Motivic Integration*

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The story of motivic integration started from Kontsevich's "Lectures at Orsay" [Kon95] on December 7, 1995. Motivic integraion was introduced to prove a conjecture by Batyrev [Bat98]. However, Kontsevich himself didn't publish any thing about the them. The theory was developed and generalized mainly in the following directions [Nic10, §4]:

- Denef and Leser [DL99] developed a theory of *geometric* motivic integration on arbitrary algebraic varieties over a field of characteristic 0.
- Loeser and Sebag [Seb04; LS03] constructed a theory of motivic integration on formal schemes over a complete discrete valuation with perfect residue field k.
- Cluckers and Loeser [CL08] built a very general framework for motivic integration theories based on model theory. And a different model-theoretic approach was developed by Hrushovski and Kazhdan [HK06].

In this talk, I will discuss (part of) the first story. The first goal is to explain what is a motivic integral. Basically, we are to understand the following notation:

$$\int_{\boxed{\text{domain of integraltion } A \subseteq \mathcal{L}(X)}} \boxed{\text{integrable function } \Psi} \, \mathrm{d} \, \boxed{\text{motivic measure } \mu_X} \in \boxed{\text{value ring of integration } \hat{\mathcal{M}}_S^{\hat{\mu}}}$$

Besides, a formula for change of variables will be introduced without proof. Finally we will introduce motivic zeta functions and their rationality.

Let k be a field. A k-variety is a *reduced separated* k-scheme of *finite type*. Many notions can be defined for arbitrary (at least perfect) field k. But to be sure, we can assume k is of characteristic 0 or simply assume $k = \mathbb{C}$.

1 Grothendieck Ring of Varieties

Let's first recall the *Grothendieck ring of k-varieties* $K_0(\mathtt{Var}_k)$.

It is the abelian group generated by isomorphism classes $[X]$ of separated k-schemes of finite type X subject to the "*scissor relations*": $[X] = [X \setminus Y] + [Y]$ if Y is a closed subscheme of X. It has a natural ring structure with multiplication given by

$$[X_1] \cdot [X_2] := [X_1 \times_k X_2]$$

for any separated k-schemes X_1 and X_2 of finite type. Put $\mathbb{L} := [A_k^1]$ and $1 = \operatorname{Spec} k$. Then denote by $\mathcal{M}_k := K_0(\mathtt{Var}_k)[\mathbb{L}^{-1}]$ the localization by inverting $\mathbb{L}$.

Remark 1 Thanks to the scissor relations, we have $[X_{\mathrm{red}}] = [X]$. Hence in the above definition, we could use "k-varieties" instead of "separated k-schemes of finite type". But one needs to be careful that if k is not perfect, then the fiber product of two reduced k-schemes may not be reduced.

*updates see `https://haoyun.github.io/files/16WS-FS-Motivic-Integration.pdf`

Remark 2 These two rings $K_0(\mathtt{Var}_k)$ and $\mathcal{M}_k$ are quite mysterious. It's known that $K_0(\mathtt{Var}_k)$ is not a domain if k is of characteristic 0 [Poo02]. It's either not known whether $\mathbb{L}$ is a zero divisor in $K_0(\mathtt{Var}_k)$.

Naumann[??] found zero-divisors in $K_0(\mathtt{Var}_k)$ for k a finite filed and these are non-zero even after localizing at $\mathbb{L}$ – thus for a finite field $\mathcal{M}_k$ is not a domain.

Example 1 $[\mathbb{P}^n] = \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + \mathbb{L} + 1$. $1 := [\mathrm{Spec}\, k]$.

Example 2 Using weak factorization theorem, it can be show $K_0(\mathtt{Var}_k)$ could be generated by $[X]$ with X smooth and $[\tilde{X}] - [\tilde{Y}] = [X] - [Y]$ [Loo02, §4].

Given a k-variety S, we can also define relative versions $K_0(\mathtt{Var}_S)$ and $\mathcal{M}_S$ in the same way. We leave out the details and introduce a more delicate one.

1.1 The $\hat{\mu}$-equivariant Grothendieck Ring

Let S be an algebraic variety over k. An S-variety X is a k-variety X equipped with a k-morphism $X \longrightarrow S$. The category of S-varieties is denoted by $\mathtt{Var}_S$, in which, the arrows are morphisms over S.

For any $n \in \mathbb{N}$, let

$$\mu_n := \mathrm{Spec}\, k[x]/(x^n - 1)$$

be the group schemeof n-th root of unity. Then $(\mu_n)_{n\in\mathbb{N}}$ form a projective system. So we can define

$$\hat{\mu} := \varprojlim \mu_n$$

the projective limit, which is a *pro-variety.*

Let X be an S-variety. A *good μ_n-action on X* is a group action

$$\alpha : \mu_n \times_k X \longrightarrow X$$

over S, such that each orbitis contained in an affine subvariety of X. The last condition is automatically satisfied if X is a quasi-projective variety.A good $\hat{\mu}$-action on X is an action of $\hat{\mu}$ on X which factors througha good μ_n-action, for some n.

