Weight function

Recall ① $X^{div} \subset X^{mon} \subset X^{bir} \subset X^{an}$

wrt a specific snc/R-model $\mathfrak{X}$:

$$\begin{array}{ccccccc} X^{div} & \subset & X^{mon} & \subset & X^{bir} & \subset & X^{an} \\ \cup & & \cup & & & & \\ \text{div wrt } \mathfrak{X} & \subset & \underline{\text{mon wrt } \mathfrak{X},} & \subset & \hat{\mathfrak{X}}_\eta & & \end{array}$$

dense in X^{an}

$p_{\mathfrak{X}}$ (retraction)

(when X proper)

i.e. $Sk(\mathfrak{X})$

$\wr\Uparrow \Phi$ (homeo)

$|\Delta(\mathfrak{X}_k)|$

$\hat{\mathfrak{X}}_\eta \xrightarrow{red_{\mathfrak{X}} \text{ (or } sp_{\mathfrak{X}}) \text{ (anti-cts)}} \mathfrak{X}_k$

We don't really use R-model.

② div / mon point wrt some data.

divisorial point (x, v_x) represented by $(\mathfrak{X}, E)$:

normal R-model

irr comp of $\mathfrak{X}_k$

~~E~~,

the birational point x on X^{an} s.t. v_x ~~[illegible]~~ on $K(X)$ with valuation ring $\mathcal{O}_{\mathfrak{X},\xi}$ (DVR), normalized to make $v_x|_K = v_K$. i.e.

$\forall f \in k(X)$, $v_x(f) = \frac{1}{N} \text{ord}_E f$

N: multiplicity of E in div $\mathfrak{X}_k$ on $\mathfrak{X}$

$\text{ord}_E f$: order of f along E.

[MN13] (2.4.2)

$\pi = u z_1^{N_1} \cdots z_r^{N_r}$

$\sum N_i \alpha_i = 1$

$\Rightarrow v_x(\pi) = 1$

monoidal point (x, v_x) rep by $(\mathfrak{X}, (E_1, \dots, E_r), (\alpha_1, \dots, \alpha_r), \xi)$

snc

some of the irr comp of $\mathfrak{X}_k$ with multiplicity $N_1, \dots, N_r$ in div $\mathfrak{X}_k$ on $\mathfrak{X}$

$\in \mathbb{R}^r_{>0}$ s.t. $\sum_{i=1}^r \alpha_i N_i = 1$

some generic point of $\bigcap_{i=1}^r E_i$

~~Q16~~ WTConfirm

不一定余一维

例如 $\{x=0\}$ $\{y=0\}$, $\{z=0\}$ 交出一点, 对于 $\mathfrak{X}$ 也余两维

reg $\Rightarrow$ normal

故离 DVR 缺"一维"条件.

prop 2.3.3 in [Survey] $\exists!$ minimal real valuation

$$v: \mathcal{O}_{\mathfrak{X},\xi} \setminus \{0\} \longrightarrow \mathbb{R}^+$$

s.t. $v(T_i) = \alpha_i$, $i = 1, \dots, r$

where T_i is the local defining equation of E_i

(Here we use "snc" assumption.)

$\boxed{\text{Construction 1}}$ Every div pt has a div rep wrt some sncd $\mathfrak{X}$.

means precisely $(\mathfrak{X}, E)$ & all $\alpha_i \in \mathbb{Q}$

Facts you need to know

1) Such a representing data is not unique; $\overbrace{\text{div wrt } \underset{\substack{\uparrow \\ \text{sncd}}}{\mathfrak{X}} \overset{(2.4.1)}{\underset{(2.4.2)}{\Rightarrow}} \text{mon wrt } \mathfrak{X}}$ $\longrightarrow \mathfrak{X}$,

but ~~div point~~ mon wrt $\mathfrak{X}$ and not div wrt $\mathfrak{X}$ $\not\Rightarrow$ not div at all

"(Any)"

[MN13] 2.4.1 Monomial point asso to $(\mathfrak{X}, (E_1, \dots, E_r), (\alpha_1, \dots, \alpha_r), \xi)$

is divisorial (possibly wrt a different model) $\Leftrightarrow$ all $\alpha_i \in \mathbb{Q}$.

