Skeleton of Berkovich spaces

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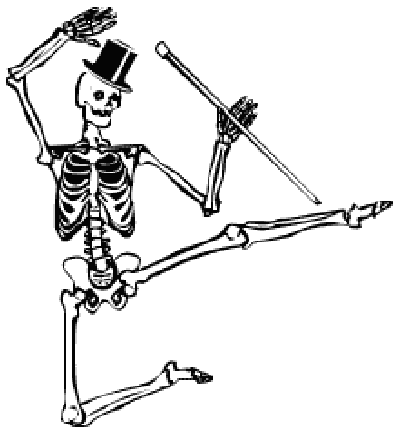
Abstract

This is the sixth talk of the Research Seminar of the Arithmetic Geometry group (FU Berlin), which is on the Winter term of 2016-17. In the seminar we want to understand the paper [NX16]. In this talk I'll try to explain the skeleton of a Berkovich space, and I will try to use this information to draw some 1-dimensional examples. Sadly I'm not able yet to make drawings in the computer, so the only skeleton that will appear is this one:



"*Tate m'a écrit de son côté sur ses histoires de courbes elliptiques, et pour me demander si j'avais des idées sur une définition globale des variétés analytiques sur des corps valués complets. Je dois avouer que je n'ai pas du tout compris pourquoi ses résultats suggéreraient l'existence d'une telle définition, et suis encore sceptique. Je n'ai pas non plus l'impression d'avoir rien compris à son théorème, qui ne fait qu'exhiber par des formules brutales un certain isomorphisme de groupes analytiques; on conçoit que d'autres formules tout aussi explicites en donneraeient un autre pas plus mauvais (sauf preuve du contraire!).* "

A. Grothendieck to J.-P. Serre, 18.08.1959[1]

[1] "*Tate has written to me about his elliptic curves stuff, and has asked me if I had any ideas for a global definition of analytic varieties over complete valuation fields. I must admit that I have absolutely not understood why his results suggest the existence of such a definition, and I remain skeptical. Nor do I have the impression of having understood his theorem at all; it does nothing more than exhibit, via brute formulas, a certain isomorphism of analytic groups; one could conceive that other equally explicit formulas might give another one which would be no worse than his (until proof to the contrary!).*"

Birational, monomial and divisorial points of Berkovich spaces

Setting: here R is a complete discrete valuation ring, with residue field k and fraction field K. We also set X to be a connected regular separated K-scheme of finite type.

Recall that we can define its Berkovich analytification, denoted X^{an}, as the set

$$\left\{ \left(x, |\cdot|_{K(x)}\right) \;\middle|\; \begin{array}{l} x \in X, \text{ and } |\cdot|_{K(x)} \text{ a valuation on the residue field } K(x) \\ \text{of the point } x \text{ extending the valuation } |\cdot|_K \end{array} \right\}$$

together with a topology and a sheaf of rings. Today we will only be interested in X^{an} as a topological space, whose properties were described in the talk of Wouter. This Berkovich analytification comes naturally with a map

$$\iota : X^{an} \to X,$$

which by the definition of the Berkovich topology is continuous.

Remark 1. If $x \in X$ is a closed point, the fibre $\iota^{-1}(x)$ consists in just one point, namely $(x, |\cdot|_{K(x)})$, because $K(x)$ is an algebraic extension (since x is closed) of a complete discrete valued field, and therefore there is a unique valuation $|\cdot|_{K(x)}$ extending $|\cdot|_K$.

In the last week, we defined the set of birational points of X^{an}, denoted X^{bir}, as the fibre $\iota^{-1}(\eta)$, where η is the generic point of X, endowed with the subspace topology.

Remark 2. If X is a curve (for example the affine line over K), then $X^{an} \setminus X^{bir}$ is in bijection with the set of closed points of X. For the affine line, those are precisely the type I points.

Before proceeding, recall that a *sncd*-model of X is a regular flat separated scheme $\mathcal{X}$ of finite type over R, together with an isomorphism between the generic fibre $\mathcal{X}_K$ and X, such that the special fibre $\mathcal{X}_k$ is a divisor with strict normal crossings.

As usual (SGA 5, 3.1.5, pg. 24), we say that a divisor D on $\mathcal{X}$ has strict normal crossings if given global sections (f_i) such that $D = \sum_i div(f_i)$, we have that for every $x \in Supp(D)$, the local restrictions $(f_i)_x$ lying in $\mathfrak{m}_x$ form a part of a regular system of parameters for the local ring $\mathcal{O}_{X,x}$.

