Monodromy conjecture and proof of Veys’ conjecture

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Notation

Today \( F \) will be a number field, \( \Omega_F^{\text{fin}} \) the set of finite places and we will denote \( \mathfrak{p} \) any element of \( \Omega_F^{\text{fin}} \). We will consider, as usual, \( f : \mathbb{A}^d_F \rightarrow \mathbb{A}^d_F \) and we will study \( X_0 := f^{-1}(0) \). We denote \( |X_0| \) the set of closed point of \( X_0 \). We will always suppose fixed a log-resolution \( h : (Y, E) \rightarrow (\mathbb{A}^d_F, X_0) \).

We recall the notation we are using in our seminar:

- We decompose \( E = \bigcup_{i \in I} E_i \), with each \( E_i \) irreducible;
- \( N_i \) will be the multiplicity of \( f \circ h \) along \( E_i \);
- \( \nu_i - 1 \) the multiplicity of \( \text{Jac}_h \), the jacobian ideal of \( h \), along \( E_i \);
- \( \lct_x := \min \{ \nu_i/N_i \mid x \in h(E_i) \} \) and \( \lct := \min_{x \in |X_0|} \lct_x \);
- For every \( \emptyset \neq J \subseteq I \), we define \( E_J := \bigcap_{j \in J} E_j \) and \( \hat{E}_J = E_J \setminus \bigcup_{j \notin J} E_j \).

1 Analytic results

1.1 Bernstein polynomials

We start defining an analytic zeta function attached to \( f \). For every \( \psi \in C_\infty^0(\mathbb{C}^d, \mathbb{R}) \) we consider

\[
Z_\psi^\infty(s) := \int_{\mathbb{C}^d} |f(z)|^{2s} \psi(z) d\mathbb{C}^d
\]

when \( \Re s > 0, s \in \mathbb{C} \).

**Theorem 1.1** (Bernstein). \( Z_\psi^\infty(s) \) admits a meromorphic continuation to \( \mathbb{C} \). The poles are negative rational numbers.

We will see how this zeta function encodes some important informations of the singularities of \( X_0 \). Before doing this we present an important tool for the studying of this function and that it is used, for example, to show the previous theorem.

Let \( \mathcal{O} := \mathbb{C}[z_1, \ldots, z_n] \) and \( f \in \mathcal{O} \), then we consider \( \mathcal{O}[s, f^{-1}] f^s \), a rank 1 free \( \mathcal{O}[s, f^{-1}] \)-module with signpost \( f^s \). Let \( D := \mathbb{C}[z_1, \ldots, z_n, \partial_{z_1}, \ldots, \partial_{z_n}] \) and \( D[s] \) the ring of polynomials in the variable \( s \) with coefficients in \( D \). We put a \( D[s] \)-action on \( \mathcal{O}[s, f^{-1}] f^s \) by setting \( \partial_{z_i}(g f^s) := (\partial_{z_i}g + g \frac{\partial_{z_i}f}{f}) f^s \) for every \( g \in \mathcal{O}[s, f^{-1}] \). We call \( D[s] f^s \) the sub-\( D[s] \)-module generated by \( f^s \) and \( D[s] f^{s+1} \) the one generated by \( ff^s \).
**Definition 1.2** (Bernstein polynomial). We define \( b_f(s) \) as the minimal polynomial of the endomorphism of the \( D[s] \)-module \( D[s]f^s / D[s]f^{s+1} \) given by the multiplication by \( s \) on the left. Equivalently \( b_f(s) \) is the minimal polynomial such that there exists a differential operator \( P \in D[s] \), such that \( b_f(s)f^s = Pf^{s+1} \).

**Theorem 1.3** (Bernstein). For every \( f \in \mathcal{O} \), \( b_f \) always exists.

You can verifying putting \( s = -1 \) that \( b_f(-1) = 0 \). In general there is a theorem of Kashiwara telling us that the roots of \( b_f(s) \) are rational numbers.

**Example 1.4.** If \( f = z_1^{N_1} z_2^{N_2} \), with \( N_1, N_2 \in \mathbb{N} \) it is easy to show that

\[
b_f(s) = \prod_{i=1}^{2} \prod_{j=0}^{N_i-1} \left( s + 1 - \frac{j}{N_i} \right)
\]

and it satisfies the differential equation

\[
b_f(s)f^s = \frac{1}{N_1^{N_1} N_2^{N_2}} \partial_{z_1}^{N_1} \partial_{z_2}^{N_2} f^{s+1}
\]

The computation of Bernstein polynomials in the general case is instead very difficult. Toshinori Oaku found an algorithm [Oak97] that gives \( b_f(s) \) for every \( f \) using an analogue of Grobner basis for differential operators.