2) $x \in X^{mon}$. Then $x \in X^{div} \Leftrightarrow v_x$ is discrete [MN13] (2.4.11)

$\Leftrightarrow$ rat'l rank 1 $\langle \alpha_1, \dots, \alpha_n \rangle_{\mathbb{Q}} \subset \mathbb{R}$ is 1-dim'l

③ $\boxed{\text{Used several times!}}$

[MN13] 3.1.7 $h: \mathfrak{X}' \to \mathfrak{X}$ R-morphism of sncd-models of X.

a) $\widehat{\mathfrak{X}'_\eta} \subset \widehat{\mathfrak{X}_\eta}$, $(\rho_{\mathfrak{X}} \circ \rho_{\mathfrak{X}'})(x) = \rho_{\mathfrak{X}}(x)$, $\forall x \in \widehat{\mathfrak{X}'_\eta}$

b) If h proper (eg. when $\mathfrak{X}'$ and $\mathfrak{X}$ are both proper R-models [redacted])

$$\begin{tikzcd} Sk(\mathfrak{X}) \subset Sk(\mathfrak{X}') \arrow[l, bend right, "\rho_{\mathfrak{X}}"'] \subset \widehat{\mathfrak{X}'_\eta} \arrow[l, bend right, "\rho_{\mathfrak{X}'}"'] \subsetneq \widehat{\mathfrak{X}_\eta} \arrow[lll, bend right, "\rho_{\mathfrak{X}}"'] \end{tikzcd}$$

[redacted]

(3.5) Although $Sk(\mathfrak{X}) \subset Sk(\mathfrak{X}')$ can be strict inclusion for proper h in general,

it performs good for blow-ups:

[MN13] 3.1.9 $h: \mathfrak{X}' \to \mathfrak{X}$ be the blowup of $\mathfrak{X}$ at the closure of $\{\xi\}$ — some generic point of $\bigcap_{i=1}^{r} E_i$

Then $Sk(\mathfrak{X}') = Sk(\mathfrak{X})$.

We can also build strict inclusion: $\boxed{\text{Construction 2}}$

④ Under the assumption: char $k = 0$ or X is a curve

$\boxed{\text{Construction 1}}$ [MN13] 4.4.2 $x \in X^{an}$. Then every proper R-model $\overline{\mathcal{Y}}$ of $\overline{X}$ (smooth compactification of X) can be

dominated by a proper sncd-model $\overline{\mathfrak{X}}$ s.t.

a) $\exists$ open subscheme $\mathfrak{X} \hookrightarrow \overline{\mathfrak{X}}$ s.t. $\mathfrak{X}/X$ sncd

b) $\widehat{\mathfrak{X}_\eta}$ is a nbhd of x in X^{an}.

If $x \in X^{mon}$ can moreover arrange $x \in Sk(\mathfrak{X})$.

$$\begin{tikzcd} \mathfrak{X} \arrow[r, hook] \arrow[d, dashed, no head] \arrow[dr, phantom, "a)"] & \overline{\mathfrak{X}} \arrow[d] & \text{proper sncd} \\ & \overline{\mathcal{Y}} & \text{proper } R \\ X \arrow[u, dashed, no head] \arrow[r, hook] & \overline{X} \arrow[u] & \end{tikzcd}$$

& $x \in \widehat{\mathfrak{X}_\eta} \subset X^{an}$ nbhd of x

Application: X and its sncd model can be "simultaneously compactified" so that we can apply §4.3 [MN13].

Sth similar: $\mathfrak{X}, \mathfrak{X}'$ both proper $\underset{\text{sncd}}{R}$-model $\rightsquigarrow$ can find one proper $\underset{\text{sncd}}{R}$ s.t. dominate both (See 2.3.1 [MN13])

4.2 Weight function on X^{div}

X conn, sm, sep k-sch, dim n

$\omega \in H^0(U, \omega_{X/k}^{\otimes m})$ rational section.