For a fixed k-variety S,the *monodromic Grothendieck group* $K_0^{\hat{\mu}}(\mathtt{Var}_S)$ is the abelian group generated by symbols $[X, \hat{\mu}]$(sometimes also denoted by $[X/S, \hat{\mu}]$), where X is an S-variety with a good $\hat{\mu}$-action, subject to relations:

1. $[X, \hat{\mu}] = [Y, \hat{\mu}]$ if X and Y are isomorphic as S-varieties with μ-actions,
2. $[X, \hat{\mu}] = [Y, \hat{\mu}] + [X \setminus Y, \hat{\mu}]$, whenever Y is a closed subvariety with the $\hat{\mu}$-action induced from that on X, and
3. $[S \times_k V, \hat{\mu}] = [S \times_k \mathbb{A}^n_k, \mu]$ where V is the affine n-space with any linera $\hat{\mu}$-action and $\mathbb{A}^n_k$ is that with the trivial action.

Similar to $K_0(\mathtt{Var}_k)$, the group $K_0^{\hat{\mu}}(\mathtt{Var}_S)$ has a ring structure induced by fiber product.

Remark 3 The above construction also applies to general groups instead of $\hat{\mu}$.

Denote by $\mathcal{M}_S^{\hat{\mu}}$ the localization $K_0^{\hat{\mu}}(\mathtt{Var}_S)[\mathbb{L}^{-1}]$, where $\mathbb{L}$ is the class in $K_0^{\hat{\mu}}(\mathtt{Var}_S)$ of $S \times_k \mathbb{A}^1_k$ with the trivial $\hat{\mu}$-action.

Example 3 If $Y \longrightarrow X$ is a (Zariski)-locally trivial fibration with fiber Z, then $[Y] = [X] \cdot [Z]$.

Let X be an S-variety with a good $\hat{\mu}$-action, which factors through a good μ_m-action). Denote by $\bar{X} := X/\mu_m$. Then there is a well defined map [Loo02, Lemma 5.1]

$$\mathcal{M}_S^{\mu} \longrightarrow \mathcal{M}_S, \quad X \longrightarrow \bar{X}$$

This works because μ_m is a fintie abelian group.

1.2 Completed Grothendieck Ring

The completed Grothendieck ring $\hat{\mathcal{M}}_k$ was also first introduced by Kontsevich [Kon95].

For any $m \in \mathbb{Z}$, let $F_m\mathcal{M}_k$ be the subgroup of $\mathcal{M}_k$ spanned by $[Z]\mathbb{L}^{-r}$ with dim $\leq m + r$. One checkes that $F_m\mathcal{M}_k \cdot F_n\mathcal{M}_k \subseteq F_{m+n}\mathcal{M}_k$. So we can complete $\mathcal{M}$ by setting

$$\hat{\mathcal{M}}_k := \varprojlim \mathcal{M}_k/F_m\mathcal{M}_k.$$

This is called the *dimensional completion*, and is also a ring.

There is another way to define the completion (see [Bli11, §2.1] and [Bat98]). In $\mathcal{M}_k$, we define $\mathbb{L}^{-1}$ to have dimension -1. Consider the composition of $\dim : \mathcal{M}_k \longrightarrow \mathbb{Z} \cup \{-\infty\}$ with $\exp : \mathbb{Z} \cup \{-\infty\} \longrightarrow \mathbb{R}_{\geq 0}$ (set $\exp(-\infty) := 0$). It defines a map

$$\delta_k : \mathcal{M}_k \longrightarrow \mathbb{R}_{\geq 0},$$

which can be verified to be a *non-archimedean* norm. Then $\hat{\mathcal{M}}_k$ is the competion with rspect to this norm. This viewpoint makes the convergence in $\hat{\mathcal{M}}_k$ easier: a series converges if and only if the norm of the summands goes to 0, in other words, the dimension goes to $-\infty$.

$\hat{\mathcal{M}}_S^{\hat{\mu}}$ dimensional completion and completion with non-archimedean norm.

Remark 4 It is not known whether the natural map $\mathcal{M}_k \longrightarrow \hat{\mathcal{M}}_k$ is injective (see [Loo02, footnote 3] and [DL01, §4.3]).

Example 4 In $\hat{\mathcal{M}}_k$, it holds that [Bli11, Exercise 2.6]

$$\sum_{i \geq 0} \mathbb{L}^{-\nu i} = \frac{1}{1 - \mathbb{L}^{-\nu}}.$$

The completion works in relative case and equivariant case. Details are left out.

2 Arc Spaces

Suppose that X is k-variety of finite type.Consider the functor (Weil restriction)

$$\mathtt{Var}_k \longrightarrow \mathtt{Set}, \quad T \mapsto \mathrm{Hom}_k(T \times_k k[t]/t^{n+1}, X)$$

This functor is represented by a variety $\mathcal{L}_n(X)$ of finite type over k. In other words, we have bijection

$$\mathrm{Hom}_k(T, \mathcal{L}_n(X)) = \mathrm{Hom}_k(T \times_k k[t]/t^{n+1}, X).$$

