Freie Universität Berlin
Forschungsseminar SS 15

# Rational sections and Serre's conjecture

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## Introduction

Recall the following conjecture of Serre.

**Conjecture.** *Let $K$ be a perfect field of cohomological dimension $\leq 2$, $G$ be a semisimple and simply connected algebraic group over $K$. Then $H^1(K,G) = 0$.*

The goal of this seminar is to prove a version of this conjecture following [JHS] when[1] $K = k(S)$, where $S$ is a surface over an algebraically closed field $k$.

For simplicity we may first assume that $G$ is defined over $k$ (we have to wait until the last talk to discuss the general situation). Now suppose we are given an element $x \in H^1(K,G)$, then $x$ corresponds to a $G$-torsor $\pi : P \to \mathrm{Spec}(K)$ in the étale topology[2]. Thus $x = 0 \Leftrightarrow P$ is a trivial $G$-torsor $\Leftrightarrow \pi$ admits a section. We may shrink $S$ a little bit to get a model $\pi_S : P_S \to S$ of $\pi$ which is still an étale $G$-torsor. Then $\pi$ admits a section $\Leftrightarrow \pi_S$ admits a rational section, the latter means that there is an dense open $U \subseteq S$ and a $k$-morphism $s_U : U \to P_S$ such that $\pi_S \circ s_U = \mathrm{id}_U$. Now the algebraic problem is converted into a problem which is of strong geometric flavor – finding rational sections. Thus we can use geometric tools such as moduli spaces and Hodge theory to attack the conjecture. This is what A. J. de Jong, X. He and J. M. Starr did in their paper [JHS].

Here are the main theorems which lead to a solution of Serre's conjecture in the case of the function fields of surfaces.

[1] Note that $K = k(S)$ is not perfect, so technically speaking, what we are going to prove is a special case of a stronger form of Serre's conjecture. We have to wait until the last talk for a precise formulation.

[2] Here one can also take the flat topology because $G$ is always assumed to be a *smooth* linear algebraic group.

**Theorem 2.** *Let $X \to C$ be a morphism of smooth projective varieties over an algebraically closed field $k$ of characteristic 0 with $C$ a curve. Let $\mathscr{L}$ be an invertible sheaf on $X$ which is ample on each fibre of $X \to C$. Assume the geometric generic fibre of $X \to C$ is rationally simply connected by chains of free lines and contains a very twisting scroll. In this case for $e$ sufficiently large there exists a canonically defined irreducible component $Z_e \subseteq Sections^e(X/C/k)$ so that the restriction of*

$$\alpha_{\mathscr{L}} : Sections^e(X/C/k) \longrightarrow \underline{Pic}^e_{C/k}$$

*to $Z_e$ has rationally connected fibres.*

In the theorem the space $\mathrm{Sections}^e(X/C/k)$ parametrizes sections $\sigma : C \to X$ of degree $e$ (with respect to $\mathscr{L}$) and the map $\alpha_{\mathscr{L}}$ assigns to a section $\sigma$ the point of $\underline{\mathrm{Pic}}^e_{C/k}$ corresponding to the invertible sheaf $\sigma^*\mathscr{L}$. This theorem plus the main result of [GHS] lead to the following:

**Theorem 1.** *Let $f : X \to S$ be a faithfully flat morphism between smooth projective varieties over an algebraically closed field $k$ of characteristic 0, where $S$ is a connected surface. Suppose that $\mathscr{L}$ is an $f$-ample invertible sheaf and the geometric generic fibre of $f$ satisfies the following two conditions*

1. *the space $X_{\bar{\eta}}$ is rationally simply connected by chains of free lines;*
2. *the space $X_{\bar{\eta}}$ has very twisting scroll,*

*then there exists a rational section of $f$.*

Amazingly one is able to deduce from 1, which is only proved over fields of characteristic 0, the following theorem in arbitrary characteristic:

