Models and monomial points

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1 Notation

1. R is a complete DVR, with maximal ideal $\mathfrak{m}$, residue field k, and fraction field K. The valuation $v_K : K \to \mathbb{Z} \cup \{\infty\}$ on R gives rise to an absolute value $|\cdot|_K = e^{v(\cdot)} : K \to \mathbb{R}$.

   A special case of interest is $R = k[[t]]$ or even more special $R = \mathbb{C}[[t]]$.

2. X is a geometrically connected, smooth, proper K-variety. In particular, X is regular. Sometimes it is enough that X be finite type and normal, or every just integral, but we assume the stronger hypotheses anyway.

3. X^{an} is the associated K-analytic space. As a set, its points are pairs $(x, |\cdot|)$ consisting of a point $x \in X$ and an absolute value $|\cdot| : k(x) \to \mathbb{R}$ on the residue field $k(x)$ which extends the absolute value $|\cdot|_K$ on R. The local ring at such a point $(x, |\cdot|)$ is the completion $\mathscr{H}(x)$ of $k(x)$ with respect to $|\cdot|$. The valuation ring of $\mathscr{H}(x)$ is denoted $\mathscr{H}(x)^\circ$.

4. $i : X^{\mathsf{an}} \to X$ is the canonical morphism of locally ringed spaces. It sends $(x, |\cdot|)$ to x. The topology on X^{an} is the coarsest topology such that i is continuous, and such that for every open subset U of X, and every regular function $f \in \mathcal{O}_X(U)$, the map

   $$i^{-1}(U) \to \mathbb{R}_{\geq 0}; \qquad (y, |\cdot|) \to |f(y)|$$

   is continuous.

5. $X^{\mathsf{bir}} = i^{-1}(\eta)$ where $\eta \in X$ is the generic point. It is given the topology induced by the inclusion $X^{\mathsf{bir}} \subseteq X^{\mathsf{an}}$.

Remark 1. There is a bijection (of sets)

$$X^{\mathsf{bir}} \cong \left\{ \text{real valuations } K(X) \to \mathbb{R}\cup\{\infty\} \text{ extending } v_K : K \to \mathbb{R}\cup\{\infty\} \right\}$$

$$(\eta, |\cdot|) \mapsto \Big(-\ln|\cdot| : k(X) \to \mathbb{R}\cup\{\infty\} \Big)$$

Remark 2. Any birational morphism $X \to Y$ induces a bijection $X^{\mathsf{bir}} \cong Y^{\mathsf{bir}}$.

2 Formal R-models and the specialisation map

Definition 3. An *R-model* $\mathscr{X}$ for X is a flat separated R-scheme of finite type equipped with an isomorphism of K-schemes $\mathscr{X}_K \xrightarrow{\sim} X$.

Remark 4. Sometimes it is assumed to be normal, but it is not assumed to be proper over R. This is important as we will be removing divisors in the closed fibre $\mathscr{X}_k$.

A *morphism* of models is an R-morphism compatible with the isomorphisms to X. There exists at most one morphism $\mathscr{X} \to \mathscr{X}'$ between any two models (R-flat implies reduced, and all generic points lie in X), and if one exists, we say that $\mathscr{X}$ *dominates* $\mathscr{X}'$.

1. $\widehat{\mathscr{X}}$ is the $\mathfrak{m}$-adic completion of $\mathscr{X}$. It is a flat separated formal R-scheme of finite type.
2. $\widehat{\mathscr{X}_K}$ denotes its generic fibre in the category of K-analytic spaces. This is a compact analytic domain in X^{an}.

$$\widehat{\mathscr{X}_K} = \Big\{ (x, |\cdot|) \in X^{\mathsf{an}} : \mathsf{Spec}(\mathscr{H}(x)) \to X \text{ extends to } \mathsf{Spec}(\mathscr{H}(x)^\circ) \to \mathscr{X} \Big\}.$$

Example 5. If $\mathscr{X}$ is proper, then $X^{\mathsf{an}} = \widehat{\mathscr{X}_K}$.