The scheme $\mathcal{L}_n(X)$ is called the *space of n-jets of X*. By the very definition, we have for any $m \geq n$, there is a natural morphism $\pi_m^n : \mathcal{L}_m(X) \longrightarrow \mathcal{L}_n(X)$. It can be verified that π_n^m is an affine morphism for each $m \geq n$. Therefore, the projective limit

$$\mathcal{L}(X) := \mathcal{L}_\infty(X) := \varprojlim \mathcal{L}_m(X)$$

exists([Stacks, Tag 01YX]) as a k-scheme and is reduced and separated if X is. But it's no longer of finite type in general.This scheme is called the *arc space of X*. By definition, for any field K/k, we have bijection

$$\mathrm{Hom}_k(K, \mathcal{L}_\infty(X)) = \mathrm{Hom}_k(K[\![t]\!], X).$$

Suppose that $Y \longrightarrow X$ is a morphism of k-schemes. There are obviously natural maps $\mathcal{L}_n(Y) \longrightarrow \mathcal{L}_n(X)$, forall $n \in \mathbb{N}$, and $\mathcal{L}_\infty(Y) \longrightarrow \mathcal{L}_\infty(X)$. Note also that any morphism $f : Y \longrightarrow X$ induces natural morphisms $f_* : \mathcal{L}_\infty(Y) \longrightarrow \mathcal{L}_\infty(X)$ and $f_n : \mathcal{L}_n(Y) \longrightarrow \mathcal{L}_n(X)$. Clearly, $\mathcal{L}_0(X) = X$ and $\mathcal{L}_1(X) = \mathcal{T}X$.

Any point $\gamma \in \mathcal{L}_\infty(X)$, usually called an *arc*, corresponds to a map $\operatorname{Spec} \kappa(\gamma) \longrightarrow \mathcal{L}_\infty(X)$, where $\kappa(\gamma)$ is the residue field at γ. Hence by definition, it further corresponds to a map $\operatorname{Spec} \kappa(\gamma)[[t]] \longrightarrow X$. We will not distinguish this map with γ from now on. The image $\gamma(0)$ of $0 = (t) \in \operatorname{Spec} \kappa(\gamma)[[t]]$ is usually called the *base point* or *origin* of the arc.

For any poser series $\alpha(t) \in K[[t]]$, we define the order of $\alpha(t)$ to be the maximal $e \in \mathbb{N} \cup \{\infty\}$ such that $t^e | \alpha(t)$. If $s \in \mathcal{O}_{X,\gamma(0)}$, we define $\operatorname{ord}_{s,t}(\gamma) := \operatorname{ord}_t(\gamma^{-1}s)$, where $\gamma : \mathcal{O}_{X,\gamma(0)} \longrightarrow \mathcal{O}_{\operatorname{Spec} \kappa(\gamma)[[t]],0} = \kappa(\gamma)[[t]]$ is the inverse image map. If $\mathcal{I} \subseteq \mathcal{O}_X$ is an ideal, we define

$$\operatorname{ord}_{\mathcal{I},t}(\gamma) := \operatorname{ord}_{\mathcal{I}}(\gamma) = \min \left\{ \operatorname{ord}_{s,t} \gamma \ : \ s \in \mathcal{I}_{\gamma(0)} \right\}.$$

Thus we get a map

$$\operatorname{ord}_{\mathcal{I}} : \mathcal{L}_\infty(X) \longrightarrow \mathbb{N} \cup \{\infty\}$$

and we can see that $\operatorname{ord}_{\mathcal{I}}(\gamma) = \infty$ if and only if the image of γ lies inside the support of $\mathcal{I}$.In the same way we get maps for all $1 \leq\in \mathbb{N}$,

$$\operatorname{ord}_{\mathcal{I}} : \mathcal{L}_\infty(X) \longrightarrow \mathbb{N} \cup \{0, 1, 2, \dots, n, \infty\}$$

If $E \subseteq X$ is an effective Cartier divisor, we write ord_E for the order map defined by the ideal sheaf corresponding to E.

Example 5 If $X = \operatorname{Spec} k[x_1, \dots, x_n] = \mathbb{A}^n_k$ is the affine n-space over k. Then $\mathcal{L}_m(X) = \mathbb{A}^{n(m+1)}_k$. [Bli11, example 2.7].

Proposition 1 *Let $X \longrightarrow Y$ be a étale morphism of k-varieties, then*

$$\mathcal{L}_m(X) \cong \mathcal{L}_m(Y) \times_Y X.$$

Proof Follows from formally étaleness and direct computation [Bli11, Proposition 2,9].

Proposition 2 *Suppose that X is* smooth *of pure dimension d over k. Then $\mathcal{L}_m(X)$ is an $\mathbb{A}^{dm}$-bndle over X. In particular, $\mathcal{L}_m(X)$ is smooth of pure dimension $d(m+1)$. In the same way, $\mathcal{L}_{m+1}(X)$ is a $\mathbb{A}^d$-bundle over $\mathcal{L}_m(X)$.*

Proof This is Smooth schemes are étale-locally like affine spaces [Stacks, Tag 054L]. Then use the previous example and proposition.

A Generalization – Greenberg Scheme See [Seb04; Loo02; Gre61; Gre63] for original discussion. [NS11] is a good introductory text. This topic It was introduced first in [Gre61; Gre63] over any Artinian local ring as to generalize Shimura's [Shi55] "reduction modulo $\mathfrak{p}$" to "reduction modulo $\mathfrak{p}^{n+1}$" for all natural numbers n. A modern treatment on this topic could be found in [BG13]. There are also some discussion in [BLR90, §9.6].

Let $(R, \mathfrak{m})$ be a complete discrete valuation ring with *perfect* residue field k and fraction field K. For each $n \in \mathbb{N}$, set $R_n := R/\mathfrak{m}^{n+1}$. Consider the functor

$$\mathtt{Alg}_k \longrightarrow \mathtt{Alg}_{R_n}, \quad A \mapsto \mathscr{R}_n(A)$$

where $\mathscr{R}_n(A)$ is defined separately in different cases (details omit.)

- equal characteristic. $R \cong k[\![t]\!]$ once a uniformizer is chosen.
- mixed characteristics:
- absolutely unramified.
- absolutely ramified.