Definition $x \in X^{div}$. Define weight function

sncd-tuple $\boxed{x = (\mathcal{X}, (E_1, \dots, E_r), (\alpha_1, \dots, \alpha_r), \xi)}$

$$wt_\omega(x) = v_x(\mathrm{div}_{\mathcal{X}}(\omega) + m(\mathcal{X}_k)_{red}) \quad \in \mathbb{Q}$$

Notations [redacted]

$\mu_i = m +$ multiplicity of $\mathrm{div}_{\mathcal{X}}(\omega)$ along E_i

$N =$ multiplicity of E in $\mathcal{X}_k$ as a divisor on $\mathcal{X}$.

For $D \in \mathrm{Div}(\mathcal{X})$ with $|D| \not\ni i(x)$ $\quad (i : X^{an} \to X)$

$$v_x(D) = -\ln|f(x)|$$

where $f \in K(X)^\times$ is the thing that locally at $sp_{\mathcal{X}}(x) \in \mathcal{X}_k \subset \mathcal{X}$,

Q6 $D = \mathrm{div}(f)$. Only rational loc. principal Cartier div

Theorem (4.2.7) 1) If $\xi \notin \{$zeros and poles of ω on $X\}$ Q15 unsolved, then

$$wt_\omega(x) = \sum_{i=1}^{r} \alpha_i \mu_i$$

(4.2.7) If $x = (\mathcal{X}, E)$ for some regular R-model,

2) $$wt_\omega(x) = \frac{m + \text{multiplicity of } E \text{ in } \mathrm{div}_{\mathcal{X}}(\omega)}{\text{multiplicity of } E \text{ in } \mathcal{X}_k}$$

[redacted]

3) (4.2.4) [MN13] $wt_\omega(x)$ [redacted] in 2) does not depends on the choice of regular R-model $\mathcal{X}$ [Lei's talk]

Q7 3)' $wt_\omega(x)$ in Def does not depend on the choice of the sncd-tuple representing it. (See §4.3 (4.3.4) 5))

4) (4.2.5) Two formulas: [Lei proved this]

$$\boxed{\begin{aligned} wt_{\omega^{\otimes d}}(x) &= d \cdot wt_\omega(x), \quad d > 0 \\ wt_{f\omega}(x) &= wt_\omega(x) + v_x(f) \quad f \text{ nonzero rational function} \end{aligned}}$$

5) (4.2.8) $\mathcal{Y}/X$ sncd-model. Moreover we can say, $y \in X^{div} \cap \widehat{\mathcal{Y}_\eta}$. Then

$$wt_\omega(y) \geq v_y(\mathrm{div}_{\mathcal{Y}}(\omega) + m(\mathcal{Y}_k)_{red})$$

and equality holds iff $y \in Sk(\mathcal{Y})$

[" Wrong model always underestimate the weight "]

4.3 Weight function on Berkovich skeleton.

X (moreover) proper

ω

Suppose X has a proper sncd-model $\mathcal{X}$.

Q9 待补充.

Definition $x \in Sk(\mathcal{X})$. Define

$$wt_{\mathcal{X},\omega}(x) = v_x(\operatorname{div}(\omega) + m(\mathcal{X}_k)_{red})$$

Note I jump over (4.3.3). I don't feel we use it. Too less time.

Theorem 1) (2.2.4) The function

$$wt_{\omega,\mathcal{X}} : Sk(\mathcal{X}) \longrightarrow \mathbb{R}$$

is continuous.

((4.3.3) says more: it's piecewise affine and ...)

重点研究proper到底用在哪 Q12 unsolved

2) If $x \in X^{div} \cap Sk(\mathcal{X})$ then $wt_{\mathcal{X},\omega}(x) = wt_\omega(x)$. [(4.2.7)]

Q10 $X^{div} \subset \widehat{\mathcal{X}_\eta}$ $\forall$ sncd $\mathcal{X}$?

