Essential skeleton and relation to birational geometry
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The main reference is [MN13], §4.5, 4.6, 6.1, 6.3, 6.4

1. Kontsevich-Soibelman skeleton

X - smooth, conn., sep. K-schemes of dim n

ω - $\neq 0$ rational m-pluricanonical form on X, $m>0$

i.e. a rational section $((U,s)/\sim)$ of $\omega_{X/K}^{\otimes m}$

Recall: a rational section of a line bundle gives a Cartier divisor

Recall: $wt_\omega(x) = \begin{cases} \frac{\mu}{N}, \text{ if } x \in X^{div}, & \text{for } \mu - m = \text{mult. of } E \text{ in } div_{\mathfrak{X}}(\omega), \ N = \text{mult of } E \text{ in } \mathfrak{X}_k \\ v_x(div_{\mathfrak{X}}(\omega) + m(\mathfrak{X}_k)_{red}) & \text{for } x \in Sk(\mathfrak{X}) \\ \sup_{\substack{\mathfrak{X}\text{ - sncd models} \\ \text{s.t. } x \in \mathfrak{X}_\eta}} \{wt_\omega(\rho_{\mathfrak{X}}(x))\} \in \mathbb{R} \cup \{+\infty\} & \text{for } x \in X^{an} \end{cases}$

where $\rho_{\mathfrak{X}} : \hat{X}_\eta \longrightarrow Sk(\mathfrak{X})$

Def. (weight of X) $wt_\omega(X) := \inf\{wt_\omega(x) \mid x \in X^{div}\} \in \mathbb{R} \cup \{+\infty\}$

Def. $x \in X^{an}$ is ω-essential if $wt_\omega(x) = wt_\omega(X)$

Def. Kontsevich-Soibelman skeleton of (X,ω)

$Sk(X,\omega) := \{\omega\text{-essential points}\}$ X^{bir}

Prop. The skeleton $Sk(X,\omega)$ is birat. invariant.: if $h: Y \to X$ birational morphism of connected, smooth K-schemes, then $h^{an}: Y^{an} \to X^{an}$ induces a homeo between $Sk(Y, h^*\omega)$ and $Sk(X,\omega)$.

Pf: Enough for $Y = U \hookrightarrow X$.

But

$U^{Bir} = X^{Bir}$ ✓

$U^{Div} = X^{Div}$ ⟸ Because we can glue any model $\mathcal{U}$ of U to X along $\mathcal{U}_K$, obtaining a model of X.

and $wt_{h^*\omega}(y) = wt_\omega(h(y))$ for h - an open immersion (Fei's talk)

Assume now X - proper, $\mathfrak{X}$ - proper sncd-model, and ω is regular on whole X!

We will give an description of $Sk(X,\omega)$ in terms of $\mathfrak{X}$

Write $\mathfrak{X}_k = \sum_{i \in I} N_i E_i$

denote as usual $\mu_i - m :=$ mult. of E_i in $div_{\mathfrak{X}}(\omega)$.

Def. $\emptyset \neq J \subset I$, $\xi :=$ a generic point of $\bigcap_{j \in J} E_j$

ξ is ω-essential if $\forall_j \frac{\mu_j}{N_j} = \min\{\frac{\mu_i}{N_i} \mid i \in I\}$ and ξ is not in the zero locus of ω on X

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Thm. With the above assumptions:

• $wt_\omega(X) \underset{(\ast)}{=} \min\{wt_\omega(x) \mid x \in X^{div}\} \underset{(\ast\ast)}{=} \min\{wt_\omega(x) \mid x \in X^{an}\} \underset{(\ast)}{=} \min\{\frac{\mu_i}{N_i} \mid i \in I\}$

•• $Sk(X,\omega) \underset{(\ast\ast)}{=} \{x \in X^{an} \mid wt_\omega(x) = wt_\omega(X)\}$

∴ $Sk(X,\omega) \underset{(\ast\ast\ast)}{=} \bigcup$ open faces in $Sk(\mathcal{X})$ corresponding to the ω-essential points of $\mathcal{X}_k$ $\subset Sk(\mathcal{X})$

In particular does not depend on $\mathcal{X}$ (only on X, ω)

It is a non-empty compact subspace of $\mathcal{X}$.

~~Corollary~~

Pf: Observe that such a union is compact as if ~~some~~ open face belongs to it, also its ~~open~~ open faces of its boundaries belong and so on. This is because if some $J \subset I$ gives a ω-essential point of $\mathcal{X}_k$, then any subset $J' \subset J$ gives an ω-essential point as well.