Example 6. If $\mathscr{X} = \mathsf{Spec}(A)$ for some integral domain A, then $X = \mathsf{Spec}(A[(R - \{0\})^{-1}])$ and one can see that we have $\widehat{\mathscr{X}_K} = \{(x, |\cdot|) : |A| \leq 1\}$. Here, we are using the composition $A \to A[(R - \{0\})^{-1}] \to \mathscr{H}(x) \xrightarrow{|\cdot|} \mathbb{R}$.

Similarly, $\widehat{\mathscr{X}_K} = \{(x, |\cdot|) : |\mathcal{O}_{\mathscr{X},x}| \leq 1\}$ where now we use the composition $\mathcal{O}_{\mathscr{X},x} \to \mathcal{O}_{\mathscr{X},x}[(R - \{0\})^{-1}] \to \mathscr{H}(x) \xrightarrow{|\cdot|} \mathbb{R}$.

Definition 7. The reduction map

$$\mathsf{sp}_{\mathscr{X}} : \widehat{\mathscr{X}_K} \to \mathscr{X}_k$$

sends $(x, |\cdot|)$ to the image of the closed point of $\mathsf{Spec}(\mathscr{H}(x)^\circ)$ under the induced morphism $\mathsf{Spec}(\mathscr{H}(x)^\circ) \to \mathscr{X}$.

$$(x, |\cdot|) \mapsto \mathsf{im}\Big(\mathsf{Spec}(\mathscr{H}(x)^\circ/\mathfrak{m}_{\mathscr{H}(x)}) \to \mathscr{X}\Big).$$

This map is anti-continuous, meaning that the inverse image of every open is closed.

Example 8. If $X = \mathbb{A}^1_K = \mathsf{Spec}(K[T])$ and $\mathscr{X} = \mathbb{A}^1_R$, then

$$\widehat{\mathscr{X}_K} = \{x \in X^{\mathsf{an}} : |T(x)| \leq 1\}$$

and $sp_{\mathscr{X}}(x)$ is the reduction of $T(x) \in \mathscr{H}(x)^\circ$ modulo the maximal ideal of $\mathscr{H}(x)^\circ$ (viewed as a point of $\mathscr{X}_k = \mathsf{Spec}(k[T])$).

Example 9. Suppose that $R = \mathbb{C}[[t]]$ and $\mathscr{X} = \mathsf{Spec}(R[x,y]/t - xy)$. Then we have an isomorphism $X \cong \mathbb{A}^1_K - \{0\} = \mathsf{Spec}(K[x, x^{-1}])$ given by $y = \frac{t}{x}$, $x^{-1} = \frac{y}{t}$. The special fibre is $\mathscr{X}_k = \mathsf{Spec}(k[x,y]/xy)$ which has two divisors.

3 snc models over R

Definition 10. We say that $\mathscr{X}$ is a sncd-model of X if $\mathscr{X}$ is regular, and its special fibre $\mathscr{X}_k$ is a divisor with strict normal crossings.

4 Divisorial points

Let $\mathscr{X}$ be an R-model of X, and let $E \subseteq \mathscr{X}_k$ be an irreducible component with generic point ξ. The local ring $\mathcal{O}_{\mathscr{X},\xi}$ is a DVR with fraction field $K(X)$. Let $v_E : K(X)^* \to \mathbb{Z}[\frac{1}{n}]$ be the corresponding discrete valuation, normalised so that $v_E(K^*) = \mathbb{Z}$.

Definition 11. The *divisorial point* associated to $(\mathscr{X}, E)$ is $(\eta, e^{-v_E(\cdot)}) \in X^{\mathsf{an}}$ where η is the generic point of X. The pair $(\mathscr{X}, E)$ is called a *divisorial presentation* of $(\eta, e^{-v_E(\cdot)})$. The set of divisorial points is denoted by

$$X^{\mathsf{div}} \subseteq X^{\mathsf{bir}} \subseteq X^{\mathsf{an}}.$$

Remark 12. The point $(\eta, e^{-v_E(\cdot)})$ is the unique point in $\mathsf{sp}_{\mathscr{X}}^{-1}(\xi)$.

5 Monomial points

Let $\mathscr{X}$ be a sncd-model of X and let $(E_1, \ldots, E_r)$ be the irreducible components of $\mathscr{X}_k$, let $E = \cap_{i=1}^r E_i$ be the intersection.