This construction leads to a functor

$$\mathtt{Var}_k \longrightarrow \mathtt{Var}_{R_n}, \quad T = (T, \mathcal{O}_T) \mapsto (T, \mathscr{R}_n(\mathcal{O}_T)) := \mathscr{R}_n(T).$$

For any variety X over R_n, the so called *Greenberg functor* is defined as

$$\mathtt{Var}_k \longrightarrow \mathtt{Set}, \quad T \mapsto \mathrm{Hom}_{R_n}(\mathscr{R}_n(T), X)$$

It is represented by a k-variety $G_n(X)$, called the *Greenberg realization.*

Now for any R-scheme X, set

$$\mathcal{X}_n := G_n(\mathcal{X} \times_R R_n), \text{ and } \mathcal{X}_\infty := \varprojlim \mathcal{X}_n$$

One easily see that in equal characteristic, if $\mathcal{X} = X \times_k R$ for some k-variety X, then $\mathcal{X}_n = \mathcal{L}_n(X)$ and $\mathcal{X}_\infty = \mathcal{L}_\infty(X)$.

All discussion in the previous part on $\mathcal{L}_\infty(X)$ generalizes to $\mathcal{X}_\infty$.

3 Motivic Measure

3.1 The tautological measure "on X"

Recall that a subset of a variety X is *constructibe* if it is a finite union of locally closed subvarieties of X. Any constructible subset $C \subseteq X$ defines an element $[C] \in \mathcal{M}_k$. Or more precisely, we can always write C as a a finite disjoint union of locally closed subsets $U_1, \dots, U_n$ and the scissor relations imply that

$$[C] := \sum_{i=1}^{r} [U_i]$$

in $K_0(\mathtt{Var}_k)$ does not depend on the choice of the partition (recall also $[X_{\mathrm{red}}] = [X]$).

The constructible subsets of X form a *Boolean algebra*, i.e., closed under complements, finite unions and finite intersections. Hence we obtain in a tautological manner a $K_0(\mathtt{Var}_k)$-valued measure μ_X defined on this Boolean algebra.

For any morphism $f : Y \longrightarrow X$, we define we can define the direct image:

$$(f_* \mu_Y)(U) := \mu_Y(f^{-1}(U))$$

for any constructible U subset of X.

Example 6 See [Nic10, §4.2] for some specialization morphisms.

3.2 The motivic measure "on $\mathcal{L}_\infty(X)$"

Our goal is to define a measure on an interesting algebra of subsets of $\mathcal{L}_\infty(X)$ in such a way that its direct image under $\pi_X : \mathcal{L}_\infty(X) \longrightarrow \mathcal{L}_0(X) = X$ "is" the tautological measure μ_X when X is smooth.

Remark 5 As we will see later, the convention differs. Some authos, e.g. [Loo02], requires the direct image is μ_X, but others, e.g. [DL01], requires the direct image to be a multiple of μ_X.

3.2.1 Naive Motivic Measure

Assume that X is of pure relative dimension d. A subset $A \in \mathcal{L}_\infty(X)$ is *stable* if for some n,

1. $\pi_n(A)$ is constructible in $\mathcal{L}_n(X)$ and $A = \pi_n^{-1}\pi_n(A)$.
2. for all $m \geq n$, the projection $\pi_{m+1}(A) \longrightarrow \pi_m(A)$ is a locally trivial fibration.[1]

[1]piecewise-trivial fibration in [Loo02]

The second condition is superfluous in case X/k is smooth as it was pointed in Proposition 2. The collection $\tilde{\mathcal{B}}(X)$ of of stable subsets of $\mathcal{L}_\infty(X)$ is a boolean algebra. Define

$$\tilde{\mu}_X : \tilde{\mathcal{B}}(X) \longrightarrow \mathcal{M}_k, \quad A \longmapsto [\pi_m(A)]\mathbb{L}^{-(m+1)d},^{2}$$

and this is called the *naive motivic measure on* X.

We call $A \subseteq \mathcal{L}_\infty(X)$ a *cyliner*, if $A = \pi_n^{-1}B$ for some constructibe subset $B \subseteq \mathcal{L}_n(X)$. If X is smooth, all cylinders are stable. And in smooth case, the stable sets are precisely the cylinders.

3.2.2 Motivic Measure

Consider the composition $\mu_X : \tilde{\mathcal{B}}(X) \longrightarrow \mathcal{M}_k \longrightarrow \hat{\mathcal{M}}_k$, and we shall extend this map to be what we will call a *motivic measure.*

A subset $A \subseteq \mathcal{L}_\infty(X)$ is *measurable* if for every (negative) integer m there is a stable subset $A_m \subseteq \mathcal{L}_\infty(X)$ and a sequence $(C_i \subseteq \mathcal{L}_\infty(X))_{i=0}^{\infty}$ of stable subsets such that the symmetric difference[3] $A \triangle A_m$ is contained in $\cup_{i\in\mathbb{N}} C_i$ with $\dim C_i < m$ for all i and $\dim C_i = -\infty$ for $i \longrightarrow \infty$.

Proposition 3 *The measurable subsets of $\mathcal{L}(X)$ form a boolean subalgebra*

$$\mu_X(A) := \lim_{m\longrightarrow -\infty} \mu_X(A_m).$$

In particular, the above limit exists in $\hat{\mathcal{M}}_k$ and it's value only depends on A.