3) If $x \in X^{div} \cap \widehat{\mathcal{X}_\eta}$, and Q11 $sp_{\mathcal{X}}(x) \notin \overline{\{\text{poles of } \omega \text{ on } X\}}^{\mathcal{X}}$, then

$$wt_\omega(x) \geq wt_{\mathcal{X},\omega}(\rho_{\mathcal{X}}(x))$$

$\Rightarrow Supp(\operatorname{div}_{\mathcal{X}}(\omega) + m(\mathcal{X}_k)_{red}) \not\ni x$ and $\rho_{\mathcal{X}}(x)$

Equality holds iff $x \in Sk(\mathcal{X})$.

["Wrong model always under estimate the weight" for divisorial points"]

4) If $y \in X^{mon} \cap \widehat{X_\eta}$, and $sp_{\mathcal{X}}(x) \notin \overline{\{\text{poles of } \omega \text{ on } X\}}$, then

?. Every mon pt is in some $Sk(\mathcal{Y})$ for some proper $\mathcal{Y}$?

Suppose $y \in Sk(\mathcal{Y}) \cap \widehat{X_\eta}$ for some proper sncd-model $\mathcal{Y}$ of X

$$wt_{\mathcal{Y},\omega}(y) \geq wt_{\mathcal{X},\omega}(\rho_{\mathcal{X}}(y))$$

Equality can only occur if $y \in Sk(\mathcal{X})$ [$> \Rightarrow y \notin Sk(\mathcal{X})$]

["Underestimate the weight" ~~Wrong model $\mathcal{X}$ in most~~ can only happen for a wrong model.]

5) $\mathcal{X}, \mathcal{Y}$ two proper sncd-models, $x \in Sk(\mathcal{X}) \cap Sk(\mathcal{Y})$. Then

$$wt_{\mathcal{X},\omega}(x) = wt_{\mathcal{Y},\omega}(x).$$

For char $k=0$ or X a curve

6) [MN13] (4.4.3) W/O "proper" requirement for $\mathcal{X}, \mathcal{Y}$, 5) still holds.

i.e. $\exists!$ function $wt_\omega : Mon(X) \longrightarrow \mathbb{Q}$ s.t.

$wt_\omega(x) = v_x(\operatorname{div}_{\mathcal{X}}(\omega) + m(\mathcal{X}_k)_{red})$

$\forall$ sncd $\mathcal{X}/X$ s.t. $x \in Sk(\mathcal{X})$.

4.4 The weight function on X^{an}

X conn sm K-sch, dim n (drop "properness !)

Assume char $k=0$ or X is a curve.

[redacted]

Definition $x \in X^{an}$. Define

$$wt_\omega(x) = \sup_{\substack{\text{sncd } \mathfrak{X}/X \\ \text{s.t. } x \in \widehat{\mathfrak{X}_\eta}}} \{ wt_\omega(\overbrace{\rho_{\mathfrak{X}}(x)}^{\in Sk(\mathfrak{X})}) \} \in \mathbb{R} \cup \{+\infty\}.$$

Theorem (4.4.5) 1) The function

$$wt_\omega : X^{an} \longrightarrow \mathbb{R} \cup \{+\infty\}$$

is lower semi-cts

2) This def agrees with the ones in §4.3 / §4.2 for mon / div points resp.

3) For $x \in$ [redacted] $\widehat{\mathfrak{X}_\eta}$ where $\mathfrak{X}/X$ sncd, [(4.4.3)]

$$wt_\omega(x) \geq wt_\omega(\rho_{\mathfrak{X}}(x))$$

Equality holds iff $x \in Sk(\mathfrak{X})$

(4) $h : Y \hookrightarrow X$ open immersion. Then $\forall y \in Y^{an}$,

$$wt_{h^*\omega}(y) = wt_\omega(h(y))$$

(5) Two formulas

$$wt_{\omega^{\otimes d}}(x) = d \cdot wt_\omega(x) \qquad d > 0$$

$$wt_{f\omega}(x) = wt_\omega(x) + v_x(f) \qquad f \neq 0 \text{ regular function.}$$

Q14

Rmk 4.4.6 1) can be non-cts. Example:

最后（若有空）展示：$\dfrac{k[x,y]}{x^{N_1}y^{N_2} - \lambda}$ 的 weight.