~~proof~~ It is clear that

$$\min\{wt_\omega(x) \mid x \in X^{an}\} = \min\{wt_\omega(y) \mid y \in Sk(\mathcal{Y}),\ \mathcal{Y}\text{-snc model},\ y \in \mathcal{Y}_\eta\}$$

As X is proper, then $\hat{\mathcal{X}}_\eta = X^{an}$

so for any $x \in X^{an}$ $wt_\omega(x) \geq wt_\omega(\rho_{\mathcal{X}}(x))$ ← [4.4.5, MN13], we use the fact that ω is regular here

so we can only look at $x \in Sk(\mathcal{X})$

Here $wt_\omega(x) = v_x(div_{\mathcal{X}}(\omega) + m(\mathcal{X}_k)_{red})$

$$\geq \sum_{i \in I} \mu_i v_x(E_i) \geq \min\{\tfrac{\mu_i}{N_i} \mid i \in I\}$$

thanks to the assumption that ω is regular on whole X!

Explanation: $div_{\mathcal{X}}(\omega) = \sum N_i E_i$ + "closure of $div_X(\omega)$"

ω-regular $\Rightarrow v_x$("closure of $div_X(\omega)$") $\geq$ basically by definition of monomial pts

On the other hand, if x corresponds to an ω-essential point, the the two inequalities are equalities

~~(That~~ (That the first inequality is equality in this case was stated in Fei's talk. ~~The second~~ ~~is just~~ We use the fact that ξ in the definition of ω-essential point is not in the zero locus of ω.

~~The~~ To see that the second inequality becomes equality is just a computation from the definition)

The above computation finishes the proof as $\sum \mu_i v_x(E_i) =$ ~~$\sum \mu_i N_i$~~

$$= \sum \frac{\mu_i}{N_i} N_i v_x(E_i) = \sum \frac{\mu_i}{N_i} N_i \alpha_i \geq \min\{\tfrac{\mu_i}{N_i} \mid i \in I\} \sum_i N_i \alpha_i$$

$$= \min\{\tfrac{\mu_i}{N_i} \mid i \in I\}$$

And the inequality is equality precisely for ω-~~maximal~~ essential points.

Ex. X - smooth, prop. $/K$. Assume $K_{X/K} = 0$ and $\mathcal{X}$ is a proper sncd model, such that $\omega_{\mathcal{X}/R}$ is trivial.

Then $Sk(X) = \bigcup$ closed faces of $Sk(\mathcal{X})$ corresp. to irred. components of $\mathcal{X}_k$ of maximal multiplicity

In particular, if $\mathcal{X}_k$ is reduced, $Sk(X_\eta) = Sk(\mathcal{X})$

Ex. It was important to assume that ω has no poles.
For example, let $X = \mathbb{P}^1_K = \mathrm{Proj}\, K[x,y]$ and $\omega = d\left(\frac{x}{y}\right)$.
Then we see that $\mathrm{wt}_\omega(X) = -\infty$ and $Sk(X,\omega) = \emptyset$.
Indeed, take as a model $\mathfrak{X} = \mathbb{P}^1_R$ and blow up a point corresponding to $y^n = 0$ at $\mathfrak{X}_k$, and denote this blow-up by $\mathcal{Y}_n$.
Then $\mathrm{wt}_\omega(y_n) = v_{y_n}(\mathrm{div}_{\mathfrak{X}}(\omega) + (\mathfrak{X}_k)_{red}) = -2n + n = -n$,
where y_n is the divisorial pt corresponding to the exceptional divisor on $\mathcal{Y}_n$

2. The essential skeleton

X - smooth, connected, proper K-variety

Def.
$$Sk(X) := \bigcup_{\substack{\omega \text{ - non-zero, regular,} \\ \text{pluricanonical forms on } X}} Sk(X,\omega) \subset X^{an}$$
"the essential skeleton"

Prop. The essential skeleton is a birational invariant.

Pf: ω's are the same on both spaces
$Sk(X,\omega)$'s are birat. invariants

ok

$Sk(\mathfrak{X})$ are highly dependent on $\mathfrak{X}$. $Sk(X)$ is an answer.

Rmk. Thm. 5.3.4 of [MN 13] ~~gives~~ is a hard theorem telling that in some cases $Sk(X,\omega)$ is connected.
We don't have time for that, however.

