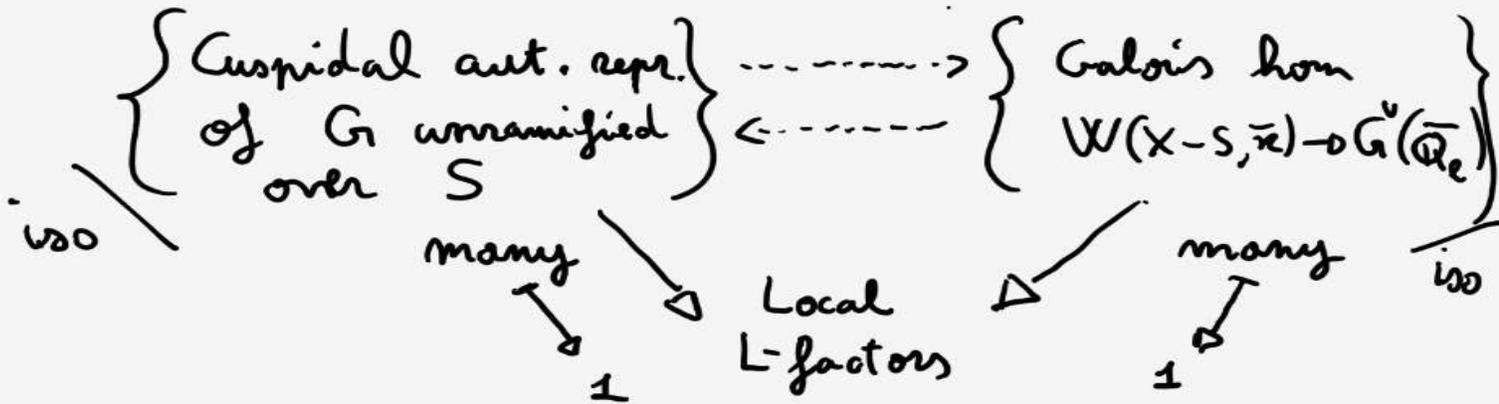
Local factors: For every reductive group G , σ a Galois hom. of G , after the choice of $\rho: G \rightarrow GL_n$ we can define for every $x \in |X - S|$,

$$L_x^\rho(\sigma, t) := \det_\rho (1 - t \sigma(\text{Frob}_x))^{-1}$$

Consequence of Čeb: Two minimal Galois representations σ_1 and σ_2 are isomorphic iff they have the same L-factors for almost every point.

Langlands correspondence

Let G be a split connected reductive group/ F , $\rho: G \rightarrow GL_n$, $\iota: \bar{\mathbb{Q}}_x \xrightarrow{\sim} \mathbb{C}$



Compatibility π cuspidal and σ Galois are said compatible if for almost every $x \in |X|$,

$$L_x^\rho(\pi, t) = \iota(L_x^\rho(\sigma, t))$$

From now on we will focus on GL_n . Let

$$A_{n,S} := \left\{ \begin{array}{l} \text{cuspidal repr} \\ \text{of } GL_n \text{ unramified} \\ \text{outside } S \end{array} \right\} \quad A_n := \bigcup_{S \subseteq X} A_{n,S}$$

and

$$G_{\mathbb{R}, S} := \left\{ \begin{array}{l} \text{irr. Galois repr.} \\ \text{of } W(X-S, \bar{\mathbb{R}}) \end{array} \right\}, \quad G_{\mathbb{R}} := \bigcup_{S \subseteq X} G_{\mathbb{R}, S} \quad \text{sim}$$

Theorem (Drinfeld, L. Lafforgue). We fix an isomorphism $c: \bar{\mathbb{A}}_{\mathbb{R}} \xrightarrow{\sim} \mathbb{C}$. For every $n \geq 1$ there exists a unique bijection

$$\begin{array}{ccc} A_n & \xrightarrow{\sim} & G_n \\ \pi & \longmapsto & \delta_{\pi} \end{array}$$

Observations:

- 1) The unicity is given by Čebotarev.
- 2) The injectivity is a consequence of the "strong multiplicity one" Theorem.
- 3) If $n=1$ the cuspidal automorphic repr. unr. outside S are the characters of $F^{\times} \backslash \mathbb{A}^{\times} / \prod_{x \notin |S|} G_x^{\times}$. At the same time

Class Field Theory is telling us that

$$F^{\times} \backslash \mathbb{A}^{\times} / \prod_{x \notin |S|} G_x^{\times} \xrightarrow{\sim} W(X-S, \bar{\mathbb{R}})^{ab}$$

hence, as a consequence, we obtain the bijection $A_1 \xrightarrow{\sim} G_1$.

