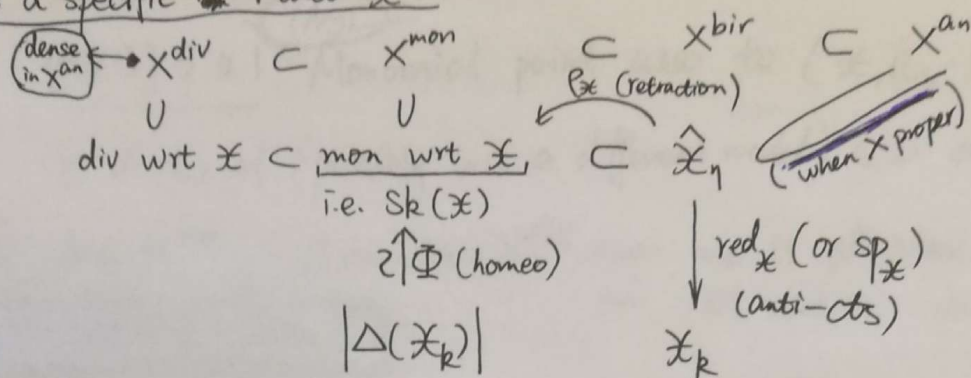


Weight function

Recall ① $X^{\text{div}} \subset X^{\text{mon}} \subset X^{\text{bir}} \subset X^{\text{an}}$

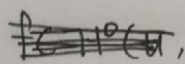
wrt a specific \mathbb{R} -model \mathbb{X} :



We don't really use \mathbb{R} -model.

② div/mon point wrt some data.

divisorial point (x, v_x) represented by (\mathbb{X}, E) :



the birational point x on X^{an} s.t. v_x ~~normalized~~ on $K(X)$ with valuation ring $\mathcal{O}_{x,3}$, normalized to make $v_x|_K = v_K$. i.e.,

[MN13] (2.4.2) $\sum N_i \alpha_i = 1 \Rightarrow \chi(x) = 1$

$\forall f \in K(X), v_x(f) = \frac{1}{N} \text{ord}_E f$

N : multiplicity of E in $\text{div } \mathbb{X}_k$ on \mathbb{X}
 $\text{ord}_E f$: order of f along E .

monoidal point (x, v_x) rep by $(\mathbb{X}, (E_1, \dots, E_r), (\alpha_1, \dots, \alpha_r), \mathbb{Z})$

\mathbb{X} (sncd) \uparrow some of the irr comp of \mathbb{X}_k with multiplicity N_1, \dots, N_r in $\text{div } \mathbb{X}_k$ on \mathbb{X}
 $(\alpha_1, \dots, \alpha_r) \in \mathbb{R}_{>0}^r$ s.t. $\sum_{i=1}^r \alpha_i N_i = 1$

prop 2.3.3 in [Survey] $\exists!$ minimal real valuation

$v: \mathcal{O}_{x,3} \setminus \{0\} \rightarrow \mathbb{R}^+$

s.t. $v(T_i) = \alpha_i, i=1, \dots, r$

where T_i is the local defining equation of E_i

(Here we use "sncd" assumption.)

some generic point of $\bigcap_{i=1}^r E_i$
 (2.6) WTC confirm 不一定余维 1 例如 $\{x=0\} \cap \{y=0\} \cap \{z=0\}$ 交出一点, 对于 \mathbb{X} 也余维 2 维

req \Rightarrow normal 故需 DVR 缺一维' 条件.

Construction 1 Every div pt has a div rep wrt some sncd X .

Facts you need to know

means precisely $(X, E) \left(\sum \alpha_i E_i \in \mathbb{Q} \right)$

1) Such a representing data is not unique; $\overbrace{\text{div wrt } X}^{(2.4.1)} \Rightarrow \text{mon wrt } X$,
 but ~~div point~~ $\text{mon wrt } X$ and not $\text{div wrt } X \not\Rightarrow$ not div at all

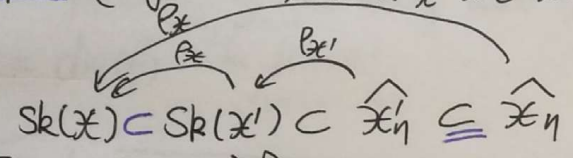
(Any)
 [MN13] 2.4.1 Monomial point asso to $(X, (E_i, -E_i), (\alpha_i, -\alpha_i), \mathbb{Z})$
 is divisorial (possibly wrt a different model) \Leftrightarrow all $\alpha_i \in \mathbb{Q}$.

