(1). • Let K be a discrete valued field with valuation ring $R \subseteq K$. Let $m \subseteq R$ be the maximal ideal, and let $k := R/m$. We use v_K to denote the standard ~~vals~~ valuation which sends a uniformizer of R to 1. Then there is an absolute value

$$|\cdot|_K : K \longrightarrow \mathbb{R}$$
$$a \longmapsto e^{-v_K(a)}$$

- Let X/K be a smooth, connected, separated scheme of dimension n $/K$.
- X^{an} is the K-analytic space associated to X. As a set X^{an} consists of pairs $(x, |\cdot|_{K(x)})$, where $x \in X$ a point and $|\cdot|_{K(x)} : K(x) \longrightarrow \mathbb{R}$ is an absolute value extending $|\cdot|_K$.
- $i : X^{an} \longrightarrow X$ is the map forgeting the valuation. X^{an} is equipped with the coarsest topology s.t. i is continu
- An R-model $\mathcal{X}$ of X is a normal flat separated R-scheme of finite type endowed with a isomorphism $\mathcal{X}_K := \mathcal{X} \otimes_R K \cong X$.

Remark: In this case $\mathcal{X}$ has to be irreducible, for if $y \in \mathcal{X}$ is another generic pt of $\mathcal{X}$ which is different from the one in X, then $y \in \mathcal{X}_k$ the special fiber, so

$$\begin{tikzcd} \operatorname{Spec}(\mathcal{O}_{\mathcal{X},y}) \arrow[r] \arrow[dr] & \mathcal{X} \arrow[r] & \operatorname{Spec} R \\ & \operatorname{Spec} k \arrow[ur] & \end{tikzcd}$$

is commutative as $\operatorname{Spec}(\mathcal{O}_{\mathcal{X},y})$ is a one pt scheme.

But $\mathrm{Spec}(\mathcal{O}_{\mathfrak{X},y}) \to \mathrm{Spec}\,R$ is flat and $\mathrm{Spec}(\mathcal{O}_{\mathfrak{X},y}) \to \mathrm{Spec}\,k$ is f.f. $\Rightarrow$ $\mathrm{Spec}\,k \to \mathrm{Spec}\,R$ is flat $\lightning$.

- Let $\mathfrak{X}_k$ be the special fiber of $\mathfrak{X}$ and $E_1, \cdots, E_r$ be its irreducible components

Claim: $E_i \subseteq \mathfrak{X}$ are of codim 1.

pf: Let $\xi_i \in E_i$ be the generic pt. We have to show that $\mathrm{Spec}\,\mathcal{O}_{\mathfrak{X},\xi_i}$ is a DVR.

The map $R \longrightarrow \mathcal{O}_{\mathfrak{X},\xi_i}$ is a flat map of local rings whose special fibre consists of only 1 pt.

$\Rightarrow$ the generic fiber is of dimension 0. But since $\mathcal{O}_{\mathfrak{X},\xi_i}$ is integral $\Rightarrow$ the generic fibre is a spectrum of an integral ring $\Rightarrow$ the generic fiber consists of only 1-point $\Rightarrow$ $\mathcal{O}_{\mathfrak{X},\xi_i}$ has dimension 1

- Divisor point: $x \in X^{an}$ is called a divisor pt if $\exists$ a model $\mathfrak{X}$ and an irreducible component $E \subseteq \mathfrak{X}_k$ with generic pt ξ s.t. $x = (y, |\cdot|_{K(X)})$ where $y \in \mathfrak{X}$ is the generic pt and

$$|\cdot|_{K(X)} : K(X) \longrightarrow \mathbb{R}$$
$$a \longmapsto e^{-v_\xi(a)}$$

where $v_\xi : K(X) \longrightarrow \mathbb{Z}$ is the valuation sending a uniformizer of $\mathcal{O}_{\mathfrak{X},\xi}$ to $\frac{1}{N}$, and N is the valuation of a uniformizer in R in $\mathcal{O}_{\mathfrak{X},\xi}$

② or equivalently, N is the multiplicity of E in $\mathcal{X}_k$.

- A cartier divisor on $\mathcal{X}$ is a global section of the sheaf $K_{\mathcal{X}}^*/\mathcal{O}_{\mathcal{X}}^*$, where $K_{\mathcal{X}}^*$ is the constant sheaf with value $K(X)$ and $\mathcal{O}_{\mathcal{X}}^* \subseteq \mathcal{O}_{\mathcal{X}}$ is the subsheaf consisting of invertible elements. A cartier divisor is represented by $\{(U_i, f_i)\}$ where $\{U_i\}$ is a covering of $\mathcal{X}$ and $f_i \in K(X)^*$ satisfy $f_i f_j^{-1} \in \mathcal{O}_{\mathcal{X}}^*(U_i \cap U_j)$.