**Theorem 3.** *Let $k$ be an algebraically closed field of any characteristic. Let $S$ be a quasi-projective surface over $k$. Let $X \to S$ be a projective morphism. Let $\bar{\eta}$ be the spectrum of the algebraic closure of the function field of $k(S)$. If $X_{\bar{\eta}}$ is of the form $G/P$ for some linear algberaic group $G$ and parabolic subgroup $P$ and $Pic(X) \to Pic(X_{\bar{\eta}})$ is surjective, then $X \to S$ has a rational section.*

With the above theorem plus a little more work and a result of [CGP] one canfinally conclude the following theorem:

**Theorem 4.** *(Serre's conjecture II for function fields) Let $k$ be an algebraically closed field. Let $k \subseteq K$ be a finitely generated field extension of transcendence degree 2. For every connected, simply connected, semisimple algebraic group $G_K$ over $K$, every $G_K$-torsor over $K$ is trivial.*

# 1 A survey on rational sections (16/04/2015)

**Aim of the talk:** In this talk we need a survey for the existence and non-existence of rational sections. You can follow [Voi].

DETAILS: Introduce the notion of Brauer-Severi varieties follow [Dix, Chapter I, §8]. Explain its relation with Azumaya algebras. Then conclude from [Dix, Chapter III, Corollaire 1.2] that there is no non-trivial Brauer-Severi varieties over an algebraic curve over an algebraically closed field. For a nice summary on Brauer-Severi varieties over fields see [Po, 4.5.1]. State [Voi, Théorème 0.1] and explain that when $d = 1$ and the hypersurfaces are smooth then the numerical condition $\sum_{i=1}^{r} d_i \leq n$ is equivalent to the condition that the fibres are Fano varieties. Now state the theorem [Voi, Théoème 1.3] and explain why it is a generalization of the previously mentioned theorems. In the end state the main theorem 1 (see also [Voi, Théorème 0.2]).

## 2 THE LANGUAGE OF STACKS (23/04/2015)

**Aim of the talk:** GIVE A *comprehensible* INTRODUCTION TO THE THEORY OF STACKS.

DETAILS: The speaker has the freedom to choose the topics. Here are only some suggestions:

- the notion of algebraic stacks (Artin stacks) and Deligne-Mumford stacks [LMB, Définition 4.1];
- the notion of an algebraic stack being locally noetherian, reduced, normal... [LMB, 4.7];
- the notion of a morphism between algebraic stacks being surjective, locally of finite type, flat, smooth... [LMB, pp. 33]
- the notion of an algebraic stack being quasi-compact [LMB, 4.16]
- the Zariski topology of an algebraic stack [LMB, §5].

## 3 THE STACK OF CURVES (30/04/2015)

**Aim of the talk:** INTRODUCE THE STACK OF CURVES AND THE STACK OF CURVES TO A TARGET.

DETAILS: Important points here are [JHS, Prop 3.3, 3.8] (explain in detail), [JHS, Prop 3.6, Def 3.1, 3.2, 3.4, 3.5, 3.7, 3.9].

## 4 FREE SECTIONS (07/05/2015)

**Aim of the talk:** INTRODUCE THE STACK OF SECTIONS, THE ABEL MAP, AND THE NOTION OF FREE SECTIONS.

DETAILS: Important points here are [JHS, Thm 4.3, Lem 4.8, Def 4.1, 4.2, 4.5, 4.6, 4.7].

## 5 KONTSEVICH STABLE MAPS (21/05/2015)

**Aim of the talk:** INTRODUCE THE KONTSEVICH STACK OF STABLE MAPS, AND DEFINE THE STACK OF STABLE SECTIONS AS A SPECIAL CASE OF THE KONTSEVICH STACK.

DETAILS: First follow [JHS, §5] to introduce the stack of Kontsevich stable maps, and then follow [JHS, §6] to introduce the stack of stable sections as a special case of Kontsevich moduli space. Also introduce combs and the Abel map [JHS, Lem 4.7].