We now explain the relation of above theory with birational geometry. We only give a sketch.

3. The weight function of a coherent ideal sheaf

F - field, char $F = 0$
X - connected, smooth F-variety
$\mathcal{I}$ - non-zero coherent ideal sheaf on X
v - divisorial valuation on $F(X)$ s.t. has a center on X and this center $\subset Z(\mathcal{I})$
Then $\exists\, h: Y \to X$ a log resol. of $\mathcal{I}$ s.t. the closure of the center of v on Y is a divisor E. We can and will assume that h is iso ~~outside~~ $\mathcal{I}$ over $X \setminus Z(\mathcal{I})$ (we will always choose log resolutions in this way)

(Def. center of v = $\{x \in X \mid \mathcal{O}_{X,x} \subset \mathcal{O}_v\}$, $\mathcal{O}_v := \{x \in F(X) \mid v(x) \geq 0\}$
It is an irreducible closed subset of X or it is empty)

Then $v = r \cdot \mathrm{ord}_E$

Denote $N := $ mult. of E in $Z(\mathcal{I}\mathcal{O}_Y)$, $\mu - 1 := $ mult. of E in $K_{Y/X}$

$\mathrm{wt}_{\mathcal{I}}(v) := \frac{\mu}{N}$ "weight of v w.r.t. X."

$\min\limits_{v \text{ as above}} (\mathrm{wt}_{\mathcal{I}}(v)) = \mathrm{lct}(X,\mathcal{I})$ $\leftarrow$ log canonical treshold

Fact: lct can be computed with a single resolution: $Y \to X$ any log resolution of $\mathcal{I}$, then if $Z(\mathcal{I}\mathcal{O}_Y) = \sum_{i\in I} N_i E_i$ and $K_{Y/X} = \sum_{i \in I} (\mu_i - 1) E_i$, then $\mathrm{lct} = \min\{\frac{\mu_i}{N_i} \mid i \in I\}$

Def: (not really important) "quasi-monomial valuation with center in $Z(\mathcal{I})$"
$\pi : Y \to X$ birational, Y - regular and connected
$y = (y_1, \dots, y_r)$ is a syst. of alg. coords at a point $\eta \in Y$
To every $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r_{\geq 0}$ one can associate a valuation on $F(X)$:
if $f \in \mathcal{O}_{Y,\eta}$ is written in $\widehat{\mathcal{O}_{Y,\eta}}$ as $f = \sum_{\beta \in \mathbb{Z}^r} c_\beta y^\beta$, with each $c_\beta \in \widehat{\mathcal{O}_{Y,\eta}}$ either zero or a unit, then
$$\mathrm{val}_\alpha(f) = \min\{\langle \alpha, \beta\rangle \mid c_\beta \neq 0\}$$

If v is a quasi-monomial val., then we set
$$wt_{\mathcal{I}}(v) = \frac{v(K_{Y/X} + Z(\mathcal{I}\mathcal{O}_Y))}{v(Z(\mathcal{I}\mathcal{O}_Y))},$$
where $h : Y \to X$ is a log resolution of $\mathcal{I}$ s.t. $(Y, Z(\mathcal{I}\mathcal{O}_Y)_{red})$ is <u>adapted to v</u>, i.e.
v can be described at some point $\eta \in Y$ with respect to coords $y_1, \dots, y_r$ such that each y_i defines at η an irreducible component of $Z(\mathcal{I}\mathcal{O}_Y)_{red}$

and where, for every effective divisor D on Y, we write
$v(D) = \min\{v(f) \mid f \in \mathcal{O}(-D)_\xi\}$
with ξ the center of v on Y.

<u>The</u> defn does not depend on the choice of h.

Y - sch. of f.t. $/X$
$\hat{Y} :=$ the formal $\mathcal{I}\mathcal{O}_Y$-adic completion of Y.

There is "$\widehat{X}_\eta$" (see [Th07, 1.7])
analytic space $/F$, F with trivial abs. val.