For example, for curves, this means that $\mathcal{X}$ is regular and that its singular points are ordinary double points.

We also defined the monomial points, that will be very important in this talk, so we also recall the definition. A *monomial* point is constructed from the data

$$(\mathcal{X}, (E_{j_1}, \dots, E_{j_r}), (\alpha_1, \dots, \alpha_r), \xi),$$

where

- $\mathcal{X}$ is an *sncd*-model of X,
- the E_i's are some of the prime divisors of the special fibre $\mathcal{X}_k = \sum_{i=1}^m N_i E_i$, the N_i's being the multiplicities of the irreducible components in the special fibre,
- the r-tuple $\alpha \in \mathbb{R}^r_{\geq 0}$ are just some real numbers such that $N_{j_1}\alpha_1 + \dots + N_{j_r}\alpha_r = 1$, and
- ξ is the generic point of one of the connected components of $E_{j_1} \cap \dots \cap E_{j_r}$.

We explained in the previous talk that every regular function $f \in \mathcal{O}_{\mathcal{X},\xi}$ can be written in $\widehat{\mathcal{O}_{\mathcal{X},\xi}}$ as a series

$$\sum_{\beta \in \mathbb{N}^r} c_\beta T^\beta,$$

with c_β a unit or 0 in $\widehat{\mathcal{O}_{\mathcal{X},\xi}}$, not necessarily in a unique way (c.f. [MN13, Lemma 2.4.4]). And from this data, we were able to associate a unique valuation defined as

$$v_\alpha := v_{(\mathcal{X},(E_{j_1},\dots,E_{j_r}),(\alpha_1,\dots,\alpha_r),\xi)} : f \mapsto \min\{\alpha \cdot \beta | \beta \in \mathbb{N}^r, c_\beta \neq 0\}$$

which is independent of the choice of the representation of f as a power series and that restricted to $|\cdot|_K$ gives our original valuation (c.f. [MN13, Prop. 2.4.6]). This valuation yields a point $(\eta, \exp(-v_\alpha))$ in X^{an}. The set of all points of this form (i.e. running through all the choices of models, divisors, parameters, etc.) is called the set of *monomial* points, and denoted X^{mon}. Given a monomial point x, we say that it is *divisorial* if there exists a model $\mathcal{X}$ such that x is associated to $(\mathcal{X}, E_i, 1/N_i, \xi)$, being ξ the generic point of E_i.

Proposition 3. *The monomial point associated to the data $(\mathcal{X}, (E_{j_1}, \dots, E_{j_r}), \alpha, \xi)$ is divisorial if and only if all the α_i are in $\mathbb{Q}_{\geq 0}$.*

The proof is in [MN13, Prop. 2.4.11]. Here we show the phenomena with a concrete example, so that we can clarify this a little bit more.

Example 4. Let $X = \operatorname{Spec} K[T_1, T_2]/(\pi - T_1T_2)$ and $\mathcal{X} = \operatorname{Spec} R[T_1, T_2]/(\pi - T_1T_2)$, where π is a uniformizer of R. Then the special fibre $\mathcal{X}_k = \operatorname{Spec} k[T_1, T_2]/(T_1T_2)$ consists of two lines, that we call E_1 and E_2, intersecting in a point that we call O. Now we can consider the monomial point $x \in X^{mon}$ associated to the data

$$(\mathcal{X}, (E_1, E_2), (1/2, 1/2), O),$$

which is not a divisorial point with respect to this model $\mathcal{X}$. But, according to the previous proposition, this is a divisorial point, so let's try to find a model $\mathcal{X}'$ so that x is divisorial (i.e. associated to a divisor) with respect to that model. For this, we blow up $\mathcal{X}$ in O and we get

$$\begin{array}{rcl} \mathcal{X}' & \to & \mathcal{X} \\ \widetilde{E_1} & \to & E_1 \\ \widetilde{E_2} & \to & E_2 \\ E & \to & O \end{array}$$

where E is the exceptional divisor mapping to the node O, and $\widetilde{E_i}$ is the strict transform of E_i. Then, one sees that the divisorial point associated to $(\mathcal{X}', E)$ is precisely x.

Remark 5. The following table explains the relation between this classification of points, and the classification of points for curves:

X^{div}	X^{mon}	X^{bir}	X^{an}
Type II	Types II and III	Types II, III and IV	Types I,..., IV

In higher dimension, there are a lot of different points. Por example, in dimension 2, there are points in $X^{an} \setminus X^{bir}$ whose image under the analytification map ι is not closed (think for example in ξ, the generic point of an irreducible divisor, together with any valuation).