Remark 13. By definition of a sncd-model, E is regular and pure of dimension $\dim X + 1 - r$, but it is not necessarily connected.

Let ξ be a generic point of E.

Definition 14. The triple $(\mathscr{X}, (E_1, \ldots, E_r), \xi)$ is called a *sncd-triple* for X.

Let $z_1, \ldots, z_r$ be a regular system of local parameters in $\mathcal{O}_{\mathscr{X},\xi}$ and

$$\pi = u z_1^{N_1} \ldots z_r^{N_r}.$$

a uniformiser of R such that, locally at ξ, the prime divisor E_i is defined by $z_i = 0$.

For every $\alpha, \beta \in \mathbb{R}^r$ define $\alpha \cdot \beta = \sum_{i=1}^r \alpha_i \beta_i$, and let $\alpha \in \mathbb{R}^r_{\geq 0}$ be such that $\alpha \cdot N = 1$ where $N = (N_1, \ldots, N_r)$.

Lemma 15 ([Mustata-Nicaise 13, Lemma 2.4.4])**.** *The morphism of sets*

$$(\widehat{\mathcal{O}}_{\mathscr{X},\xi})^*[[z_1, \ldots, z_r]] \to \widehat{\mathcal{O}}_{\mathscr{X},\xi}$$

is surjective.

Remark 16. Cf. Cohen's structure theorem: In equi-characteristic, there is an isomorphism

$$\widehat{\mathcal{O}}_{\mathscr{X},\xi} \cong k(\xi)[[z_1, \ldots, z_r]].$$

Remark 17. The map in the above lemma is plainly not injective, but one can show that for any $\sum_{\beta \in \mathbb{N}^r} c_\beta y^\beta \in (\widehat{\mathcal{O}}_{\mathscr{X},\xi})^*[[z_1, \ldots, z_r]]$, the value

$$\min\{\alpha \cdot \beta : \beta \in \mathbb{N}^r, c_\beta \neq 0\}$$

depends only on its image in $\widehat{\mathcal{O}}_{\mathscr{X},\xi}$.

Proposition 18 ([Mustata-Nicaise 13, 2.4.6, 3.1.6], [Nicaise 14 2.3.3])**.** *There exists a unique real valuation*

$$v_\alpha : K(X)^* \to \mathbb{R}$$

such that for any representative $\sum_{\beta \in \mathbb{N}^r} c_\beta y^\beta$ of an $f \in K(X)^*$ one has

$$v_\alpha(f) = \min\{\alpha \cdot \beta : \beta \in \mathbb{N}^r, c_\beta \neq 0\}.$$

(In particular, $v_\alpha(z_i) = \alpha_i$). The valuation does not depend on the choice of $z_1, \ldots, z_r$ and its restriction to K coincides with v_K (this is why we chose $\alpha \cdot N = 1$).

Definition 19. A *monomial point* of X^{bir} is any point obtained from a valuation v_α associated to data $(\mathscr{X}, (E_1, \dots, E_r), \alpha, \xi)$. The set of monomial points is denoted X^{mon}.

Remark 20. The valuation v_α defines a point x in X^{bir}, and if all α_i are non-zero, it has specialisation $\mathsf{sp}_{\mathscr{X}}(x) = \xi$. If $r = 1$ we get a divisorial point, hence, inclusions

$$X^{\mathsf{div}} \subseteq X^{\mathsf{mon}} \subseteq X^{\mathsf{bir}} \subseteq X^{\mathsf{an}}.$$

Remark 21. Given $(\mathscr{X}, (E_1, \dots, E_r), \xi)$ as above, the map

$$\{\alpha \in \mathbb{R}^r_{\geq 0} : \alpha \cdot N = 1\} \to X^{\mathsf{bir}}$$

is continuous.