- 4) Thanks to the previous theorem for any character $\chi \in A_1 = A_2$ we can twist at the same time a representation of $GL_2(\mathbb{A})$ and a Galois representation by χ . Hence we can reduce to the correspondence between cuspidal automorphic representation with finite order central character and Galois repr. with finite order (under \otimes) determinant

5) Surjectivity

"Reconstruction" Theorem (Piatetski-Shapiro): Let $n \geq 2$ be an integer, $\pi = \otimes \pi_x$ an admissible representation of $GL_n(\mathbb{A})$ that admits a Whittaker model. Suppose moreover $\exists S$ such that:

- i) The central character factors through F^\times .
- ii) $\forall r' < r \quad \forall \pi' \in A_{r'}$ unramified over S the L -functions $L(\pi \times \pi', t)$ and $L(\check{\pi} \times \check{\pi}', t)$ are polynomials and $L(\pi \times \pi', t) = \varepsilon(\pi \times \pi', t) L(\check{\pi} \times \check{\pi}', (qt)^{-1})$

Then there exists an automorphic representation (not forcedly cuspidal) that is isomorphic to $\pi_x \quad \forall x \notin |S|$.

Converse Theorem

(Surjectivity): Suppose we have already constructed $\forall r' < r \quad A_{r'} \rightarrow G_{r'}$, then $\exists G_r \rightarrow A_r$ (everything preserving local L -factors). ! The proof uses many facts we have not seen during the previous talks.

Proof: Let $\sigma \in G_{r, S}$, $\forall x \notin |S|$ we can find π_x , an unram. representation having as Hecke eigenvalues the Frobenius eigenvalues of σ at x . We can then complete the family $\{\pi_x\}_{x \notin |S|}$ obtaining an adm. representation $\pi = \bigotimes_{x \notin |S|} \pi_x$ with a Whittaker model having as central character $\prod_{x \notin |S|} \det \sigma$. We want to apply now the previous theorem. We know $\forall r' < r$ and $\pi' \in A_{r', S}$ there exists by assumption a comp. $\sigma' \in G_{r', S}$. If we choose X enough ramified on S , of finite order, $\forall x \in S, L_x(X \pi \times \pi', t) = L_x(X \otimes \sigma \otimes \sigma', t) = 1$ and the same holds for the contragredient repr. Hence

$$L(X \pi \times \pi', t) = L(X \otimes \sigma \otimes \sigma', t) \quad \text{and}$$

$$L(X' \check{\pi} \times \check{\pi}', t) = L(X' \otimes \sigma^\vee \otimes (\sigma')^\vee, t)$$

then the functional equation for the adm. repr is a consequence of the functional equation for Galois representation (Poincaré duality) once we know the (non-trivial) fact that ε -factors correspond (Langlands formula).

We have not yet finished, we need to show that π is cuspidal. Thanks to a result of Langlands, if π is not cuspidal $\exists (\pi_1, \dots, \pi_n) \in A_{r_1, S} \times \dots \times A_{r_n, S}$ s.t. $r_i < r$ and s.t. the Hecke eigenvalues of π are disjoint union of the Hecke eigenvalues of (π_1, \dots, π_n) . By the assumption $\exists (\sigma_1, \dots, \sigma_n)$ compatible Galois

representations. Hence by Čebotarev $\sigma \simeq \sigma_1 \oplus \dots \oplus \sigma_r$ and this is not possible because σ is assumed to be irreducible.

What is still missing is the proof of a map $A_2 \rightarrow G_2$ when $n \geq 2$. □

In the next 4 talks we will work with the moduli space of Shtukas. Looking at the étale cohomology we can construct a $\overline{\mathbb{Q}}_l$ -representation V_l^{coop} of the group $\text{GL}_2(\mathbb{A}) \times \mathcal{W}(\overline{\mathbb{F}}/F) \times \mathcal{W}(\overline{\mathbb{F}}/F)$.

We will show that $V_l^{\text{coop}} \simeq \bigoplus_{\pi} \pi \otimes \rho_{\pi} \otimes \rho_{\pi}^{\vee}(-1)$. The compatibility between π and ρ_{π} is proven via a point-counting + trace formulas for both sides.