The dual intersection complex

In the above setting, given a *sncd*-model $\mathcal{X}$ of X with special fibre $\mathcal{X}_k = \sum_{i\in I} N_i E_i$, we want to define a simplicial complex that encodes important information of $\mathcal{X}$. We will actually be interested just in the underlying topological space of this complex, so we construct it directly.

For any non-empty subset $J \subset I$, we denote $E_J := \bigcap_{j\in J} E_j$.

Definition 6. The *dual intersection complex* of the model $\mathcal{X}$, denoted $|\Delta(\mathcal{X}_k)|$, is defined as follows:

- The faces: for all $d \in \mathbb{N}$, we stablish a bijection

$$\begin{array}{rcl} \{\text{Simplices of dimension } d\} & \longleftrightarrow & \{\text{Connected components of } E_J,\ \text{where } |J| = d+1\} \\ \tau & \longmapsto & C_\tau. \end{array}$$

- The glueing: given τ, τ' simplices of $|\Delta(\mathcal{X}_k)|$, we stablish

$$\tau \subset \tau' \Longleftrightarrow C_{\tau'} \subset C_\tau.$$

Remark 7. Note that vertices of $|\Delta(\mathcal{X}_k)|$ correspond o irreducible components of $\mathcal{X}_k$.

Remark 8. The points of $|\Delta(\mathcal{X}_k)|$ can be seen as couples (ξ, β), where ξ corresponds to the generic point of an intersection of r distinct irreducible components of $\mathcal{X}_k$, and β is an element of

$$\Delta_\xi^o := \left\{ x \in \mathbb{R}^r_{>0} | \sum_{i=1}^r x_i = 1 \right\}.$$

This couple is called the barycentric coordinates, and we will see later how to relate them with the data that defines monomial points.

Example 9. If we have a Tate curve of type I_n (this is the Kodaira-Néron classification), with $n \geq 2$, then it has a minimal model $\mathcal{X}$ where the special fibre $\mathcal{X}_k$ is given by a loop of n copies of $\mathbb{P}^1$. Hence, its intersection complex is just a loop of n vertices.

If $n = 1$, then $\mathcal{X}_k$ has a node, so it is not a strict normal crossing model, is just normal crossing. We can still blow up at that point in order to get a model with strict normal crossings that will give us the same dual complex $|\Delta(\mathcal{X}_k)|$ as the previous case with $n = 2$.

Example 10. For a higher dimension example, we can take as $\mathcal{X}_k$ the union of the coordinate planes in the 3-dimensional space. This correspond, for example, for the degeneration of an affine cubic surface to the union of 3 planes. Here, $|\Delta(\mathcal{X}_k)|$ will be the standard 2-dimensional simplex.

Example 11. For another 2-dimensional examples, we have that type 2 degeneration of K3 surfaces give us an intersection complex $|\Delta(\mathcal{X}_k)|$ which is a triangulation of a sphere.

Skeleton of Berkovich spaces

The topology of Berkovich spaces is complicated, but their homotopy type is easier to understand. The skeleton will help us to study the homotopy type of Berkovich spaces.

Through this section, we fix an *sncd*-model of X, say $\mathcal{X}$, with special fibre $\mathcal{X}_k = \sum_{i\in I} N_i E_i$. We define a map $\Phi_{\mathcal{X}}$ from the intersection complex $|\Delta(\mathcal{X}_k)|$ to the Berkovich analytification as follows:

$$\Phi := \Phi_{\mathcal{X}} : |\Delta(\mathcal{X}_k)| \longrightarrow X^{an}$$

- A vertex v_i is sent to the divisorial point associated to $(\mathcal{X}, E_i)$, where E_i is the irreducible component of $\mathcal{X}_k$ corresponding to v_i:

$$\Phi(v_i) = (\mathcal{X}, E_i).$$

- Any other point x of $|\Delta(\mathcal{X}_k)|$ is characterized by the following data:

  1. A unique face τ of $|\Delta(\mathcal{X}_k)|$ such that x is in the interior of τ. Let (v_j) denote the set of vertex of $|\Delta(\mathcal{X}_k)|$ spanning τ (with $j \in J \subset I$).
  2. Barycentric coordinates $(\beta_j)_{j\in J}$ such that $\beta_j \geq 0$, $\sum_{j\in J} \beta_j = 1$, and $x = \sum_{j\in J} \beta_j v_j$.