$\overline{\{\xi\}} \subset \bigcap_{i=1}^{r} E_i \subset \mathfrak{X}_k$

§4.2 proof of the theorem

1) By definition of the symbols

$$\mathrm{div}_{\mathfrak{X}}(\omega) = \sum_{i=1}^{r} (\mu_i - m) E_i \qquad \boxed{\text{无条件成立}}$$

$$\Rightarrow \nu_x(\mathrm{div}_{\mathfrak{X}}(\omega) + m(\mathfrak{X}_k)_{red})$$

(Q15) 哪里用到条件 & 条件什么意思 ref [survey 2.3.3]

$$= \nu_x(\mathrm{div}_{\mathfrak{X}}(\omega)) + \nu_x(m(\mathfrak{X}_k)_{red}) = \sum(\mu_i - m)\alpha_i + \sum m\alpha_i = \sum \mu_i \alpha_i$$

2) $\underline{x = (\mathfrak{X}, (E_1, \cdots, E_r), (\alpha_1, \cdots, \alpha_r), \xi)}$ sncd-tuple

$\underline{r=1}$

$$wt_\omega(x) = \nu_x(\mathrm{div}_{\mathfrak{X}}(\omega) + m(\mathfrak{X}_k)_{red})$$

$$\overset{\text{Recall②}}{=} \frac{\mathrm{ord}_E(\mathrm{div}_{\mathfrak{X}}(\omega) + m(\mathfrak{X}_k)_{red})}{\text{multiplicity of } E \text{ in div } \mathfrak{X}_k \text{ on } \mathfrak{X}}$$

$$= \frac{\mathrm{ord}_E(\mathrm{div}_{\mathfrak{X}}(\omega)) + m}{N} = \frac{\mu}{N} \quad ✓$$

$\underline{r>1}$ We generate a process to reduce to $r=1$.

In (any) the monoidal rep of x, (2.4.11) $\Rightarrow \alpha_i \in \mathbb{Q}$, $i = 1, \cdots, r$.

We repeat the process in the proof of (2.4.11): "Every div pt must have a div rep wrt a sncd-model"

Construction 3

Permuting the indices, may assume $\alpha_1 \leq \alpha_2, \alpha_3, \cdots, \alpha_r$.

$h: \mathfrak{X}' \to \mathfrak{X}$ be the blow-up at the closure of ξ.

Denote by E_i' the strict transform of E_i, $i = 2, \cdots, r$. and E_1' the excep div.

Let ξ' be the (Q17) generic point of $\bigcap_{i=1}^{r} E_i'$. Then

Straightforward computation $\Rightarrow$ $(\mathfrak{X}', (E_1', \cdots, E_r'), \xi')$ is an sncd tuple

& $(\mathfrak{X}', (E_1', \cdots, E_r'), \underbrace{(\alpha_1, \alpha_2 - \alpha_1, \cdots, \alpha_r - \alpha_1)}_{:= \alpha'}, \xi')$ represents x.

(2.4.9) $\Rightarrow$ can eliminate zero entries in α' and adjust a bit for ξ' and $\mathfrak{X}'$ unchanged $\Rightarrow$ represent the same x.

(2.4.11): $x \in X^{div} \Leftrightarrow \langle \alpha_1, \cdots, \alpha_r \rangle_{\mathbb{Q}} \subset \mathbb{R}$ is 1-dim'l

$\Rightarrow$ After finitely many steps, we get a tuple with only one α_i, i.e. it looks like $(\mathfrak{X}, E, \alpha, \xi)$. i.e. $r=1$

α: must be 1; ξ: must be the generic pt of E

According to the rule of an sncd-tuple.

There's two repeated ~~process~~ operations in this process: <u>blow-ups</u> and <u>eliminating zero α_i's</u>. <u>WTS</u> $wt_\omega(x)$ remains the same for these operations.