2) $X \in X^{\text{mon}}$. Then $X \in X^{\text{div}} \Leftrightarrow \forall X$ is discrete [MN13] (2.4.11)

③ **Used several times!** \Leftrightarrow rat'l rank 1 $\langle \alpha_1, \dots, \alpha_r \rangle_{\mathbb{Q}} \subset \mathbb{R}$ is 1-dim'l

[MN13] 3.1.7 $h: X' \rightarrow X$ R -morphism of sncd-models of X .

- a) $\widehat{X}'_h \subset \widehat{X}_h$, $(P_{X'} \circ P_{X'})^{-1}(X) = P_X^{-1}(X)$, $\forall X \in \widehat{X}'_h$
- b) If h proper (eg. when X' and X are both proper R -~~models~~)



~~proper models and proper models~~
 for proper h

③.5 Although $\text{Sk}(X) \subset \text{Sk}(X')$ can be strict inclusion in general, it performs good for blow-ups:

[MN13] 3.1.9 $h: X' \rightarrow X$ be the blowup of X at the closure of $\{z\}$

some generic point of $\bigcap_{i=1}^r E_i$

Then $\text{Sk}(X') = \text{Sk}(X)$.

We can also build strict inclusion: **Construction 2**

④ Under the assumption: char $k = 0$ or X is a curve

smooth compactification of X

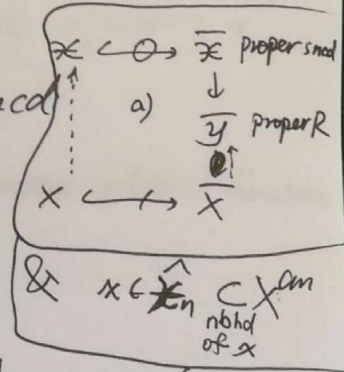
Construction 1 [MN13] 4.4.2 $X \in X^{\text{an}}$. Then every proper R -model \overline{Y} of \overline{X} can be dominated by a proper sncd-model \overline{X} s.t.

a) \exists open subscheme $X \hookrightarrow \overline{X}$ s.t. X/X sncd

b) \widehat{X}'_h is a nbhd of X in X^{an} .
 If $X \in X^{\text{an}}$ can moreover arrange $X \in \text{Sk}(X)$.

Application: X and its sncd model can be "simultaneously compactified" so that we can apply § 4.3 [MN13].

Seth similar: X, X' both proper R -model \hookrightarrow can find one proper R s.t. dominate both (See 2.3.1 [MN13])



4.2 Weight function on X^{div}

X conn, sm, sep K -sch, $\dim n$
 $w \in H^0(U, \omega_{X/K}^{\otimes m})$ rational section.

Definition $x \in X^{\text{div}}$. Define weight function

sncd-tuple

$$x = (\mathcal{X}, (E_i, -E_i), (\alpha_i, -i\alpha_i), \xi)$$

$$wt_w(x) = v_x(\text{div}_x(w) + m(\mathcal{X}_k)_{\text{red}}) \in \mathbb{Q}$$

Notations

$\mu_i = m +$ multiplicity of $\text{div}_x(w)$ along E_i

$N =$ multiplicity of E in \mathcal{X}_k as a divisor on \mathcal{X} .