The relation between Bernstein polynomials and the analytic zeta function is the following.

**Proposition 1.5.** For every \( \psi \in C^\infty_c(\mathbb{C}^n, \mathbb{R}) \) if \( s_0 \) is a pole of \( Z^\infty_\psi(s) \) with \( \text{Re}(s_0) \geq 0 \), then \( s_0 + j \) is a root of \( b_f \) for some integer \( j \) such that \( 0 \leq j \leq m \).

**Proof.** The proof proceed by induction on \( m \). If \( m = 0 \) it holds emptily because analytic zeta functions have no poles when \( \text{Re}(s) \geq 0 \).

For the inductive step we use the Berstein polynomial of \( f \). We know here exists \( P \in D[s] \) such that

\[
b_f(s)f^s = Pf^{s+1}.
\]

Applying the conjugation we also get

\[
\frac{b_f(s)}{b_f(s)}f^s = \frac{Pf}{f^{s+1}}.
\]

Hence

\[
|b_f(s)|^2 Z^\infty_\psi(s) = |b_f(s)|^2 \int_{\mathbb{C}^n} |f(z)|^{2s} \psi(z) \, dz \, d\bar{z} = \int_{\mathbb{C}^n} P \mathcal{P} \left( |f(z)|^{2(s+1)} \right) \psi(z) \, dz \, d\bar{z}.
\]

Thanks to the partial integration formula \(^1\) the RHS is equal to \( Z^\infty_{\mathcal{P}\mathcal{P}(\psi)}(s) \), thus the partial differential equation defining \( b_f(s) \) translates to

\[
|b_f(s)|^2 Z^\infty_\psi(s) = Z^\infty_{\mathcal{P}\mathcal{P}(\psi)}(s+1).
\]

If \( s_0 \) is a pole of \( Z^\infty_\psi(s) \) which is not a root of \( b_f(s) \), then \( s_0 + 1 \) is a pole of \( Z^\infty_{\mathcal{P}\mathcal{P}(\psi)}(s) \). Hence we can use the inductive hypothesis on \( Z^\infty_{\mathcal{P}\mathcal{P}(\psi)}(s) \) getting the final result.

\(^1\) Here we are strongly using the fact we are working with analytic zeta functions. Indeed this formula has no analogue for \( \mathcal{P} \)-adic and motivic zeta functions.
Proposition 1.6. \(\text{lct}_x = \sup \{ s \mid |f|^{-2s} \text{ integrable around } x \} \)

Proof. Exercise: Take a log-resolution \( h : (Y, E) \to (\mathbb{A}^d, X_0) \), then use the change of variables formula. \(\square\)

Corollary 1.7. For every \( x \in X_0 \), \(-\text{lct}_x\) is a zero of \( b_f \).

Proof. Exercise: The reasoning is analogue to the proof of Proposition 1.5. \(\square\)

1.2 Monodromy

Milnor showed that if \( f : \mathbb{C}^d \to \mathbb{C} \) is algebraic then for every \( x \) such that \( f(x) = 0 \), there exists a ball \( B \subseteq \mathbb{C}^d \) centered at \( x \) and \( A \subseteq \mathbb{C} \setminus \{0\} \) a punctured ball centered at 0 such that \( A \subseteq f(B) \) and \( f|_B \) is a locally trivial \( C^\infty \)-fibration over \( A \) with fiber \( F_x = f^{-1}(t) \cap B \) for a certain \( t \in A \). If we choose a generator of the topological fundamental group of \( A \) it induces an endomorphism \( T_x \) of \( \bigoplus_{i=0}^{2d} H^i_{\text{sing}}(F_x, \mathbb{Z}) \). We call the monodromy eigenvalues at \( x \) the eigenvalues of \( T_x \).

Theorem 1.8 (Malgrange [Mal83], Barlet [Bar84]). For every \( \alpha \in \mathbb{R} \), the class \( [\alpha] \in \mathbb{R}/\mathbb{Z} \) is represented by a root of the Bernstein polynomial if and only if \( \exp(2\pi i \alpha) \) is a monodromy eigenvalue for a certain \( x \in X_0 \).