- Let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. We define a rational section of $\mathcal{L}$ to be a pair (s, U) ~~equivalence~~ with $s \in \mathcal{L}(U)$, up to the following equivalence: $(s, U) \sim (s', U')$ iff $s|_{U \cap U'} = s'|_{U \cap U'}$

- $\Gamma(\mathcal{X}, K_{\mathcal{X}}^*/\mathcal{O}_{\mathcal{X}}^*)$ is 1-1 with $\left\{ \begin{array}{l} \text{Line bundles } \mathcal{L} \text{ on } \mathcal{X} \\ \text{equipped with a non-zero} \\ \text{rational section } (s, U) \end{array} \right\} / \cong$

Given $D = \{(U_i, f_i)\} \longmapsto \mathcal{L}$: the subsheaf of $K_{\mathcal{X}}^*$ generated by the sections $\{(U_i, f_i^{-1})\}$

the rational section is on the complement of the support of D, it is $(\mathcal{X} - |D|, 1)$

Given $(\mathcal{L}, (s, U)) \longmapsto$ $\mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_{\mathcal{X}}} K_{\mathcal{X}} \xrightarrow{\cong} K_{\mathcal{X}}$

$s \longmapsto 1$

• $Y \xrightarrow{f} Z$ a morphism of finite type between locally Noetherian schemes. We call f a local complete intersection if $\exists \{Y_i\}$ a covering of Y and a diagram

$$\begin{tikzcd} & Z_i \arrow[d, "\text{smooth}"] \\ Y_i \arrow[ur, hook, "\text{regular embedding}"] \arrow[r, "f|_{Y_i}"] & Z \end{tikzcd}$$

In particular, if Y and Z are both regular then f is always a local complete intersection.

If f is a local complete intersection ~~and quasi~~ and quasi-proj then

$$\begin{tikzcd} & Z' \arrow[d, "\text{smooth}"] \\ Y \arrow[ur, hook, "\text{regular embedding}"] \arrow[r, "f"] & Z \end{tikzcd}$$

$$\omega_{Y/Z} := \det(C_{Y/Z'})^\vee \otimes_{\mathcal{O}_Y} i^*(\det \Omega_{Z'/Z})$$

This is line bundle and it becomes the usual canonical sheaf on the smooth locus of f.

$\omega_{Y/Z}$ could also be defined without the assumption that Y/Z is quasi-proj

Def: If $\mathcal{X}$ is a regular model of X/K and if $(y, |\cdot|_{K(X)})$ is a divisor point of X^{an} with respect to $\mathcal{X}$, then we define $wt_x(\omega) := \frac{\mu}{N}$ where N is the multiplicity of the irr component $E \subseteq \mathcal{X}_k$ defining $(y, |\cdot|_{K(X)})$, $\overset{\circ}{\omega} \in \omega^m_{X/K} = \omega^m_{\mathcal{X}/R}|_X$ is a rational section

③ $N - m$ is the multiplicity of the divisor $\mathrm{div}_{\mathcal{X}}(\omega)$ corresponding to $(\omega_{\mathcal{X}/R}^{\otimes m}, \omega)$

Lemma. $\mathrm{wt}_x(\omega)$ depends only on X, ω, and $x := (\eta, |\cdot|_{K(x)}) \in X^{an}$

Pf: Let $\mathcal{Y}$ be another regular model of X/K. Suppose $(\eta, |\cdot|_{K(x)}) \in X$ is defined by ~~another~~ the component $F \subseteq \mathcal{Y}_k$.

Let $\xi_1 \in E$, $\xi_2 \in F$ be the generic points. Since they define the same valuation on $K(X)$, we have $\mathcal{O}_{\mathcal{X},\xi_1} = \mathcal{O}_{\mathcal{Y},\xi_2}$ as subrings of $K(x)$. Since N is the valuation of a uniformizer of $R \subseteq \mathcal{O}_{\mathcal{X},\xi_1} = \mathcal{O}_{\mathcal{Y},\xi_2}$ it does not depend on $\mathcal{X}$ or $\mathcal{Y}$. Since $\mathcal{X}$, $\mathcal{Y}$ are all schemes of finite type $/R$, $\exists\, \xi_1 \in U \subseteq \mathcal{X}$, $\xi_2 \in V \subseteq \mathcal{Y}$

s.t. $\underset{\xi_1 \longleftrightarrow \xi_2}{U \cong V}$ as models of X/K.

Now we are reduced to the case when $\mathcal{Y} = U$.

We have a line bundle $\mathcal{L} = \omega_{\mathcal{X}/R}^{\otimes m}$ on $\mathcal{X}$, a codim 1 pt $\xi_1 \in \mathcal{X}$, an open $U \subseteq \mathcal{X}$ containing ξ_1, $X \subseteq U$ an open and a non-zero rational section $\omega \in \omega_{\mathcal{X}/R}^{\otimes m}(U')$, $U' \subseteq X$

$\Rightarrow \mathrm{div}_{\mathcal{X}}(\omega)|_U = \mathrm{div}_U(\omega)$.

