Epsilon factors, Spring 2018
Programme

Michael Groechenig

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1 Introduction

1.1 The determinant of cohomology

Let $\mathcal{F}$ be a constructible ℓ-adic sheaf on a smooth proper curve $X/\mathbb{F}_q$. The determinant of the cohomology of $\mathcal{F}$ is the 1-dimensional ℓ-adic vector space

$$\det(X, \mathcal{F}) = \bigotimes_{i=0}^{2} \left(\bigwedge^{top} H^i(X_{\bar{\mathbb{F}}_q}, \mathcal{F})\right)^{(-1)^i}.$$

This ℓ-adic line is endowed with an action of the Frobenius, and the corresponding eigenvalue $\epsilon(\mathcal{F})$ is an important arithmetic invariant: Recall that the L-function $L(\mathcal{F}, t)$ is equal to the product

$$L(\mathcal{F}, t) = \prod_{i=0}^{2} \det(1 - t \cdot F_X, H^i(X_{\bar{\mathbb{F}}_q}, \mathcal{F}))^{(-1)^{i+1}}.$$

By definition, $L(\mathcal{F}, t)$ is a rational function in t (with ℓ-adic coeffcients) and Poincaré duality implies the functional equation

$$L(\mathcal{F}, \frac{1}{qt}) = \epsilon(\mathcal{F}) \cdot L(\mathcal{F}^\vee, t).$$

The constant $\epsilon(\mathcal{F})$ arising in this functional equation is equal to the Frobenius eigenvalue on the determinant of cohomology.

1.2 The local epsilon factors

The constant $\epsilon(\mathcal{F})$ is also known as the *global epsilon factor*. A formalism of *local epsilon factors* allows one to associated to a non-zero rational 1-form ω on X and a closed point $x \in X$ an (invertible) ℓ-adic number $\epsilon_{\omega,x}(\mathcal{F})$ which is equal to 1 for almost all closed points x, such that we have a *product formula*

$$\epsilon(\mathcal{F}) = \prod_{x \in X_{cl}} \epsilon_{\omega,x}(\mathcal{F}). \tag{1}$$

Furthermore, the quantity $\epsilon_{\omega,x}(\mathcal{F})$ is only to depend on the constructible sheaf $\mathcal{F}$ (and the form ω) near x, that is, the restriction to the formal completion of X at x.

If we assume that $\mathcal{F}$ is generically of rank 1, methods akin to class field theory and Tate's thesis can be used to build a theory of local ϵ-factors in this case.

It was conjectured by Deligne [Del73] that a formalism of local ϵ-factors should exist even in the higher rank case. This was proven by Laumon [Lau87] using global methods (and therefore in some sense indirectly). A purely local definition of $\epsilon_{\omega,x}(\mathcal{F})$ remains elusive and would be highly desirable for the tales it might tell about wild ramification.

1.3 The de Rham analogue

Replacing finite fields by the complex numbers $\mathbb{C}$ (or a field of characteristic zero) and ℓ-adic sheaves by holonomic D-modules, the determinant line of de Rham cohomology remains an interesting invariant (e.g. due to its connection to period matrices). It's been observed by Deligne (unpublished) and Beilinson–Bloch–Esnault [BBE02] that one can define lines (i.e. 1-dimensional vector spaces) $\mathcal{E}_{\omega,x}(\mathcal{F})$ for a holonomic D-module $\mathcal{F}$ and a non-zero rational 1-form on X, such that we have the following analogue of formula (1):

$$\det(X,\mathcal{F}) \simeq \bigotimes_{x\in X_{cl}} \mathcal{E}_{\omega,x}(\mathcal{F}).$$

Furthermore, this isomorphism is canonical (up to signs which can be dealt with). The fascinating aspect to their construction is that it's inherently local and surprisingly elementary. The main goal is to understand this theory of de Rham epsilon factors in depth. If time permits we will also discuss epsilon periods following Beilinson.

2 Programme

The following is a rough outline of how I plan to structure the course and might be revised later on.

1. (19.04) *Motivation I: geometric class field theory and epsilon-factors.* The highlight will be a proof of (1) for a lisse rank 1 local system on X.
2. (26.04 - 2 pm!) *Motivation II: more on ℓ-adic epsilon factors after Deligne and Laumon.*
3. (03.05) *Holonomic D-modules and irregular flat connections on curves.*
4. (??.05 - the 10th is a holiday) *Graded lines and determinants.* We will learn how to deal with sign issues in the theory in a natural way and discuss the connection with algebraic K-theory.
5. (17.05) *Good lattice pairs.* We construct de Rham epsilon factors using Deligne's good lattice pairs.
6. (24.05) *Infinite-dimensional vector spaces and determinantal theories.* This serves as a preparation to define de Rham epsilon factors à la Beilinson-Bloch-Esnault.
7. (31.05) *de Rham epsilon factors and the epsilon connection.*
8. (07.06) *Complimentary material.* I will discuss the de Rham analogue of local class field theory and sketch Patel's higher-dimensional theory of epsilon-factors.

9. (14.06) *Betti epsilon factors.* We briefly discuss the Betti analogue of epsilon factors which is needed in order to define epsilon periods.

10. (21.06) *Beilinson's unicity theorem.* We discuss Beilinson's result [Bei09] that there's is essentially only one theory of epsilon factors over the field of complex numbers. This involves an ergoditiy theorem for mapping class group actions by Goldman and Pickrell–Xia.

11. (28.06) *Epsilon periods.* We will take a look at sample computations.

References

[BBE02] Alexander Beilinson, Spencer Bloch, and Hélène Esnault, *ϵ-factors for Gauss-Manin determinants*, Mosc. Math. J. **2** (2002), no. 3, 477–532, Dedicated to Yuri I. Manin on the occasion of his 65th birthday. MR 1988970 (2004m:14011) 1.3

[Bei09] Alexander Beilinson, *$\mathcal{E}$-factors for the period determinants of curves*, Motives and algebraic cycles, Fields Inst. Commun., vol. 56, Amer. Math. Soc., Providence, RI, 2009, pp. 15–82. MR 2562452 (2011h:14023) 10

[Del73] Pierre Deligne, *Les constantes des équations fonctionnelles des fonctions l*, Lecture Notes in Math **349** (1973), 501–597. 1.2

[Lau87] G. Laumon, *Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil*, Inst. Hautes Études Sci. Publ. Math. (1987), no. 65, 131–210. MR 908218 (88g:14019) 1.2