Hence as a consequence we obtain the main result of this section.

Theorem 1.9. If for some \( \psi \in C_\infty^\infty(\mathbb{C}^n, \mathbb{R}) \), complex number \( s_0 \) is a pole of \( Z_\infty^\psi(s_0) \), then \( \exp(2\pi i s_0) \) is a monodromy eigenvalue for a certain \( x \in X_0 \).

2 \(\mathfrak{p}\)-adic monodromy conjecture

We now switch to the \(\mathfrak{p}\)-adic zeta function defined during Tanya’s talk. For simplicity we will only work with

\[
Z^\mathfrak{p}(s) := \int_{\mathbb{Q}_\mathfrak{p}} |f|^s dx.
\]

We have seen the following theorem due to Igusa.

Theorem 2.1 (Igusa). \( Z^\mathfrak{p}(s) \) is rational in the variable \( t = q^{-s} \). If \( s_0 \) is a pole of \( Z^\mathfrak{p}(s) \), then

\[
s_0 \in -\nu_i \frac{2\pi i}{\ln q} \mathbb{Z}
\]

for some \( i \in I \).

Even if \( \text{Re}(s_0) = -\nu_i/N_i \) for some \( i \in I \), many numerical examples show that some possibilities are never taken. The data collected suggest a behavior similar to the analytic zeta function (Theorem 1.9).

Conjecture 2.2 (\(\mathfrak{p}\)-adic monodromy conjecture). For almost every \( \mathfrak{p} \in \Omega_\mathfrak{p}^{fin} \), if \( s_0 \) is a pole of \( Z^\mathfrak{p}(s) \) then \( \exp(2\pi i \text{Re}(s_0)) \) is a monodromy eigenvalue for some \( x \).

Let’s try to understand now how to attack this conjecture. We define the monodromy zeta function at \( x \) as

\[
\zeta_x(t) := \prod_{n=0}^{2d} \det(1 - t T_x | H^i_{\text{sing}}(F_x, \mathbb{Z}))(1 - ^{-1})^{n+1}.
\]

There is an explicit formula of this function using the log-resolution of \( (\mathbb{A}^d, X_0) \).
Theorem 2.3 (A’ Campo’s formula [A’C75]).

\[ \zeta_x(t) = \prod_{i \in I} (1 - t^{N_i})^{-\chi_{\text{top}}(\hat{E}_i \cap h^{-1}(x))} \]

It may happen that a monodromy eigenvalue at \( x \) is not a zero or a pole of \( \zeta_x(t) \) because of some unlucky cancellation. Nevertheless Denef [Den93] has proven, thanks to the perversity of the nearby cycles complex, that every eigenvalue of the monodromy operator of a certain point \( x \) is a zero or a pole of \( \zeta_y(t) \) for a certain point \( y \), maybe different from \( x \).

Meanwhile even the \( \mathcal{P} \)-adic zeta function has an explicit formula using the log-resolution. We take a model \( \mathcal{Y} \) of the log-resolution \( Y \) and we call \( \hat{\mathcal{E}}_J \) the closure of \( \mathcal{E}_J \) in \( \mathcal{Y} \) for every \( \emptyset \neq J \subseteq I \). Denote \( k_P \) the residue field at \( P \).

Theorem 2.4 (Denef’s formula).

For almost every \( P \),

\[ Z^\mathcal{P}(s) = q^{-d} \sum_{\emptyset \neq J \subseteq I} |\hat{\mathcal{E}}_J(k_P)| \prod_{j \in J} (q - 1)^{-N_j s - \nu_j} (1 - q^{-N_j s - \nu_j}) \]

where \( q \) is the cardinality of \( k_P \) and \( |\hat{\mathcal{E}}_J(k_P)| \) is the cardinality of the \( k_P \)-points of \( \hat{\mathcal{E}}_J \).

Proof. [Den87, Theorem 3.1] \qed

Now we have explicit formulas for the monodromy eigenvalues and of the \( \mathcal{P} \)-adic zeta function, using a log-resolution.

The main difficulty from here to prove the monodromy conjecture is the configuration of the irreducible components \( E_i \) in \( Y \).