  By definition of $|\Delta(\mathcal{X}_k)|$, τ corresponds to a connected component C_τ of E_J, with generic point ξ. Then, we define the image of x as:

$$\Phi(x) = \left(\mathcal{X}, E_J, \alpha = \left(\frac{\beta_j}{N_j}\right)_{j\in J}, \xi\right).$$

Remark 12. Here we are abusing notation, since we are identifying the monomial point of X^{an} with the data that determines it.

Remark 13. This map is injective. Indeed, different points of $|\Delta(\mathcal{X}_k)|$ will yield different data $\left(\mathcal{X}, E_J, \alpha = \left(\frac{\beta_j}{N_j}\right)_{j\in J}, \xi\right)$, and this data will define different points of X^{an}.

Definition 14. The *Berkovich skeleton* of $\mathcal{X}$, denoted $\mathrm{Sk}(\mathcal{X})$, is the image of the above map. We endow the topology induced by the one in X^{an}.

Remark 15.

- Note that this definition depends on the chosen *sncd*-model $\mathcal{X}$ of X. In the next talks, we will define the essential skeleton of X, which will be independent of the chosen *sncd*-model.
- The Berkovich skeleton of $\mathcal{X}$ is contained in $\widehat{\mathcal{X}_\eta}$, because all monomial points are. This comes from the fact that if $x \in X^{mon}$ is represented by $\left(\mathcal{X}, E_J, \alpha = \left(\frac{\beta_j}{N_j}\right)_{j\in J}, \xi\right)$, then its image under the reduction map is precisely ξ.
- A monomial point on X^{an} lies on $\mathrm{Sk}(\mathcal{X})$ if and only if the *sncd*-model $\mathcal{X}$ is adapted to x, in the sense that it can be represented by $\left(\mathcal{X}, E_J, \alpha = \left(\frac{\beta_j}{N_j}\right)_{j\in J}, \xi\right)$, with $\mathcal{X}$ the fixed model.

Proposition 16. *The map $\Phi : |\Delta(\mathcal{X}_k)| \to \mathrm{Sk}(\mathcal{X})$ is a homeomorphism.*

Before proving the proposition, let's check the skeleton of our favourite example.

Example 17. Let $\mathcal{X} = \operatorname{Spec} R[T_1, T_2]/(\pi - T_1^{N_1} T_2^{N_2})$, as in the previous examples, but now with some multiplicities in the special fibre. The proposition is telling us that $\mathrm{Sk}(\mathcal{X})$ is homeomorphic to $|\Delta(\mathcal{X}_k)|$, so let's try to make this explicit.

First of all, $|\Delta(\mathcal{X}_k)|$ is the standard 1-simplex, which can be seen with vertices $v_1 = (1,0)$, $v_2 = (0,1)$, which correspond to the prime divisors E_1 and E_2, and the whole simplex as

$$\{(\lambda, 1-\lambda)|\ \ 0 \leq \lambda \leq 1\}.$$

Now, the map

$$\Phi : |\Delta(\mathcal{X}_k)| \to \widehat{\mathcal{X}_\eta}$$

maps $(1,0)$ to the divisorial point associated with $(\mathcal{X}, E_1)$, and $(0,1)$ to the one associated with $(\mathcal{X}, E_2)$. For $0 < \lambda < 1$, we have

$$\Phi(\lambda, 1-\lambda) = \left(\mathcal{X}, E_{12}, \alpha = \left(\frac{\lambda}{N_1}, \frac{1-\lambda}{N_2}\right), O\right).$$

Proof of Prop. 16. Surjectivity is automatic, since $\mathrm{Sk}(\mathcal{X})$ is by definition the image of Φ. We saw in a previous remark that the map is injective. Since $|\Delta(\mathcal{X}_k)|$ is compact, closed subsets are again compact, so their image under a continuous map is again compact. Since X^{an} is Hausdorff, they will be closed. This means that if we prove that $\mathrm{Sk}_{\mathcal{X}}$ is continuous, then it will also be a closed map, and since it is a bijection, we get a homeomorphism.

So let's prove the continuity of Φ.

First we reduce to the affine case, by choosing a finite cover of affine open subsets $\mathcal{U}$ of $\mathcal{X}$, which induce a closed covering of $|\Delta(\mathcal{X}_k)| = \bigcup |\Delta(\mathcal{U}_k)|$ compatible with the $\Phi_{\mathcal{X}}$ and $\Phi_{\mathcal{U}}$.