Proof See [Loo02, Prop. 2.2]

So a countable union of stable sets $A = \cup_{n\in\mathbb{N}} A_n$ with $\lim_{i\longrightarrow\infty} \dim A_n = -\infty$ is measurable and

$$\mu_X(A) = \lim_{n\longrightarrow\infty} \mu_X(\cup_{k\le n} A_k).$$

The boolean algebra of measurable subsets of $\mathcal{L}_\infty(X)$ is denoted by $\mathcal{B}(X)$. Thus the *motivic measure* is the map

$$\mu_X : \mathcal{B}(X) \longrightarrow \hat{\mathcal{M}}_k, \quad A \longmapsto \mu_X(A).$$

4 Transformation Rule

A function $\Psi : \mathcal{L}_\infty(X) \longrightarrow \hat{\mathcal{M}}_k$ is *measurable* or *integrable* if each fibre of Ψ is measurable and the sum $\sum_a \mu_X(\Psi^{-1}(a))a$ converges. That is to say, there are at most countably many nonzero terms $\mu_X(\Psi^{-1}(a_i))a_i$, $i \in \mathbb{N}$, and $\mu_X(\Psi^{-1}(a_i))a_i \in F_{m_i}\hat{\mathcal{M}}_k$ with $\lim m_i = -\infty$. The *motivic integration* of Ψ is then by definition the value of this series:

$$\boxed{\int_{\mathcal{L}_\infty(X)} \Psi \, \mathrm{d}\mu_X := \sum_i \mu_X\left(\Psi^{-1}(a_i)\right) a_i}$$

Of course, we can define the integrable function $\Psi : A \longrightarrow \hat{\mathcal{M}}_k$ and its itegration over A.

There is a special kind of measurable functions given by ideals $\mathcal{I} \subseteq \mathcal{O}_X$:

$$\mathbb{L}^{-\operatorname{ord}_{\mathcal{I}}} : \mathcal{L}_\infty(X) \longrightarrow \hat{\mathcal{M}}_k.$$

(For any $n \in \mathbb{N}$, the condition $\operatorname{ord}_{\mathcal{I}}(\gamma) = n$ depends only on on the n jet of γ and it defines a constructible $C_n = \{\gamma \in \mathcal{L}_n(X) : \operatorname{ord}_{\mathcal{I}}(\gamma) = n\}$. Hence the fibers $\mathbb{L}^{-\operatorname{ord}_{\mathcal{I}}}$ are measurable. One can show that $\mu_X(C_n) \cdot \mathbb{L}^{-n}$ has dimension no more than $d - n$, hence the series converges. See [Bli11, §2.4–2.5] for details.) By definition, we have

$$\int_{\mathcal{L}_\infty(X)} \mathbb{L}^{-\operatorname{ord}_{\mathcal{I}}} \mathrm{d}\mu_X = \sum_{n=0}^{\infty} \mu_X(C_n)\mathbb{L}^{-n}.$$

[2] as pointed out before, [Loo02] uses the factor $\mathbb{L}^{-md}$

[3] $A \triangle B := (A \cup B) \setminus (A \cap B)$.

Lemma 1 *Let R be a ring. Let M be a finite R-module. Choose a presentation*

$$R^{\oplus J} \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0$$

of M. Let $A = (a_{ij})_{i=1,\dots,n;\, j\in J}$ be the matrix of the first map, Then the ideal $\operatorname{Fit}_k(M)$ generated by the $(n-k)\times(n-k)$ minors of A is independent of th the choice of the presentation.

Proof See [Stacks, Tag 07Z8] for a proof. The remark [Stacks, Tag 02FD] is also interesting.

Let $h : Y \longrightarrow X$ be a morphism of k-varieties.The *Jacobian ideal* $\mathcal{J}_h \subseteq \mathcal{O}_Y$ of h is the 0-th Fitting ideal $\operatorname{Fitt}_0(\Omega^1_{\mathcal{Y}/\mathcal{O}_X})$.

Proposition 4 ([Loo02, Prop. 3.2]) *Let $h : Y \longrightarrow X$ be a proper birational k-morphism of pure dimensional k-varieties such that Y is smooth over k. If A is a measurable subset of $\mathcal{L}_\infty(Y)$ with $H|_A$ injective, where $H : \mathcal{L}_\infty(Y) \longrightarrow \mathcal{L}_\infty(X)$ is the morphism induced by h, then $H(A)$ is measurable and*

$$\mu_X\big(H(A)\big) = \int_A \mathbb{L}^{-\operatorname{ord}\mathcal{J}_h}\, \mathrm{d}\mu_Y.$$

5 Motivic Zeta Function

Let k be any field, and X a smooth k-variety of pure dimension d. Let $f : X \longrightarrow \mathbb{A}^1$ be a k-morphism and $n \geq 1$ be an integer.

View f as a global section of $\mathcal{O}_X$, we have

$$\operatorname{ord}_t : \mathcal{L}_n(X) \longrightarrow \{1,\dots,n,\infty\}, \quad \text{and} \quad \operatorname{ord}_t : \mathcal{L}_n(X) \longrightarrow \mathbb{N}\cup\{\infty\},$$

defined by (f) as discussed before.[4] Set

$$\mathcal{X}_n := \{\gamma \in \mathcal{L}_n(X) : \operatorname{ord}_t \gamma = n\}$$

to be the locally closed sub-variety of $\mathcal{L}_n(X)$ consist of arcs exactly of order n.