Cases when the conjecture is known are:

- \( n = 2 \);
- \( n = 3 \) and \( f \) homogeneous;
- some nice classes of singularities.

For references about these cases and some other facts about the \( \mathcal{P} \)-adic monodromy conjecture you can look at [Nic09, Section 3].

3 Motivic zeta function

We have defined in the previous talk the naive motivic zeta function.

\[ Z^{\text{naive}}(s) := \int_{\mathcal{L}(X_0)} \mathbb{L}^{-\text{ord}_1(f)} s \, d\mu = \sum_{n=1}^{\infty} \mathbb{L}^{-ns-d(n+1)} [X_n/X_0] \in \mathcal{M}_{X_0}(\mathbb{L}^{-s}) \]

where \( X_n := \mathcal{L}_n(X_0) \cap \text{ord}_1^{-1}(n) \). We will be interested in studying \( Z^{\text{naive}}_x(s) := Z^{\text{naive}}(s) \times_{X_0} x \in \mathcal{M}_x(\mathbb{L}^{-s}) \) where the fiber product is done on the coefficients of the series and the topological zeta function \( Z^{\text{top}}_x(s) := \chi_{\text{top}}(Z^{\text{naive}}_x(s)) \), defined taking the Euler characteristic of the coefficients.

We have seen the following formula without a proof.
Theorem 3.1 (Denef-Loeser’s formula).
\[ Z^{naive}(s) = \mathbb{L}^{-n} \sum_{\emptyset \neq J \subseteq I} \frac{1}{(\mathbb{L} - 1)^{|J|}} [\hat{E}/X_0] \prod_{j \in J} \frac{\mathbb{L}^{-N_j s - \nu_j}}{1 - \mathbb{L}^{-N_j s - \nu_j}} \]

Sketch of the proof. One can reduce to the case when \( X_0 \) is an snc divisor, taking a log-resolution and using the change of variables formula. Then the computation becomes easier thanks to the local description of the divisor \( E \) as the zero locus of monomials. If you want to see how to do concretely this last computation I added in the Appendix A an example.

As a consequence we also have a formula for the topological zeta function:
\[ Z^{top}_x(s) = \sum_{\emptyset \neq J \subseteq I} \chi_{top}(\hat{E} \times X_0 x) \prod_{j \in J} \frac{1}{N_j s + \nu_j}. \] (3.1.1)

We finally have all the tools to prove Veys’ conjecture.

Theorem 3.2 (Veys’ conjecture). If \( s_0 \) is a pole of order \( d \) of \( Z^{top}_x(s) \) then \( s_0 = - \lct_x \).

Proof. If for some \( J, \circ E_J \neq \emptyset \) then the cardinality of \( J \) is at most \( d \) by dimension reasoning on \( E_J \), using that \( E \) is an snc divisor. Hence by the formula 3.1.1, if \( s_0 \) is a pole of order \( d \), there exists \( J_0 \subseteq I \) such that \( |J_0| = d, \ E_{J_0} \cap h^{-1}(x) \neq \emptyset \) and for every \( j \in J_0 \), \( -\nu_j/N_j = s_0 \). In particular, \( J_0 \) is maximal with this property. Hence we can apply the Main Theorem of Michael’s talk. Namely by the maximality of \( J_0, \nu_j/N_j = \lct_x \) for every \( j \in J_0 \). This proves the theorem.

We can ask for the motivic zeta function an analogous of the \( \mathcal{P} \)-adic monodromy conjecture. Here talking about poles is more delicate because \( M_{X_0} \) is not a domain [Poo02].

Conjecture 3.3 (Motivic monodromy conjecture). There exists a finite subset \( S \subseteq \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \) such that
\[ Z^{naive}(s) \in M_{X_0} \left[ \mathbb{L}^{-s}, \frac{1}{1 - \mathbb{L}^{-as-b}} \right] \subset M_{X_0}[[\mathbb{L}^{-s}]] \]
and such that \( (a, b) \in S \) implies \( \exp(-2\pi ib/a) \) is a monodromy eigenvalue for some \( x \in X_0 \).