Now, for $\mathcal{X}$ affine, recall that the Berkovich topology on X^{an} is the finest topology making the maps

$$(x, |\cdot|_x) \mapsto |f(x)|_x$$

continuous for every regular function f on $X = \mathcal{X}_K$. Hence, we just need to prove that

$$\begin{array}{ccc} |\Delta(\mathcal{X}_k)| & \longrightarrow & \mathbb{R}_{\geq 0} \\ x & \longmapsto & |f(\Phi(x))|_\alpha \end{array}$$

is continuous. But as we mentioned, the valuation defined by the data $(\mathcal{X}, E_J, \alpha, \xi)$ varies continuously on the parameter α. This gives the continuity in the interior of the faces of the complex.

In order to glue the faces in a continuous way, we have to do the following remark: given $(E_1, \ldots, E_r)$ and ξ the generic point of a connected component of $\bigcap E_j$, there exists a unique ξ', generic point of a connected component of $\bigcap_{j\neq i} E_j$, such that its closure $\{\xi'\}$ in X contains ξ. Then, the point in X^{mon} defined by

$$(\mathcal{X}, E_J, \alpha = (\alpha_1, \ldots, \alpha_{i-1}, 0, \alpha_{i+1}, \ldots, \alpha_r), \xi)$$

is the same one as the point defined by

$$\left(\mathcal{X}, E_{J\setminus\{i\}}, \alpha = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_r), \xi'\right),$$

which is just saying that when we regard a point $x \in |\Delta(\mathcal{X}_k)|$, instead on the unique face τ where it is in the interior, in a bigger face containing τ, everything fits together. This allow us to glue the faces in a continuous way, and therefore we have continuity in the whole $|\Delta(\mathcal{X}_k)|$. $\square$

We can relate this skeleton $\mathrm{Sk}(\mathcal{X})$ with $\widehat{\mathcal{X}_\eta}$. Recall that $\widehat{\mathcal{X}_\eta}$ is the subset of points of X^{an} where we can define the reduction map. As a set, this is just

$$\widehat{\mathcal{X}_\eta} = \left\{ \big(x, |\cdot|_{K(x)}\big) \in X^{an} \;\middle|\; \begin{tikzcd} \operatorname{Spec}\mathcal{H}(x) \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}\mathcal{H}(x)^o \arrow[r, dashed] & \mathcal{X} \end{tikzcd} \right\},$$

so to say, where the valuative criterion of properness holds. Recall also that if X and $\mathcal{X}$ are proper, then $X^{an} = \widehat{\mathcal{X}_\eta}$.

Example 18. For $\mathcal{X} = \mathbb{A}^1_R = \operatorname{Spec} R[T]$, a point $x \in X^{an}$ belongs to $\widehat{\mathcal{X}_\eta}$ if and only if we have a lift

$$\begin{tikzcd} K[T] \arrow[r] & \mathcal{H}(x) \\ R[T] \arrow[u] \arrow[r, dashed] & \mathcal{H}(x)^o \arrow[u] \end{tikzcd}$$

and obviously this happens if and only if the image of T lies in $\mathcal{H}(x)^o$, or in other words, if and only if $|T(x)|_{K(x)} \leq 1$:

$$\widehat{\mathcal{X}_\eta} = \{(x, |\cdot|_{K(x)} | |T(x)|_{K(x)} \leq 1\}.$$

Remark 19. In the next talk, we will construct a retraction map $\rho_{\mathcal{X}} : \widehat{\mathcal{X}_\eta} \to \mathrm{Sk}(\mathcal{X})$. In particular, if X and $\mathcal{X}$ are proper, then $X^{an} = \widehat{\mathcal{X}_\eta}$, so we see that $\mathrm{Sk}(\mathcal{X})$ controls the homotopy type of compact Berkovich analytic spaces.

Note that if $\mathcal{X}$ is not proper, then there is no chance of having an homotopy equivalence, since we could consider something like $\mathcal{X} \setminus \mathcal{X}_k$, which has empty skeleton.