If we set $X_0 := f^{-1}(0) \subseteq X$ to be the closed subvariety defined by f. Then $\pi_0^n(\mathcal{X}_n) \subseteq X_0$ where $\pi_0^n : \mathcal{L}_n(X) \longrightarrow X$. So the restriction of π_0^n to $\mathcal{X}_n$ gives $\mathcal{X}_n$ an X_0-scheme structure. Moreover, there is natural map

$$\operatorname{ac} : \mathcal{X}_n \longrightarrow \mathbb{G}_{\mathrm{m},k} = \mathbb{A}^1_k - \{0\}$$

that sends each $\gamma \in \mathcal{X}_n$ to the coefficients of t^n of the corresponding (truncated) power series. There is natural $\mathbb{G}_{\mathrm{m},k}$-action on $\mathcal{X}_n$ given by $\gamma\cdot a = \gamma(at)$, where $\gamma(t)$ is the power series corresponding to γ. Clearly, $\operatorname{ac}(\gamma\cdot a) = a^n \operatorname{ac}(\gamma)$. If we set $\mathcal{X}_{n,1}$ to be the fiber over 1 under the map ac, then the $\mathbb{G}_m$-action on $\mathcal{X}_n$ induces a good μ_m-action on $\mathcal{X}_{n,1}$, hence a good action of $\hat{\mu}$. We can also consider $\mathcal{X}_n$ as a $X_0 \times_k \mathbb{G}_{\mathrm{m},k}$-scheme via

$$(\pi_0^n, \operatorname{ac}) : \mathcal{X}_n \longrightarrow X_0 \times_k \mathbb{G}_{\mathrm{m},k}.$$

Now it makes sense to talk about

$$[\mathcal{X}_{n,1}/X_0, \hat{\mu}] \in \mathcal{M}^{\hat{\mu}}_{X_0} \quad \text{and} \quad [\mathcal{X}_n/X_0] \in \mathcal{M}_{X_0}.$$

The *motivic zeta function* of $f : X \longrightarrow \mathbb{A}^1_k$ is the power series

$$Z_f(T) := \sum_{n\geq 1} [\mathcal{X}_{n,1}/X_0, \hat{\mu}]\mathbb{L}^{-nd}T^n \in \mathcal{M}^{\hat{\mu}}_{X_0}[\![T]\!],$$

[4] There is an alternative way (essentially the same) to define the order. Note that the image of γ under the induced map $\mathcal{L}_\infty(X) \longrightarrow \mathcal{L}_\infty(\mathbb{A}^1_k)$ is an arc in $\mathbb{A}^1_k$, which is canonically identified with a power series. So the order of γ with respect to f can be defined as the order or this power series.

and the *native motivic zeta function* of f is defined as the power series

$$Z_f^{\mathrm{naive}}(T) := \sum_{n\geq 1}[\mathcal{X}_n/X_0]\mathbb{L}^{-nd}T^n \in \mathcal{M}_{X_0}[\![T]\!].$$

One can verify that $\mathcal{X}_n$ as a $(X_0 \times \mathbb{G}_{\mathrm{m},k})$-variety, is isomorphic to the quotient of $\mathcal{X}_{n,1} \times_k \mathbb{G}_{\mathrm{m},k}$ unde the $\hat{\mu}_n$-action defined by $a \cdot (\phi, b) = (\phi \cdot a, a^{-1}b)$. The projection $\mathcal{X}_n \longrightarrow \mathcal{X}_{n,1}/\mu_n$ gives μ_n a $\mathbb{G}_m$ bundle structure.

Assume the following setting as usual. Let (Y, h) be a resolution of f. This means that Y is a smooth irreducible k-variety, $h : Y \longrightarrow X$ is proper, such that its restriction $Y \setminus h^{-1}(X_0) \longrightarrow X \setminus X_0$ is an isomorphism and that $h^{-1}(X_0)$ has only normal crossings. Suppose $E = \mathrm{div}(f \circ h) = \sum N_i E_i$ with E_i being the irreducible components, and $\mathrm{div}(\mathcal{J}_h) = \sum(\nu_i - 1)E_i$.For each $I \subseteq \mathrm{Irr}(E)$, set $E_I = \cap_{i\in I}E_i$ and $E_I^0 = E_I \setminus \cup_{j\notin I}E_j$. Set $N_I := \gcd(N_i)$.

Moreover, we constucut two kinds of auxilary schemes.

1. For each $i \in \mathrm{Irr}(E)$, let U_i be the normal bundle of E_i minus the zero section. Hence U_i is a $\mathbb{G}_m$ bundle over E_i. For each $I \subseteq \mathrm{Irr}(E)$, set U_I to be the product of bundles $U_i|_{E_I^0}$, which is a $\mathbb{G}_m^{|I|}$ bundle over E_I^0.
2. Consider the folloing construction:

$$\begin{tikzcd}
\widetilde{E_I^0} \arrow[rr] \arrow[d] & & E_I^0 \arrow[d, hook] \\
\tilde{Y} \arrow[r, "\text{normalization}"] & Y' \arrow[r] \arrow[dd] \arrow[dr, phantom, "\square"] & Y \arrow[d, "h"] \\
& & X \arrow[d, "f"] \\
& \mathbb{A}^1_k \arrow[r, "m\text{ power}"] & \mathbb{A}^k_1
\end{tikzcd}$$

Make a base change of $f \circ h$ along the m-th power map of $\mathbb{A}^1_k$, and normalize, then we get a μ_m covering $\tilde{Y} \longrightarrow Y$. Take $\widetilde{E_I^0}$ to be a connected component of the preimage of E_I^0 in $\tilde{Y}$. Then $\widetilde{E_I^0} \longrightarrow E_I^0$ is a Galois covering with Galois group a subgroup of μ_{N_I}. If $I \neq \emptyset$, $\widetilde{E_I^0}$ lies over X_0. So $[\tilde{E}_I^0/X_0, \mu_{N(I)}]$ is well defined element in $\mathcal{M}_{X_0}^{\hat{\mu}}$.