6 Density of divisorial points and the Zariski-Riemann space

Proposition 22 ([MN13, 2.4.11])**.** *Let $x \in X^{mon}$ be associated to a valuation v_α as discussed above. Then x is divisorial if and only if $v_\alpha(K(X)^*) \otimes \mathbb{Q} = \mathbb{Q} \subset \mathbb{R}$.*

Proof. Let $(\mathscr{X}, (E_1, \dots, E_r), \xi)$ and $\alpha \in \mathbb{R}^r_{\geq 0}$ be a sncd-triple and a tuple representing x. Since $v_\alpha(K(X)^*) \otimes \mathbb{Q} \cong \mathbb{Q}$, we must have $\alpha \in \mathbb{Q}^r_{\geq 0}$. Permuting the indices, we may assume that $\alpha_1 \leq \alpha_i$ for all i.

Let $h : \mathscr{X}' \to \mathscr{X}$ be the blowup at $\overline{\{\xi\}} \subseteq \mathscr{X}$. Let E'_i be the strict transform of E_i for $i \in \{2, \dots, r\}$, and let E'_1 be the exceptional divisor of the blow-up. Let ξ' be the generic point of $E'_1 \cap \dots E'_r$. Then one can see that

$$(\mathscr{X}', (E'_1, \dots, E'_r), \xi')$$

is still an sncd-triple, and together with

$$\alpha' = (\alpha_1, \alpha_2 - \alpha_1, \dots, \alpha_r - \alpha_1)$$

represents the point x. Next, notice, that if we remove any or all E'_i with $\alpha'_i = 0$, we get the same valuation. So remove any zeros.

Now choose a rational number $q \in \mathbb{Q}_{>0}$ such that

$$\alpha = (n_1 q, \dots, n_r q), \qquad n_i \in \mathbb{N}$$

By induction, the above procedure of subtracting the smallest α_i then removing any zeros, produces an α of the form $\alpha = (nq)$ for some $n \in \mathbb{N}_{\geq 0}$. Hence, after finitely many blowups we obtain a divisorial representation for x. $\square$

Definition 23. Let $\mathcal{M}$ denote the category of proper R-models of X.

The Zariski-Riemann space of X is the inverse limit (in the category of locally ringed spaces) of the special fibres over all proper R-models

$$X^{\mathsf{ZR}} = \varprojlim_{\mathscr{X} \in \mathcal{M}_X} \mathscr{X}_k.$$

A point of X^{ZR} is a coherent sequence of points $(x \in \mathscr{X})_{\mathscr{X} \in \mathcal{M}_X}$. The canonical morphism induced by the valuative criterion for properness is denoted

$$j : X^{\mathsf{an}} \to X^{\mathsf{ZR}}.$$

Remark 24. If we have resolution of singularities then we can take the limit over all sncd-models, but even if not, we can restrict to those proper models which are normal.

Proposition 25 ([MN13, 2.3.2])**.** *The map $j : X^{an} \to X^{ZR}$ is injective, and admits a continuous retraction $r : X^{ZR} \to X^{an}$, such that the topology on X^{an} is the quotient topology.*

Proposition 26 ([MN13, 2.4.12])**.** *The set X^{div} is dense in X^{an}.*

Proof. We can skip the first have of the proof in [MN13] because we have assumed from the beginning that X is proper.

It suffices to show that the (images of) divisorial points are dense in X^{ZR}. Indeed, if $U \subseteq X^{\mathsf{an}}$ is an open that doesn't contain any divisorial points, then so is $r^{-1}(U)$. But this implies $r^{-1}(U)$ is empty, and therefore so is $U = r(r^{-1}(U))$.

Let $\mathscr{X}$ be a proper R-model of X and consider the projection morphism $p : X^{\mathsf{ZR}} \to \mathscr{X}_k$. For every generic point of $\mathscr{X}_k$, the fibre $p^{-1}(\xi)$ consists of a unique point, lets say ξ^{ZR} (The local ring $\mathcal{O}_{\mathscr{X},\xi}$ is a dvr, and proper birational morphisms with integral source towards dvr's are isomorphisms, so any $\mathscr{X}' \to \mathscr{X}$ is an isomorphism in an open neighbourhood of ξ).

In every $\mathscr{X}_k$ the generic points are clearly dense, and so since X^{ZR} has the limit topology (all projections are continuous), the points of the form ξ^{ZR} form a dense subset of X^{ZR} as $\mathscr{X}$ varies over the class of proper R-models of X. $\square$