## 6 CHAINS OF FREE LINES (28/05/2015)

**Aim of the talk:** STUDY FREE LINES AND FREE CHAINS OF LINES ON A FAMILY OF PROJECTIVE VARIETIES.

DETAILS: Follow [JHS, §7]. The Hypothesis 7.8 is very important to us, please explain it clearly.

## 7 SPACES OF SECTIONS OF A FIBRATION WHOSE GENERAL FIBRE IS RATIONALLY SIMPLY CONNECTED BY CHAINS OF FREE LINES (04/06/2015)

**Aim of the talk:** STUDY THE SPACES OF SECTIONS OF $f : X \to C$, WHERE $C$ IS A CURVE AND THE GENERAL FIBRE OF $f$ IS RATIONALLY SIMPLY CONNECTED BY CHAINS OF FREE LINES.

DETAILS: Following [JHS, §8,§9] introduce porcupines and construct a sequence of irreducible components of $\Sigma(X/C/k)$ for a given irreducible component of $Z \subseteq \Sigma(X/C/k)$. The main points are [JHS, Lem 8.4, 8.9, Cor 9.8].

## 8 VERY TWISTING SCROLL (11/06/2015)

**Aim of the talk:** INTRODUCE THE NOTION OF *very twisting scrolls* AND PROVE: 1) THE PROJECTIVE LINE $\mathbb{P}^1$ CONNECTS $t \in \sum(X/C/k)$, A POINT IN A BOUNDARY STRATUM, TO A MODULI POINT IN THE INTERIOR, THE COMPLEMENT OF THE BOUNDARY DIVISOR; 2) A FAMILY $X \to C$ CONTAINS VERY TWISTING SCROLLS IF THE GEOMETRIC GENERIC FIBRE HAS A VERY TWISTING SCROLL.

DETAILS: For the notion of *very twisting scrolls* see [JHS, Def 12.3, 12.7]. Then explain [JHS, Lem 12.5, Prop 12.12, Cor 12.13].

## 9 MAIN THEOREM 2 (18/06/2015)

**Aim of the talk:** PROVE MAIN THEOREM 2.

DETAILS: Explain [JHS, Thm 13.1]. You can also compare this with the stack free version [Voi, §2.2, §2.3].

## 10 MAIN THEOREM 1 (25/06/2015)

**Aim of the talk:** PROVE MAIN THEOREM 1.

DETAILS: Explain [JHS, Cor 13.2, Lem 13.3, 13.4]. You can also compare this with the stack free version [Voi, §2.1].

## 11 MAIN THEOREM 3 (02/07/2015)

**Aim of the talk:** SKETCH THE PROOF OF MAIN THEOREM 3.

DETAILS: Explain [JHS, Lem 16.1, 16.2, 16.3, Thm 16.5, 16.6] in detail. It is very interesting to see how one could lift from characteristic $p > 0$ to characteristic 0.

## 12 SERRE'S CONJECTURE (09/07/2015)

**Aim of the talk:** IN THIS TALK WE NEED A SURVEY OF THE RECENT PROGRESSES OF SERRE'S CONJECTURE AND A COMPLETE PROOF OF SERRE'S CONJECTURE IN THE CASE OF FUNCTION FIELDS OF SURFACES USING *Main Theorem 3.* A NICE REFERENCE WOULD BE [GI].

DETAILS: The full statement of Serre's conjecture is in [Gi, Conjecture 2.5]. For the proof in the case of function fields of surfaces see [JHS, Thm 1.4, Thm 1.5]. You can also follow [Gi, §6]. If you are very interested in the topic and there is not enough time for your talk, then you can move part of your talk to the day 16/07/2015. But a lot of people in our group will be in the US on that day, so in that case please do the more important things on 09/07/2015.

## REFERENCES


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[GHS] T. Graber, J. Harris & J. M. Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. 16, pp. 57-67, 2003.

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[LMB] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 39, Springer, Berlin, 2000.

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