Specialisation to \( \mathcal{P} \)-adic world

Take the ring
\[ \mathcal{Z}_p := \mathbb{Q} \left[ |k_p|^{-as-b} \right]_{(a,b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}} \]
where \( |k_p| \) is the cardinality of the residue field at \( p \). We denote \( \mathcal{Z} \) as the ring obtained via the quotient of \( \prod_{p \in \Omega_F^{fin}} \mathcal{Z}_p \) by the ideal \( \bigoplus_{p \in \Omega_F^{fin}} \mathcal{Z}_p \). We can define a morphism of rings
\[ \mathcal{N} : M_{X_0} \left[ \mathbb{L}^{-as-b}, \frac{1}{1 - \mathbb{L}^{-as-b}} \right]_{(a,b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}} \to \mathcal{Z} \]
in the following way: for every variety \( T \) we take a model \( \hat{T} \) over \( \Omega_F \) and we send the class \( [T/X_0] \in M_{X_0} \) to the class \( [(\hat{T}(k_p))]_{p \in \Omega_F^{fin}} \) where \( \hat{T}(k_p) \) is the number of \( k_p \)-points of the model \( \hat{T} \). The morphism \( \mathcal{N} \) is a well defined morphism of rings because two different models of \( T \) are isomorphic for almost every \( \mathcal{P} \). Putting together Denef and Loeser’s formulas for \( \mathcal{P} \)-adic and motivic zeta function we obtain the following.
Theorem 3.4 (Denef-Loeser).

\[ \mathcal{N}(Z^{\text{naive}}(s)) = \left[ \left[ Z^{\mathfrak{F}}(s) \right]_{\mathfrak{F} \in \Omega_f^{\text{fin}}} \right]. \]

In particular, as a consequence of this, the motivic monodromy conjecture implies the \( \mathfrak{F} \)-adic monodromy conjecture for almost every \( \mathfrak{F} \).
A An example

We want to compute the naive motivic zeta function

\[ Z^{\text{naive}}(s) := \sum_{n=1}^{\infty} \mathbb{L}^{-ns-d(n+1)}[X_n/X_0] \]

when \( f = x^{N_1}y^{N_2} \). We can decompose \( X_0 \) as a disjoint union \( \hat{E}_1 \sqcup \hat{E}_2 \sqcup \hat{E}_{12} \) with \( \hat{E}_1 = \{ x = 0, y \neq 0 \} \), \( \hat{E}_2 = \{ x \neq 0, y = 0 \} \) and \( \hat{E}_{12} = \{ x = 0, y = 0 \} \).

To compute the motivic zeta function we need to understand \([X_n/X_0]\) for every \( n \). We recall that \( X_n \) is the subscheme of \( \mathcal{L}_n(\mathbb{A}^d) \) with \( \mathbb{C} \)-points the \( n \)-jets with order \( n \). The \( \mathbb{C} \)-points of \( \mathcal{L}_n(\mathbb{A}^d) \) corresponds to \( \text{Hom}_{\text{Ring}}(\mathbb{C}[x, y], \mathbb{C}[t]/t^{n+1}) \), hence they are determined by the images of \( x \) and \( y \), namely a couple \((a_0 + a_1 t + \cdots + a_n t^n, b_0 + b_1 t + \cdots + b_n t^n)\) with \( a_i \) and \( b_i \) complex numbers. The order with respect to \( t \) of a certain \( \gamma_n \in \text{Hom}_{\text{Ring}}(\mathbb{C}[x, y], \mathbb{C}[t]/t^{n+1}) \) is given by \( v_t(\gamma_n(x^{N_1}y^{N_2})) \), where \( v_t \) is the standard valuation on \( \mathbb{C}[t]/t^{n+1} \) with \( v_t(t) = 1 \). The previous decomposition translates in a decomposition \( X_n = \hat{E}_{1,n} \sqcup \hat{E}_{2,n} \sqcup \hat{E}_{12,n} \) where \( \hat{E}_{J,n} := (\pi^n)^{-1}(E_J) \cap X_n \).