Reduction map

Recall also that given a proper model $\mathcal{X}$, the classical reduction map

$$\pi : X^o \to \mathcal{X}_k^o.$$

defined between the closed points of the generic and the special fibre (this is denoted by the upper index o) by sending a point x to $\overline{\{x\}} \cap \mathcal{X}_k$, can be extended to all the points of the special fibre:

$$\mathrm{sp}_{\mathcal{X}} : X^{an} \to \mathcal{X}_k.$$

If the model is not proper, then we still have

$$\mathrm{sp}_{\mathcal{X}} : \widehat{\mathcal{X}_\eta} \to \mathcal{X}_k.$$

Remark 20. We mentioned last week that this map is anticontinuous (meaning that the inverse image of open sets in closed). The idea behind this fact is that given a regular function $f \in \mathcal{O}_{\mathcal{X}}(X)$ and a point $x \in \mathcal{X}_\eta$, then

$$|f(x)|_{K(x)} < 1 \iff f(x) \in \mathcal{H}(x)^{oo} \iff f(\mathrm{sp}_{\mathcal{X}}(x)) = 0,$$

and we see that an open condition in the Berkovich space corresponds to a closed condition in the special fibre.

Let now X be a smooth, proper and geometrically integral curve over K, and $\mathcal{X}$ a semistable (proper) formal model. Then, we can describe explicitly the fibres of the reduction morphism.

In the classical setting, we have that for $z \in \mathcal{X}_k^o$,

- if z is a nonsingular point of $\mathcal{X}_k$, then the formal fibre $X(z)$ is analytically isomorphic to the open unit disc $B(0,1)^- \subset K$ (seen as a rigid analytic space);
- if z is a singular point (which will be an ordinary double point from the semistable assumption), then the formal fibre $X(z)$ is analytically isomorphic to a standard openn annulus[2] $A(\alpha)^-$ for some unique $\alpha \in |K^*|$ with $0 < \alpha < 1$.

See for example [BL85, Prop. 2.2 and 2.3].

Now, we can ask how do the fibres of the reduction morphism

$$\mathrm{sp}_{\mathcal{X}} : X^{an} \to \mathcal{X}_k$$

look like. So let $z \in \mathcal{X}_k$, then

- if z is a closed nonsingular point of $\mathcal{X}_k$, then the formal fibre $\mathrm{sp}_{\mathcal{X}}^{-1}(z)$ is isomorphic to $\mathcal{B}(0,1)^- = \left\{(x, |\cdot|_{K(x)}) \in \mathbb{A}_K^{1,an} \middle|\ |T(x)|_{K(x)} < 1\right\}$;
- if z is a closed singular point of $\mathcal{X}_k$, then the fibre $\mathrm{sp}_{\mathcal{X}}^{-1}(z)$ is isomorphic to $\mathcal{A}(\alpha)^- = \left\{(x, |\cdot|_{K(x)}) \in \mathbb{A}_K^{1,an} \middle|\ \alpha < |T(x)|_{K(x)} < 1\right\}$;
- if z is the generic point of the irreducible component E_i of $\mathcal{X}_k$, then $\mathrm{sp}_{\mathcal{X}}^{-1}(z) = (\mathcal{X}, E_i)$, where we are again identifying the divisorial point of X^{an} with its representation.

This gives us the complete picture, and we can use it in order to draw Berkovich curves. Further details can be found on [Bak08, Section 5].

Drawing an elliptic curve

Exercise: to draw a contractible elliptic curve from its skeleton using the description of the fibres of $\mathrm{sp}_{\mathcal{X}}$. See [Bak08, Section 5] for more hints, but it is worth to give it a try first.

References

[Bak08] Matthew Baker, "An introduction to Berkovich analytic spaces and non-Archimedean potential theory on curves", in: *p-adic geometry*, vol. 45, Univ. Lecture Ser. Amer. Math. Soc., Providence, RI, 2008, pp. 123–174.

[BL85] Siegfried Bosch and Werner Lütkebohmert, "Stable reduction and uniformization of abelian varieties. I", in: *Mathematische Annalen* 270.3 (1985), pp. 349–379.

[MN13] Mircea Mustaţă and Johannes Nicaise, *Weight functions on non-archimedean analytic spaces and the Kontsevich-Soibelman skeleton*, Available at arXiv:1212.6328, (2013).

[NX16] Johannes Nicaise and Chenyang Xu, "Poles of maximal order of motivic zeta functions", in: *Duke Mathematical Journal* 165.2 (2016), pp. 217–243.

[2] Recall that $A(\alpha)^- = \{z \in B(0,1) |\ \alpha < |z|_K < 1\}$, which is the inverse image in $A(\alpha) = \mathrm{Sp}\, K\langle T, \alpha T^{-1}\rangle$ of the singular point of $\mathrm{Spec}\, k[T_1, T_2]/(T_1 T_2)$ under the reduction map.