For $D \in \text{Div}(\mathcal{X})$ with $|D| \not\ni x$ ($i: X^{\text{an}} \rightarrow X$)

$$v_x(D) = -\ln|f(x)|$$

where $f \in K(X)^{\times}$ is the thing that locally at $\text{sp}_x(x) \in \mathcal{X}_k \subset \mathcal{X}$,

$D = \text{div}(f)$.
only rational
loc. principal
Cartier div

Theorem

(4.2.7) $\{ \xi \notin \{\text{zeros and poles of } w \text{ on } X\} \}$, then

Q15 unsolved

(4.2.7) If $x = (\mathcal{X}, E)$ for some regular R -model, $\sum_{i=1}^r \alpha_i \mu_i$

$$2) \quad wt_w(x) = \frac{m + \text{multiplicity of } E \text{ in } \text{div}_x(w)}{\text{multiplicity of } E \text{ in } \mathcal{X}_k}$$

~~...~~

3) (4.2.4) [MV13] $wt_w(x)$ ~~...~~ does not depend on the choice of regular R -model \mathcal{X} [Lei's talk]

3') $wt_w(x)$ ^{in Def} does not depend on the choice of the sncd-tuple representing it. (see §4.3 (4.3.4) 5)

4) Two formulas:

[Lei proved this]

$$\begin{cases} wt_{w^d}(x) = d \cdot wt_w(x), \\ wt_{fw}(x) = wt_w(x) + v_x(f) \end{cases}$$

$d > 0$

f nonzero rational function

(4.2.8) Moreover we can say,

5) y/X sncd-model. $y \in X^{\text{div}} \cap \widehat{\mathcal{Y}}_y$. Then

$$wt_w(y) \geq v_y(\text{div}_y(w) + m(\mathcal{Y}_k)_{\text{red}})$$

and equality holds iff $y \in \text{Sk}(\mathcal{Y})$

["Wrong model always underestimate the weight"]

4.3 Weight function on Berkovich skeleton.

X (moreover) proper

ω

Suppose X has a proper sncd-model \mathcal{X} .

(Q9) 待补充.

Definition $x \in \text{Sk}(\mathcal{X})$. Define

$$wt_{\mathcal{X}, \omega}(x) = v_x(\text{div}(\omega) + m(\mathcal{X}_k)_{\text{red}})$$

Note I jump over (4.3.3). I don't feel we use it. Too less time.

Theorem (2.2.4) 1) The function

$$wt_{\omega, \mathcal{X}} : \text{Sk}(\mathcal{X}) \longrightarrow \mathbb{R}$$

is continuous.

(4.3.3) says more: it's piecewise affine and ...

2) If $x \in X^{\text{div}} \cap \text{Sk}(\mathcal{X})$ then $wt_{\mathcal{X}, \omega}(x) = wt_{\omega}(x)$. [(4.2.7)]

3) If $x \in X^{\text{div}} \cap \widehat{X}_\eta$, and $\text{sp}_{\mathcal{X}}(x) \notin \overline{\{\text{poles of } \omega \text{ on } X\}}^{\mathcal{X}}$, then

Equality holds iff $x \in \text{Sk}(\mathcal{X})$. $\Rightarrow \text{supp}(\text{div}(\omega) + m(\mathcal{X}_k)_{\text{red}}) \not\ni x$ and $\text{sp}_{\mathcal{X}}(x)$
 [Wrong model always under estimate the weight for divisorial points"]

4) If $y \in X^{\text{mon}} \cap \widehat{X}_\eta$, and $\text{sp}_{\mathcal{X}}(x) \notin \overline{\{\text{poles of } \omega \text{ on } X\}}^{\mathcal{X}}$, then

Suppose $y \in \text{Sk}(\mathcal{Y}) \cap \widehat{X}_\eta$ for some proper sncd-model \mathcal{Y} of X
 $wt_{\mathcal{Y}, \omega}(y) \geq wt_{\mathcal{X}, \omega}(\text{sp}_{\mathcal{X}}(y))$

Equality can only occur if $y \in \text{Sk}(\mathcal{X})$ [$> \Rightarrow y \notin \text{Sk}(\mathcal{X})$]

"Underestimate the weight"
 [Wrong model ~~X~~ in most ^{proper} can only happen for a wrong model.]