<u>eliminating zero α_i's</u> $\quad \mathrm{div}_{\mathcal{X}}(\omega)$ unchanged ; $v_x(m(\mathcal{X}_k)_{red}) = m\sum\alpha_i$ thus also unchanged.

<u>Blow-ups</u> <u>WTS</u> $v_x(\mathrm{div}_{\mathcal{X}}(\omega)) + m\sum_{i=1}^{r}\alpha_i = v_x(\mathrm{div}_{\mathcal{X}'}(\omega)) + m\sum_{i=1}^{r}\alpha_i'$

<u>Note</u> that the relative canonical divisor $K_{\mathcal{X}'/\mathcal{X}} = (r-1)E_1'$ (Q18) Hart II 8.5 b)

(4.2.2) gives

$$h^* \mathrm{div}_{\mathcal{X}}(\omega) = \mathrm{div}_{\mathcal{X}'}(\omega) - m(r-1)E_1'$$

also Prop 8.20 (Q19)

$$\Rightarrow v_x(\mathrm{div}_{\mathcal{X}'}(\omega)) + m\sum_{i=1}^{r}\alpha_i' = \boxed{v_x(h^*\mathrm{div}_{\mathcal{X}}(\omega))} + m(r-1)\alpha_1 + m\sum_{i=1}^{r}\alpha_i'$$

$$= \boxed{v_x(\mathrm{div}_{\mathcal{X}}(\omega))} + m\sum_{i=1}^{r}\alpha_i$$

(2.2.3), h proper, $\mathcal{X}'$, $\mathcal{X}$ regular. (Q20)

5) 略. 有时间加例子.

$\xi \in \bigcap^{r} E_i$

$\uparrow$ $r = \mathrm{codim}\{\xi\}$ in $\mathcal{X}$

∵是 simple normal crossing, 每加一个除子降一维

$$\begin{array}{ccc} \mathfrak{X}_k & \rightarrow & k \\ \downarrow & & \downarrow \\ \mathfrak{X} & \rightarrow & R \\ \uparrow & & \uparrow \\ X & \rightarrow & K \end{array}$$

proof of §4.3

1) By (2.2.4), $\forall D \in \mathrm{Div}(\mathfrak{X})$,

$$\{x \in \hat{\mathfrak{X}}_\eta \mid i(x) \notin \mathrm{Supp} D\} \longrightarrow \mathbb{R}$$
$$x \longmapsto v_x(D)$$

is cts.

$\subset \mathfrak{X}$ ~~Q21~~ cf 2.2.3 笔记

NTS $Sk(\mathfrak{X}) \subset \{x \in \hat{\mathfrak{X}}_\eta \mid \underbrace{i(x)}_{\subset X = \mathfrak{X}_k \subset \mathfrak{X}} \notin \overline{\mathrm{Supp}\, D}\}$

$D = \mathrm{div}_{\mathfrak{X}}(\omega) + m(\mathfrak{X}_k)_{red}$

But this is obvious since $Sk(\mathfrak{X})$.

3) 由 §4.2 Thm 5)

$$wt_\omega(x) \geq v_x(\mathrm{div}_{\mathfrak{X}}(\omega) + m(\mathfrak{X}_k)_{red})$$
$$= v_x\Big(\sum_{i \in I}(\mu_i - m)E_i + m(\mathfrak{X}_k)_{red}\Big)$$
$$= v_{\rho_{\mathfrak{X}}(x)}(\quad\underline{\qquad\qquad}\quad)$$
$$= wt_{\mathfrak{X},\omega}(\rho_{\mathfrak{X}}(x))$$

(3.1.5): $v_x(E_i) = v_{\rho_{\mathfrak{X}}(x)}(E_i)$ for every irr comp E_i of $\mathfrak{X}_k$

4) 不证. Denote

$$P = \{x \in \mathfrak{X}_k \mid x \in \overline{\{\text{poles of } \omega \text{ on } X\}}\} \subset \mathfrak{X}_k$$ closed

$$Z = \{y' \in Sk(\mathcal{Y}) \cap \hat{\mathfrak{X}}_\eta \mid sp_{\mathfrak{X}}(y') \notin P\} \subset Sk(\mathcal{Y}) \cap \hat{\mathfrak{X}}_\eta$$

closed $\xleftarrow{\ sp_{\mathfrak{X}}\ }$ anti-ctinuity of $sp_{\mathfrak{X}}$

and we have $y \in Z$.