Theorem 1 *It holds that*

$$\boxed{Z_f(T) = \sum_{\emptyset\neq I\subseteq \mathrm{Irr}(E)} (\mathbb{L}-1)^{|I|-1}[\widetilde{E_I^0}/X_0; \mu_{N(I)}]\prod_{i\in I}\frac{\mathbb{L}^{-\nu_i}T^{N_i}}{1-\mathbb{L}^{-\nu_i}T^{-}N_i} \in \mathcal{M}_{X_0}^{\hat{\mu}}[\![T]\!]}$$

and

$$\boxed{Z_f^{naive}(T) = \sum_{\emptyset\neq I\subseteq \mathrm{Irr}(E)} (\mathbb{L}-1)^{|I|}[E_I^0/X_0]\prod_{i\in I}\frac{\mathbb{L}^{-\nu_i}T^{N_i}}{1-\mathbb{L}^{-\nu_i}T^{-}N_i} \in \mathcal{M}_{X_0}[\![T]\!]}$$

Proof (ideas) Recall that we have

$$[\mathcal{X}_{n,1}] = \mathbb{L}^{(n+1)d}\mu_X\left(\pi_n^{-1}(\mathcal{X}_{n,1})\right) \in \mathcal{M}_k$$

where $\pi_n^{-1}(\mathcal{X}_{n,1}) \in \mathcal{L}_\infty(X)$ is a cylinder. The next step is to compute $\mu_X\left(\pi_n^{-1}(\mathcal{X}_{n,1})\right)$ using transformation rule:

$$\mu_X\left(\pi_n^{-1}(\mathcal{X}_{n,1})\right) = \int_{H^{-1}\pi_n^{-1}(\mathcal{X}_{n,1})} \mathbb{L}^{-\operatorname{ord}\mathcal{J}_H}\,\mathrm{d}\mu_Y$$

This can be computed using U_I constructed above: If we set $U_I(1)$ to be the fiber of U_I over 1 under the natural map $U_I \longrightarrow \mathbb{G}_m$ induced by the natural projection composed with $f \circ h$, then the covering map $\widetilde{E_I^0} \longrightarrow E_I^0$ factors through $U_I(1)$ and moreover, $U_I(1)$ is a $\mathbb{G}_m^{|I|-1}$-bundle over $\widetilde{E_I^0}$ — that's where the factor $(\mathbb{L}-1)^{|I|-1}$ comes.

Recall that $\mathcal{X}_n$ is a $\mathbb{G}_m$-bundle over $\mathcal{X}_{n,1}/\mu_n$. So under the map $\mathcal{M}_{X_0}^{\hat{\mu}} \longrightarrow \mathcal{M}_{X_0}$, $[\mathcal{X}_{n,1}, \mu_n]$ has image $(\mathbb{L}-1)^{-1}[\mathcal{X}_n]$ in $\mathcal{M}_{X_0}$. Hence the image of $(\mathbb{L}-1)Z_f(T)$ in $\mathcal{M}_{X_0}[\![T]\!]$ is $Z_f^{\text{naive}}(T)$. Meanwile, the image of $[\widetilde{E_I^0}/X_0; \mu_{N(I)}]$ in $\mathcal{M}_{X_0}$ is exactly $[E_I^0/X_0]$. That is why we have one more $(\mathbb{L}-1)$ in the explicit expression of $Z_f^{\text{naive}}(T)$.

See [DL02, Theorem 2.4] and [Loo02, Theorem 5.4].

6 The Topological Zeta Functions

Let $\mathcal{M}_{S,\text{loc}}$ resp. $\mathcal{M}_{S,\text{loc}}^{\hat{\mu}}$, be the ring obtained from $\mathcal{M}_S$, resp. $\mathcal{M}_S^{\hat{\mu}}$ by inverting the elements[5] $[\mathbb{P}_k^i \times_k S] = 1 + \mathbb{L} + \mathbb{L}^2 + \cdots + \mathbb{L}^i$ for all $i \in \mathbb{N}$.

Let $k = \mathbb{C}$. Evaluating $Z_f^{\text{naive}}(T)$ at $\mathbb{L}^{-s}$, and apply the topological Euler characteristic χ_{Top}, we get (recall $\chi_{\text{Top}}(\mathbb{P}_{\mathbb{C}}^n) = n+1$)

$$Z_{\text{Top}}(f, s) := \chi_{\text{Top}}\left(Z_f^{\text{naive}}(\mathbb{L}^{-s})\right) = \sum_{\emptyset \neq I \subseteq \text{Irr}(E)} \chi_{\text{Top}}(E_I^0) \prod_{i \in I} \frac{\chi_{\text{Top}}(X_0)}{sN_i + \nu_i} \in \mathbb{Q}(s)^6$$

This is called the untwisted topological zeta function of f.