5) \mathcal{X}, \mathcal{Y} two sncd-models, $x \in \text{Sk}(\mathcal{X}) \cap \text{Sk}(\mathcal{Y})$. Then

$$wt_{\mathcal{X}, \omega}(x) = wt_{\mathcal{Y}, \omega}(x).$$

6) [MV13] (4.4.3) w/o "proper" requirement for \mathcal{X}, \mathcal{Y} , 5) still holds.
 i.e. $\exists!$ function $wt_{\omega} : \text{Mon}(X) \rightarrow \mathbb{Q}$ s.t.

$$wt_{\omega}(x) = v_x(\text{div}_{\mathcal{X}}(\omega) + m(\mathcal{X}_k)_{\text{red}})$$

\forall sncd \mathcal{X}/X s.t. $x \in \text{Sk}(\mathcal{X})$.

重点研究 proper 到底用在哪

(Q12) unsolved

(Q10) $X^{\text{div}} \subset \widehat{X}_\eta$
 \forall sncd \mathcal{X} ?

? Every mon pt is in some $\text{Sk}(\mathcal{Y})$ for some proper \mathcal{Y} ?

For char=0 or X a curve

4.4 The weight function on X^{an}

X conn sm K -sch, $\dim n$ (drop "properness!")

Assume $\text{char } k = 0$ or X is a curve.

Definition

$x \in X^{\text{an}}$. Define $\text{wt}_w(x) = \sup_{\substack{\text{sncd } \mathbb{X}/X \\ \text{s.t. } x \in \widehat{\mathbb{X}}_\eta}} \{ \text{wt}_w(\overbrace{P_{\mathbb{X}}(x)}^{eSk(\mathbb{X})}) \} \in \mathbb{R} \cup \{+\infty\}$.

Theorem (4.4.5)

1) The function

$$\text{wt}_w : X^{\text{an}} \longrightarrow \mathbb{R} \cup \{+\infty\}$$

is lower semi-cts

2) This def agrees with the ones in §4.3 / §4.2 for mon/div points resp. [(4.4.3)]

3) For $x \in \widehat{\mathbb{X}}_\eta$ where \mathbb{X}/X sncd,

$$\text{wt}_w(x) \geq \text{wt}_w(P_{\mathbb{X}}(x))$$

Equality holds iff $x \in eSk(\mathbb{X})$

(4) $h: Y \hookrightarrow X$ open immersion. Then $\forall y \in Y^{\text{an}}$,

$$\text{wt}_{h^*w}(y) = \text{wt}_w(h(y))$$

(5) Two formulas

$$\text{wt}_{w \otimes d}(x) = d \cdot \text{wt}_w(x) \quad d > 0$$

$$\text{wt}_{fw}(x) = \text{wt}_w(x) + v_x(f) \quad f \neq 0 \text{ regular function. } \textcircled{Q14}$$

Rmk 4.4.6 1) can be non-cts. Example:

最后(若有空) 展示: $\frac{k[x,y]}{x^{n_1}y^{n_2}-z}$ 的 weight.

§4.2 proof of the theorem

$$\overline{\{ \}} \subset \bigcap_{i=1}^r E_i \subset X_k$$

1) By definition of the symbols

$$\text{div}_X(w) = \sum_{i=1}^r (u_i - m) E_i \quad \boxed{\text{无条件成立}}$$

$$\begin{aligned} \Rightarrow v_X(\text{div}_X(w) + m(X_k)_{\text{red}}) & \quad \text{(Q15) 哪里用到条件 及条件什么意思 ref [survey] 2.3.3} \\ &= v_X(\text{div}_X(w)) + v_X(m(X_k)_{\text{red}}) = \sum (u_i - m) \alpha_i + \sum m \alpha_i = \sum u_i \alpha_i \end{aligned}$$

2) $X = (X, (E_1, \dots, E_r), (\alpha_1, \dots, \alpha_r), \xi)$ sncd-tuple

$$\begin{aligned} \underline{r=1} \quad \text{wt}_w(X) &= v_X(\text{div}_X(w) + m(X_k)_{\text{red}}) \\ &\stackrel{\text{Recall } \textcircled{e}}{=} \frac{\text{ord}_E(\text{div}_X(w) + m(X_k)_{\text{red}})}{\text{multiplicity of } E \text{ in } \text{div } X_k \text{ on } X} \\ &= \frac{\text{ord}_E(\text{div}_X(w)) + m}{N} = \frac{u}{N} \quad \checkmark \end{aligned}$$

$r > 1$ We generate a process to reduce to $r=1$.