$\widehat{X}_\eta = $ (generic fiber of $\hat{X}$) $\setminus$ points that lie on the analyt. of $Z(\mathcal{I}) \subset \hat{X}$

- $\mathrm{red}_{\hat{X}} : \hat{X}_\eta \to Z(\mathcal{I})$

- $\left\{\begin{array}{l}\text{quasi-monomial valuations} \\ \text{on } X \text{ with center in } Z(\mathcal{I})\end{array}\right\} \subset \hat{X}_\eta$ "quasi-monomial points"

- each log resol. $Y \to X$ of $\mathcal{I}$ gives rise to $Sk(\hat{Y}) \subset \hat{X}_\eta$
$=$ quasi-monomial pts such that Y is adapted to v_x

- contraction $\rho_Y : \hat{X}_\eta \to Sk(\hat{Y})$ that can be extended to a strong deform.

- $wt_{\mathcal{I}}(x) \geq wt_{\mathcal{I}}(\rho_Y(x))$ for every quasi-monomial pt x on $\hat{X}_\eta$

- define $wt_{\mathcal{I}} : \hat{X}_\eta \to \mathbb{R} \cup \{+\infty\}$, $x \mapsto wt_{\mathcal{I}}(x) = \sup_{\substack{h : Y \to X \\ \text{log resol. of } \mathcal{I}}} wt_{\mathcal{I}}(\rho_Y(x))$

4. Comparision of the weight functions (we only give an overview of what is true)

$R := F[[t]]$, $K := F((t))$

X - conn., sm. <u>F - variety</u> of dim $n+1$, $\mathcal{I}$ - coherent id. sh. on X.

$$Z(\mathcal{I}\mathcal{O}_Y) = \sum_{i\in I} N_i E_i, \quad K_{Y/X} = \sum_{i\in I} (\mu_i - 1) E_i$$

Let $\xi \in Z(\mathcal{I}\mathcal{O}_Y)$. Locally at ξ (say in $U \ni \xi$) $\mathcal{I}\mathcal{O}_U$ is gen. by a regular function f. (by a defn of log resolution of $(X,\mathcal{I})$ $\mathcal{I}\mathcal{O}_Y$ defines a divisor)

$$f : U \longrightarrow \mathbb{A}^1_F = \operatorname{Spec} F[t]$$

define $\mathcal{U}$ by

$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d] & U \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \operatorname{Spec} F[t] \end{tikzcd}$$

$\mathcal{U}$ is an sncd model for $\mathcal{U}_K$

Choose some volume form $\phi \in \Gamma(V, \Omega^{n+1}_{X/k})$, V - a nbhd of $h(\xi)$
Shrink U and get $U \subset h^{-1}(V)$ and $U \setminus Z(f)$ smooth over V

~~ϕ on V induces~~

$h^*\phi|_{U\setminus Z(f)}$ induces a volume form $\omega \in \Omega^n_{U/\mathbb{A}^1}(U \setminus Z(f))$
~~(locally due to~~ that is unique that satisfy $\omega \wedge df = h^*\phi$ in $\Omega^{n+1}_{U/\mathbb{A}}(U \setminus Z(f))$

It induces a volume form $\omega \in \Omega^n_{\mathcal{U}_K}$ (some open)

(*) we have $\operatorname{div}_{\mathcal{U}}(\omega) = \sum_{i\in I} (\mu_i - 1)(E_i \cap U)$ (see [NS07, thm. 9.6])

$\hat{\mathcal{U}}_\eta$ can be identified with the subspace of $\hat{O}_\eta := \operatorname{red}_\varphi^{-1}(U \cap Z(\mathcal{I}\mathcal{O}_Y)) \subset \hat{X}_\eta$

This embedding has an retraction

$$\begin{tikzcd} \hat{\mathcal{U}}_\eta \arrow[r, phantom, "\subset"] & \hat{U}_\eta \arrow[l, bend left, "r_{\mathcal{U}}"] \end{tikzcd}$$

given by: $r_{\mathcal{U}}(x)$ is the unique point in $\operatorname{red}_{\hat{\mathcal{U}}}^{-1}(\operatorname{red}_\varphi(x))$ such that $\forall g \in \mathcal{O}_{Y,\operatorname{red}_\varphi(x)}$ $|g(r_{\mathcal{U}}(x))| = |g(x)|^{1/\ln|f(x)|}$

One can prove using (*) that

$$wt_{\mathcal{I}}|_{O_\eta} = wt_\omega \circ r_{\mathcal{U}},$$

where wt_ω is associated to $(\mathcal{U}_K, \omega)$

References: [MN13] Mustata, Nicaise Weight functions on non-archim. ~
[Th07] A. Thuillier. Geometrie toroidale et geometrie analytique
[NS07] Nicaise, Sebag The motivic Serre invariant