Claim w/o proof: Divisorial pts are dense in Z.

5). Enough to prove $wt_{\mathcal{Y},\omega}(x) \geq wt_{\mathfrak{X},\omega}(x)$.

Notice that by multiply a nonzero rational function f on X we reduce to the case where $sp_{\mathfrak{X}}(x) \notin \overline{\{\text{poles of } \omega \text{ on } X\}}$.

Then 5) follows from 4).

proof of §4.4

1) (lower semicts at x_0 : If $f(x_0) < +\infty$: $\forall \varepsilon > 0$, $\exists U \ni x_0$ nbhd s.t. $f(x) \geq f(x_0) - \varepsilon$, $\forall x \in U$. If $f(x_0) = +\infty$ then $f(x) \to +\infty$ as $x \to x_0$)

Let $x \in X^{an}$. By 4.4.2 (See Recall ⑤), can find sncd $\mathfrak{X}/X$ s.t. $\hat{\mathfrak{X}}_\eta \ni x$ is a nbhd. $\forall y \in \hat{\mathfrak{X}}_\eta$

$$wt_\omega(y) = \sup_{\left\{\mathfrak{X}' \,\middle|\, \substack{\mathfrak{X}' \to \mathfrak{X} \text{ proper} \\ R\text{-mor of sncd's}}\right\}} \{ wt_\omega(\rho_{\mathfrak{X}'}(y)) \} \in \mathbb{R} \cup \{+\infty\}.$$

与$\mathfrak{X}$无关，再用sup.

and $wt_{\omega,\mathfrak{X}'} \geq wt_{\omega,\mathfrak{X}}$ on Skeleton

and two sncd's can be dominated by a common sncd

$$\begin{array}{ccc} & \text{sncd} & \\ & | & \\ & R & \\ \text{sncd} & & \text{sncd} \end{array}$$

3) Inequality follows directly from def. Only need to show when equality holds.

§4.3 Thm 4) $\Rightarrow$ if $y \in (\hat{\mathfrak{X}}_\eta \setminus Sk(\mathfrak{X})) \cap X^{mon}$, then

$$wt_\omega(y) > wt_\omega(\rho_{\mathfrak{X}}(y))$$

So ~~we~~ our aim is to "make x monoidal": Find a proper morphism $\mathfrak{X}' \to \mathfrak{X}$ of sncd-models of X

$$\begin{tikzcd} Sk(\mathfrak{X}) \subset Sk(\mathfrak{X}') \subset \hat{\mathfrak{X}}'_\eta \arrow[r, equal] & \hat{\mathfrak{X}}_\eta \arrow[l, bend right, "\rho_{\mathfrak{X}'}"'] \arrow[l, bend left, "\rho_{\mathfrak{X}}"] \end{tikzcd}$$

($x' \leftarrow x$)

s.t. $x' = \rho_{\mathfrak{X}'}(x) \notin Sk(\mathfrak{X})$. i.e. ~~But~~ $x' \in (\hat{\mathfrak{X}}_\eta \setminus Sk(\mathfrak{X})) \cap X^{mon}$.

In this case, $wt_\omega(x) \underset{\text{def}}{\geq} wt_\omega(\underbrace{\rho_{\mathfrak{X}'}(x)}_{=x'}) > wt_\omega(\underbrace{\rho_{\mathfrak{X}}(x')}_{=\rho_{\mathfrak{X}}(x)}) = wt_\omega(\rho_{\mathfrak{X}}(x))$

$\boxed{\text{Construction 2}}$ seems not depending on "chark=0 or X=curve"

4) Let $\mathcal{Y}/Y$ sncd s.t. $y \in \widehat{\mathcal{Y}_\eta}$.