In ^(any) the monoidal rep of X , (2.4.11) $\Rightarrow \alpha_i \in \mathbb{Q}, i=1, \dots, r$.

We repeat the process in the proof of (2.4.11): "Every div pt must have a div rep wrt a sncd-model"

Construction 3

Permuting the indices, may assume $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_r$.

$h: X' \rightarrow X$ be the blow-up at the closure of ξ .

Denote by E_i' the strict transform of $E_i, i=2, \dots, r$ and E_1' the excep div.

Let ξ' be ^(Q17) the generic point of $\bigcap_{i=1}^r E_i'$. Then

Straightforward computation $\Rightarrow (X', (E_1', \dots, E_r'), \xi')$ is an sncd tuple
 $\& (X', (E_1', \dots, E_r'), (\alpha_1, \alpha_2 - \alpha_1, \dots, \alpha_r - \alpha_1), \xi')$ represents X .
 $:= \alpha'$

(2.4.9) \Rightarrow can eliminate zero entries in α' and adjust a bit for ξ' and X' unchanged \Rightarrow represent the same X .

(2.4.11): $X \in X^{\text{div}} \Leftrightarrow \langle \alpha_1, \dots, \alpha_r \rangle_{\mathbb{Q}} \subset \mathbb{R}$ is 1-dim'l

\Rightarrow After finitely many steps, we get a tuple with only one α_i , i.e. it looks like (X, E, α, ξ) . i.e. $r=1$

must be 1 \leftarrow must be the generic pt of E
 According to the rule of an sncd-tuple.

There's two repeated ~~process~~ operations in this process: blow-ups and eliminating zero α_i 's. WTS $wt_w(x)$ remains the same for these operations.

eliminating zero α_i 's $div_x(w)$ unchanged; $v_x(m(x_p)_{red}) = m \sum \alpha_i$
~~is~~ thus also unchanged.

Blow-ups WTS $v_x(div_x(w)) + m \sum_{i=1}^r \alpha_i = v_x(div_{x'}(w)) + m \sum_{i=1}^r \alpha_i'$

Note that the relative canonical divisor $K_{x'}/K_x = (r-1)E_1'$ (Q18)
 Hart II 8.5 b) also Prop 8.20

(4.2.2) gives $h^* div_x(w) = div_{x'}(w) - m(r-1)E_1'$ (Q19)

$$\Rightarrow v_x(div_{x'}(w)) + m \sum_{i=1}^r \alpha_i' = \underbrace{v_x(h^* div_x(w))}_{= v_x(div_x(w))} + m(r-1)\alpha_1 + m \sum_{i=1}^r \alpha_i'$$

(2.2.3), h proper, x', x regular. (Q20)

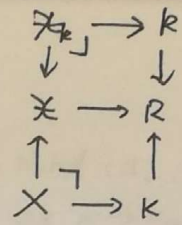
5) 略. 有时间加例子.

$$\{3\} \in \bigcap E_i$$

$$\uparrow r = \text{codim}\{3\} \text{ in } X$$

\therefore 是 simple normal crossing, 每加一个除子降一维

proof of §4.3



1) By (2.2.4), $\forall D \in \text{Div}(X)$,

$$\{x \in \hat{X}_\eta \mid i(x) \notin \text{Supp } D\} \rightarrow \mathbb{R}$$

$$x \mapsto v_x(D)$$

is obs.

NTS $\text{Sk}(X) \subset \{x \in \hat{X}_\eta \mid i(x) \notin \text{Supp } D\}$ (Q21) cf 2.2.3 & 10

$$C_X \otimes \mathbb{A}_k = \text{div}_X(\omega) + m(\mathbb{A}_k)_{\text{red}}$$

But this is obvious since $\text{Sk}(X)$.