Remark 6 There is also a version of twisted topological zeta function defined by evaluating $(\mathbb{L}-1)Z(T)$ at $T = \mathbb{L}^{-s}$ and applying the equivarianttopological Euler characteristic. See [DL01, §3.4].

References

[Bat98] Victor V. Batyrev. "Stringy Hodge numbers of varieties with Gorenstein canonical singularities." English. In: *Integrable systems and algebraic geometry. Proceedings of the 41st Taniguchi symposium, Kobe, Japan, June 30–July 4, 1997, and in Kyoto, Japan, July 7–11 1997.* Singapore: World Scientific, 1998, pp. 1–32 (cit. on pp. 1, 3).

[BG13] A. Bertapelle and C. D. Gonzalez-Aviles. "The Greenberg functor revisited". In: *ArXiv e-prints* (Oct. 2013). arXiv: `1311.0051 [math.NT]` (cit. on p. 4).

[Bli11] Manuel Blickle. "A short course on geometric motivic integration." English. In: *Motivic integration and its interactions with model theory and non-archimedean geometry. Vol. I.* Cambridge: Cambridge University Press, 2011, pp. 189–243 (cit. on pp. 3, 4, 6).

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models.* English. Berlin etc.: Springer-Verlag, 1990, pp. x + 325 (cit. on p. 4).

[CL08] Raf Cluckers and François Loeser. "Constructible motivic functions and motivic integration." English. In: *Invent. Math.* 173.1 (2008), pp. 23–121 (cit. on p. 1).

[DL01] Jan Denef and François Loeser. "Geometry on arc spaces of algebraic varieties." English. In: *3rd European congress of mathematics (ECM), Barcelona, Spain, July 10–14, 2000. Volume I.* Basel: Birkhäuser, 2001, pp. 327–348 (cit. on pp. 3, 5, 9).

[DL02] Jan Denef and François Loeser. "Lefschetz numbers of iterates of the monodromy and truncated arcs." English. In: *Topology* 41.5 (2002), pp. 1031–1040 (cit. on p. 9).

[DL99] Jan Denef and François Loeser. "Germs of arcs on singular algebraic varieties and motivic integration." English. In: *Invent. Math.* 135.1 (1999), pp. 201–232 (cit. on p. 1).

[Gre61] M.J. Greenberg. "Schemata over local rings." English. In: *Ann. Math. (2)* 73 (1961), pp. 624–648 (cit. on p. 4).

[5] in the origianl paper [Nic10], it was written $\mathbb{P}_k^i$, but this does not in the relative Grothendieck ring.

[6] the factor $\chi_{\text{Top}}(X_0)$ is missing in [DL01], Usually, one specialize the zeta function to a point $x \in X_0$, in which case, we do not have such a factor.

[Gre63] M. Greenberg. "Schemata over local rings. II." English. In: *Ann. Math. (2)* 78 (1963), pp. 256–266 (cit. on p. 4).

[HK06] Ehud Hrushovski and David Kazhdan. "Integration in valued fields." English. In: *Algebraic geometry and number theory. In Honor of Vladimir Drinfeld's 50th birthday.* Basel: Birkhäuser, 2006, pp. 261–405 (cit. on p. 1).

[Kon95] Maxim Kontsevich. *Lectures at Orsay.* Nov. 1995 (cit. on pp. 1, 3).

[Loo02] Eduard Looijenga. "Motivic measures." English. In: *Séminaire Bourbaki. Volume 1999/2000. Exposés 865–879.* Paris: Société Mathématique de France, 2002, 267–297, ex (cit. on pp. 2–7, 9).

[LS03] François Loeser and Julien Sebag. "Motivic integration on smooth rigid varieties and invariants of degenerations." English. In: *Duke Math. J.* 119.2 (2003), pp. 315–344 (cit. on p. 1).

[Nic10] Johannes Nicaise. "An introduction to p-adic and motivic zeta functions and the monodromy conjecture." English. In: *Algebraic and analytic aspects of zeta functions and L-functions. Lectures of the French-Japanese winter school, Miura, Japan, January 8–11, 2008.* Tokyo: Mathematical Society of Japan, 2010, pp. 141–166 (cit. on pp. 1, 5, 9).

[NS08] Johannes Nicaise and Julien Sebag. "Motivic Serre invariants and Weil restriction." English. In: *J. Algebra* 319.4 (2008), pp. 1585–1610.

[NS11] Johannes Nicaise and Julien Sebag. "Motivic invariants of rigid varieties, and applications to complex singularities." English. In: *Motivic integration and its interactions with model theory and non-archimedean geometry. Vol. I.* Cambridge: Cambridge University Press, 2011, pp. 244–304 (cit. on p. 4).

[Poo02] Bjorn Poonen. "The Grothendieck ring of varieties is not a domain." English. In: *Math. Res. Lett.* 9.4 (2002), pp. 493–497 (cit. on p. 2).

[Seb04] Julien Sebag. "Intégration motivique sur les schémas formels." French. In: *Bull. Soc. Math. Fr.* 132.1 (2004), pp. 1–54 (cit. on pp. 1, 4).

[Shi55] Goro Shimura. "Reduction of algebraic varieties with respect to a discrete valuation of the basic field." English. In: *Am. J. Math.* 77 (1955), pp. 134–176 (cit. on p. 4).

[Stacks] The Stacks Project Authors. *Stacks Project.* `http://stacks.math.columbia.edu`. 2015 (cit. on pp. 3, 4, 7).