Glue $\mathcal{Y}_k (\cong Y)$ to X by the open immersion h

(Q22)? $\rightsquigarrow$ get an sncd $\mathfrak{X}/X$ s.t $h(y) \in \widehat{\mathfrak{X}_\eta}$ $= \mathcal{Y}_k \cup \mathcal{Y}_k$.

glue in the cat of R-schemes

Note that $X \to \operatorname{Spec} K \to \operatorname{Spec} R$!

In this case $\mathfrak{X} := X \cup_{\mathcal{Y}_K} \mathcal{Y}$ WTS $\mathfrak{X}$ is a R-scheme

~~$\mathfrak{X} \to \operatorname{Spec} R$~~ $\Leftrightarrow R \to \Gamma(\mathfrak{X})$

$K \to \Gamma(X)$, $\Gamma(\mathcal{Y})$, $\Gamma(\mathcal{Y}_K)$ Second way, already had

$$wt_{h^*\omega}(\rho_{\mathcal{Y}}(y)) \underset{\substack{\uparrow \\ \text{def of } \mathfrak{X} \\ \text{cf (Q22)}}}{=} wt_\omega(\rho_{\mathfrak{X}}(h(y))) \leq wt_\omega(h(y))$$

$\mathcal{Y} \longrightarrow \mathfrak{X}$

$Sk(\mathcal{Y})$ $Sk(\mathfrak{X}) \subset \widehat{\mathfrak{X}_\eta} \subset \widehat{\mathcal{Y}_\eta}$

This holds for all sncd-models $\mathcal{Y}/Y$ with $y \in \widehat{\mathcal{Y}_\eta}$, thus

$$wt_{h^*\omega}(y) \leq wt_\omega(h(y))$$

Conversely, let $\mathfrak{X}/X$ sncd s.t. $h(y) \in \widehat{\mathfrak{X}_\eta}$. Then apply

Construction 1 :

(Q23)?

$$\begin{array}{ccccccc} \exists \mathcal{Y} \text{ sncd} & \hookrightarrow & \exists \mathfrak{X}' \text{ sncd} & \hookrightarrow & \overline{\mathfrak{X}}' & & \\ \& \ y \in \widehat{\mathcal{Y}_\eta} & & \downarrow \text{proper} & \square & \downarrow & & \\ & & \mathfrak{X} & \hookrightarrow & \boxed{\overline{\mathfrak{X}}} & \longrightarrow & \operatorname{Spec} R \\ \vdots & & \uparrow & & \uparrow & & \\ \boxed{Y} & \hookrightarrow & X & \hookrightarrow & \overline{X} & & \\ & & & & \downarrow \text{proper} & & \\ & & & & \operatorname{Spec} K & & \end{array}$$

take generic fibre later

$\widehat{\mathcal{Y}_\eta} \supset \widehat{\mathfrak{X}'_\eta}$

. ? ~~$\overline{\mathfrak{X}}$ (compactification of $\mathfrak{X}$) is a proper R-model of $\overline{X}$~~

We have

~~(Q24)?~~

$$wt_\omega(\rho_{\mathfrak{X}}(h(y))) \overset{\text{def}}{\leq} wt_\omega(\rho_{\mathfrak{X}'}(h(y))) \underset{\substack{\uparrow \\ \text{def of } \mathfrak{X} \,\&\, \mathcal{Y}}}{=} wt_{h^*\omega}(\rho_{\mathcal{Y}}(y)) \overset{\text{def}}{\leq} wt_{h^*\omega}(y).$$

$\rho_{\mathfrak{X}}(\rho_{\mathfrak{X}'}(h(y)))$

$\square$

$$\boxed{Sk(\mathfrak{X}) \subset Sk(\mathfrak{X}') \subset \widehat{\mathfrak{X}_\eta} = \widehat{\mathfrak{X}_\eta}}$$

$\rho_{\mathfrak{X}}$, $\rho_{\mathfrak{X}'}$

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