Let’s study one piece at a time. The variety \( \hat{E}_{1,n} \) is a locally trivial fibration of \( \hat{E}_1 \). We want to understand the fiber. We fix a point \((\pi_0^n, \overline{b}_0) \in \hat{E}_1 \), hence \( \pi_0^n = 0 \) and \( \overline{b}_0 \neq 0 \). The points \( \gamma_n = (\pi_0^n + a_1 t + \cdots + a_n t^n, \overline{b}_0 + b_1 t + \cdots + b_n t^n) \) over \((\pi_0^n, \overline{b}_0)\) are precisely given by the condition \( v_t(\gamma_n(x^{N_1}y^{N_2})) = n \). The element \( \gamma_n(y) \) is invertible as \( \overline{b}_0 \neq 0 \), thus \( v_t(\gamma_n(x))N_1 = n \). In particular \( N_1|n \) and if we denote \( a_1 := n/N_1 \), then \( a_1 = \cdots = a_{\alpha_1 - 1} = 0, a_{\alpha_1} \neq 0 \). There are no conditions on the other \( b_i \), hence \( \hat{E}_{1,n} \) is a \( (\mathbb{G}_m \times \mathbb{A}^{n-\alpha_1} \times \mathbb{A}^n) \)-bundle over \( \hat{E}_1 \) when \( N_1|n \) and it’s empty if \( N_1 \nmid n \).

Thus if \( N_1|n \), \( \hat{E}_{1,n}/X_0 = (\mathbb{L} - 1)^{2n-\alpha_1}[\hat{E}_1/X_0] \) and

\[
\sum_{n=1}^{\infty} \mathbb{L}^{-ns}\mathbb{L}^{-2(n+1)}[\hat{E}_{1,n}/X_0] = \sum_{\alpha_1|1} \mathbb{L}^{-ns}\mathbb{L}^{-2(n+1)}(\mathbb{L} - 1)^{2n-\alpha_1}[\hat{E}_1/X_0] =
\]

\[
\mathbb{L}^{-2(\mathbb{L} - 1)}[\hat{E}_1/X_0] \sum_{\alpha_1|1} \mathbb{L}^{-\alpha_1(-N_1s-1)} =
\]

\[
\mathbb{L}^{-2(\mathbb{L} - 1)}[\hat{E}_1/X_0] \frac{\mathbb{L}^{-N_1s-1}}{1 - \mathbb{L}^{-N_1s}}.
\]

The same reasoning applies to \( \hat{E}_2 \).

The case \( \hat{E}_{12,n} \) is slightly different. The scheme \( \hat{E}_{12,n} \) consists only of one point \((0, 0)\). The \( n \)-jets \( \gamma_n = (a_1 t + \cdots + a_n t^n, b_1 t + \cdots + b_n t^n) \) over \((0, 0)\) with order \( n \) are again given by the condition \( \gamma_n(x^{N_1}y^{N_2}) = n \), thus we have \( v_t(\gamma_n(x))N_1 + v_t(\gamma_n(y))N_2 = n \). For every choice of \((\alpha_1, \alpha_2) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \) such that \( \alpha_1 N_1 + \alpha_2 N_2 = n \), the \( n \)-jets with \( v_t(\gamma_n(x)) = \alpha_1 \) and \( v_t(\gamma_n(y)) = \alpha_2 \) give a variety isomorphic to \( (\mathbb{G}_m^n \times \mathbb{A}^{n-\alpha_1} \times \mathbb{A}^{n-\alpha_2}) \) over \( \hat{E}_{12} \). In other words, if \( \alpha_1 N_1 + \alpha_2 N_2 = n \), \( [\hat{E}_{12,n}/X_0] = (\mathbb{L} - 1)^{2n-\alpha_1-\alpha_2}[\hat{E}_{12}/X_0] \) and
\[
\sum_{n=1}^{\infty} L^{-nsL^{-2(n+1)}} [\hat{\mathcal{E}}_{12,n}/X_0] = \sum_{n=1}^{\infty} \sum_{\alpha_1, \alpha_2 \in \mathbb{Z}_{n=0}} L^{-nsL^{-2(n+1)}(\mathbb{L} - 1)^2 \mathbb{L}^{2n-\alpha_1-\alpha_2} [\hat{\mathcal{E}}_{12}/X_0] = \\
= L^{-2}(\mathbb{L} - 1)^2 [\hat{\mathcal{E}}_{12}/X_0] \sum_{\alpha_1, \alpha_2 \in \mathbb{Z}_{>0}} L^{\alpha_1(-N_s-1)} L^{\alpha_2(-N_s-1)} = \\
= L^{-2}(\mathbb{L} - 1)^2 [\hat{\mathcal{E}}_{12}/X_0] L^{-N_1s-1} L^{-N_2s-1}. 
\]

Now you can put the three pieces together and compare the result with Theorem 3.1. Recall that in our case \( \nu_1 = 1 \).

References