Since U contains ξ, the multiplicity of ξ in $\mathrm{div}_{\mathcal{X}}(\omega)$ and $\mathrm{div}_{\mathcal{X}}(\omega)|_U$ are equal. $\square$

Lemma: (1). $\mathrm{wt}_{\omega^{\otimes d}}(x) = d \cdot \mathrm{wt}_\omega(x)$ $\quad \omega^{\otimes d} \in \omega_{X/K}^{\otimes dm}$

(2). $\mathrm{wt}_{f\omega}(x) = \mathrm{wt}_\omega(x) + v_x(f)$ for $f \neq 0 \in K($

$\qquad\qquad\qquad\qquad\qquad\qquad v_x(f) = -\ln|f|_{k(x)}$

~~rational~~

Pf: (1). Because $\mathrm{div}_{\mathcal{X}}(\omega^{\otimes d}) = d \cdot \mathrm{div}_{\mathcal{X}}(\omega)$

as $(\omega_{\mathcal{X}/R}^{\otimes md}, \omega^{\otimes d}) = (\omega_{\mathcal{X}/R}^{\otimes m}, \omega)^{\otimes d}$.

(2) ω lives on $U \subseteq X$, one may shrink U a bit so that f becomes an element in $\mathcal{O}_X^*(U)$

$\Rightarrow$ $f\omega$ is a non-zero rational section of $\omega_{\mathcal{X}/R}^{\otimes m}$

$$\mathrm{wt}_{f\omega}(x) = \frac{\mu_{f\omega}}{N} = \frac{\mathrm{div}_{\mathcal{X}}(f\omega)|_E + m}{N} = \frac{\mathrm{div}_{\mathcal{X}}(\omega)|_E + (f)|_E + m}{N}$$

$$= \frac{\mathrm{div}_{\mathcal{X}}(\omega)_E + N \cdot v_x(f) + m}{N}$$

$$= \mathrm{wt}_\omega(x) + v_x(f).$$

- Recall for $(x, |\cdot|_{k(x)}) \in X^{an}$ we denote $\mathcal{H}(x)$ the completion of $k(x)$ w.r.t. the absolute value $|\cdot|_{k(x)}: k(x) \to \mathbb{R}$, $\mathcal{H}(x)^\circ$ its valuation ring.
- $\exists\ \hat{\mathcal{X}}_\eta \subseteq X^{an}$ a subspace consisting of points

④
$$\begin{tikzcd} \operatorname{Spec}\mathcal{H}(x) \arrow[r] \arrow[d] & X \subseteq \mathfrak{X} \arrow[d] \\ \operatorname{Spec}\mathcal{H}(x)^\circ \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} R \end{tikzcd}$$

which admit the broken arrow.

~~E.g.~~

If $x \in \hat{X}_\eta$, then $sp_{\mathfrak{X}}(x)$ is the image of the special pt of $\mathcal{H}(x)^\circ$ under the broken arrow

Let D a Cartier divisor on $\mathfrak{X}$ whose support $|D|$ does not contain $i(x)$. (Recall $i: X^{an} \xrightarrow{i} X$ the forget map), the we set $v_x(D) = -\ln|f(x)|_{K(x)}$, where

$f \in K(X)^*$ s.t. locally at $sp_{\mathfrak{X}}(x)$, $D = \mathrm{div}(f)$.

Remk: If Z is an irr component of the support $|D|$ with generic point ξ, then either $\xi \in \overline{\{i(x)\}}$ in which case $i(x)$ is the generic pt, or $\xi \notin \overline{\{i(x)\}}$ in which case locally at $sp_{\mathfrak{X}}(x)$ f is a regular function on $\mathfrak{X}$, so $f(x)$ makes sense!

Prop: If y is a divisor pt, and $y \in \hat{\mathcal{Y}}_y$, where

$\mathcal{Y}$ is an sncd model of X/K, then

$$wt_w(y) \geq v_y(\mathrm{div}_y(w) + m(\mathcal{Y}_k)_{red})$$

and the equality holds iff $y \in Sk(\mathcal{Y})$

Def: Now suppose $\mathcal{X}/R$ is a proper sncd model

$x \in Sk(\mathcal{X})$ then

$$wt_{\mathcal{X},w}(x) := v_x(\mathrm{div}(w) + m(\mathcal{X}_k)_{red})$$

Prop: (1). $wt_{\mathcal{X},w^{\otimes d}}(x) = d \cdot wt_{\mathcal{X},w}(x)$

(2). $wt_{\mathcal{X},fw}(x) = wt_{\mathcal{X},w}(x) + v_x(f)$

(3). If $\mathcal{Y}$ is another proper sncd model and if $x \in Sk(\mathcal{X}) \cap Sk(\mathcal{Y})$

$\Rightarrow$ $wt_{\mathcal{X},w}(x) = wt_{\mathcal{Y},w}(x)$.