3) 由 §4.2 Thm 5)

$$\begin{aligned}
 \text{wt}_\omega(x) &\geq v_x(\text{div}_X(\omega) + m(\mathbb{A}_k)_{\text{red}}) \\
 &= v_x\left(\sum_{i \in I} (m_i - m) E_i + m(\mathbb{A}_k)_{\text{red}}\right) \\
 &= v_{\mathbb{A}_k(x)}(\dots) \\
 &= \text{wt}_{X, \omega}(P_X(x))
 \end{aligned}$$

(3.1.5):
 $v_x(E_i) = v_{\mathbb{A}_k(x)}(E_i)$
 for every irr comp E_i
 of \mathbb{A}_k

4) 71 证. Denote

$$\begin{aligned}
 P &= \{x \in \mathbb{A}_k \mid x \in \{\text{poles of } \omega \text{ on } X\}\} \text{ closed} \\
 Z &= \{y' \in \text{Sk}(\mathcal{O}_y) \mid \text{sp}_{X'}(y') \notin P\} \subset \text{Sk}(\mathcal{O}_y) \cap \hat{X}_\eta
 \end{aligned}$$

↑ $\text{sp}_{X'}$
 closed ← anti-continuity of $\text{sp}_{X'}$

and we have $y \in Z$.

Claim w/o proof Divisorial pts are dense in Z .

5). Enough to prove $\text{wt}_{\mathcal{O}_y, \omega}(x) \geq \text{wt}_{X, \omega}(x)$.

Notice that by multiply a nonzero rational function f on X , we reduce to the case where $\text{sp}_{X'}(x) \notin \{\text{poles of } \omega \text{ on } X\}$

Then 5) follows from 4).

proof of §4.4

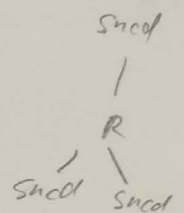
1) lower semicont at x_0 : If $f(x_0) < +\infty$: $\forall \varepsilon > 0, \exists U \ni x_0$ nbhd s.t. $f(x) \geq f(x_0) - \varepsilon, \forall x \in U$. If $f(x_0) = +\infty$ then $f(x) \rightarrow +\infty$ as $x \rightarrow x_0$.

Let $x \in X^{an}$. By 4.4.2 (See Recall ⑤), can find sncd \mathbb{X}/X s.t. $\hat{\mathbb{X}}_\eta \ni x$ is a nbhd. $\forall y \in \hat{\mathbb{X}}_\eta$

$$wt_\omega(y) = \sup_{\left\{ \mathbb{X}' \mid \begin{array}{l} \mathbb{X}' \rightarrow \mathbb{X} \text{ proper} \\ R\text{-mor of sncd's} \end{array} \right\}} \{ wt_\omega(P_{\mathbb{X}'}(y)) \} \in \mathbb{R} \cup \{+\infty\}.$$

与 X 无关, 再用 sup.

and $wt_{\omega, \mathbb{X}'} \geq wt_{\omega, \mathbb{X}}$ on Skeleton
and two sncd's can be dominated by a common sncd



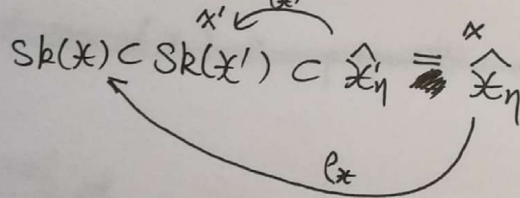
3) Inequality follows directly from def. Only need to show when equality holds.

§4.3 Thm 4) \Rightarrow if $y \in (\hat{\mathbb{X}}_\eta \setminus Sk(\mathbb{X})) \cap X^{mon}$, then

$$wt_\omega(y) > wt_\omega(P_{\mathbb{X}}(y))$$

So our aim is to "make X monoidal": Find a proper morphism

$\mathbb{X}' \rightarrow \mathbb{X}$ of sncd-models of X



s.t. $x' = P_{\mathbb{X}'}(x) \notin Sk(\mathbb{X})$. i.e. $x' \in (\hat{\mathbb{X}}_\eta \setminus Sk(\mathbb{X})) \cap X^{mon}$.

In this case, $wt_\omega(x) \geq wt_\omega(P_{\mathbb{X}'}(x)) > wt_\omega(P_{\mathbb{X}}(x')) = wt_\omega(P_{\mathbb{X}}(x))$

Telet
was at first

Construction 2 seems not depending on $char = 0$ or $X = \text{curve}$

4) Let \mathcal{Y}/\mathcal{Y} sncd s.t. $y \in \hat{\mathcal{Y}}_\eta$.

Glue $\mathcal{Y}_k (\cong \mathcal{Y})$ to X by the open immersion h

(Q22)? \leadsto get an sncd \mathcal{X}/X s.t. $h(y) \in \hat{\mathcal{X}}_\eta$.

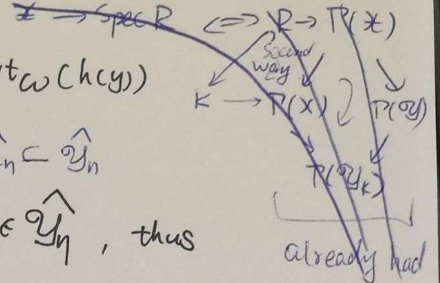
In this case

$$\mathcal{X} = X \cup_{\mathcal{Y}_k} \mathcal{Y} \quad \text{WTS } \mathcal{X} \text{ is a } R\text{-scheme}$$

glue in the cat of R -schemes
Note that $X \rightarrow \text{Spec } R \rightarrow \text{Spec } R$!

$$\text{wt}_{h^* \omega} (P_{\mathcal{Y}}(y)) \stackrel{\text{def of } \mathcal{X}}{=} \text{wt}_{\omega} (P_{\mathcal{X}}(h(y))) \leq \text{wt}_{\omega} (h(y))$$

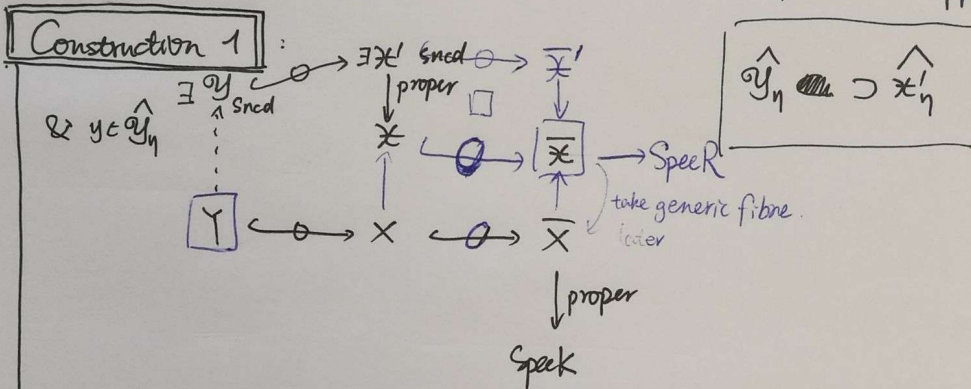
cf (Q22) $\mathcal{Y} \rightarrow \mathcal{X}$
 $\text{Sk}(\mathcal{Y}) \text{Sk}(\mathcal{X}) \subset \hat{\mathcal{X}}_\eta \subset \hat{\mathcal{Y}}_\eta$



This holds for all sncd-models \mathcal{Y}/\mathcal{Y} with $y \in \hat{\mathcal{Y}}_\eta$, thus

$$\text{wt}_{h^* \omega} (y) \leq \text{wt}_{\omega} (h(y))$$

Conversely, let \mathcal{X}/X sncd s.t. $h(y) \in \hat{\mathcal{X}}_\eta$. Then apply



(Q23)?

? \mathcal{X} (compactification of \mathcal{X}) is a proper R -model of \mathcal{X}

We have

$$\text{wt}_{\omega} (P_{\mathcal{X}}(P_{\mathcal{X}}(h(y)))) \stackrel{\text{def}}{\leq} \text{wt}_{\omega} (P_{\mathcal{X}'}(h(y))) = \text{wt}_{h^* \omega} (P_{\mathcal{Y}}(y)) \stackrel{\text{def}}{\leq} \text{wt}_{h^* \omega} (y).$$

def of \mathcal{X} & \mathcal{Y} . □

