BANACH RINGS AND BERKOVICH SPACES.

Def

A seminorm on $\mathcal{A}$ is a map $|\ |: \mathcal{A} \to \mathbb{R}_{\geq 0}$ s.t.

1) $|0| = 0$; $|1| = 1$
2) $|f-g| \leq |f| + |g|$
3) $|f \cdot g| \leq |f| \cdot |g|$

$\boxed{1}$

It is called
- multiplicative if $|f \cdot g| = |f| \cdot |g|$
- non Archimedean if $|f-g| \leq \max\{|f|, |g|\}$.
- norm if $|f| = 0 \Rightarrow f = 0$

Rmk
- $|\ |$ induces a topology on $\mathcal{A}$.

  Hausdorff $\Leftrightarrow$ $|\ |$ norm.
- A seminormed ring can always be completed.

Def

A Banach ring (BR) is a complete normed ring.

Ex.
- Completions of seminormed rings.
- All rings $\mathcal{A}$ with $|a|_0 = \begin{cases} 1 & a \neq 0 \\ 0 & a = 0. \end{cases}$
- $(\mathbb{Z}, |\ |_\infty)$
- Valuation fields = field complete by a multiplicative norm

non archimedean field: " " " " non Archimedean " "

$(\mathbb{R}, |\ |_\infty)$, $(\mathbb{C}, |\ |_\infty)$ valuation fields

$\mathbb{Q}_p$ = completion of $\mathbb{Q}$ with $\left|\frac{a}{b}p^n\right|_{p,\varepsilon} = \varepsilon^n$ $\quad (a,p)=(b,p)=1$, $n\in\mathbb{Z}$.

- $\mathcal{A}$ BR and $I \subseteq \mathcal{A}$ closed ideal then

$$\left(\mathcal{A}/_I, |\ |\right) \qquad |\bar{a}| = \inf_{i\in I} |a+i|$$

is a BR.

~~Def~~

~~$\mathcal{A}$ BR, $\mathcal{A}$ seminorm $|$~~

<u>Notation</u> x seminorm on $\mathcal{A}$ $\rightsquigarrow$ $|f|_x$

<u>Def</u>

A seminorm x on $\mathcal{A}$ is bounded if

$$\exists\, C>0 \quad |\ |_x \le C|\ |.$$

<u>rmk</u>

x mult semin on $\mathcal{A}$ bounded $\Leftrightarrow$ $|\ |_x \le |\ |$

<u>pf</u>

$$\forall n\in\mathbb{N} \quad |f|_x^n = |f^n|_x \le C|f^n| \le C|f|^n$$

$$\Rightarrow |f|_x \le C^{\frac{1}{n}}|f| \xRightarrow{n\to+\infty} |f|_x \le |f|.$$

<u>Def</u>

The spectrum of a BR $\mathcal{A}$ is

$$\mathcal{M}(\mathcal{A}) = \left\{ \begin{array}{c} \text{bounded multiplicative} \\ \text{seminorms on } \mathcal{A} \end{array} \right\} \subseteq \mathrm{Hom}(\mathcal{A}, \mathbb{R}_{\ge 0})$$

with the weakest topology making

$$\begin{aligned} \mathcal{M}(\mathcal{A}) &\longrightarrow \mathbb{R}_{\ge 0} \\ x &\longmapsto |f|_x \end{aligned} \qquad f\in\mathcal{A}$$

continue

Def

A map of seminormed rings $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ is bounded if $\exists C>0$ $\quad |\varphi(a)| \le C|a| \quad \forall a \in \mathcal{A}$.

$\boxed{2}$

~~If In this:~~

if $\mathcal{A} \to \mathcal{B}$ is a bounded map of BR

$$\Rightarrow \mathcal{M}(\mathcal{B}) \longrightarrow \mathcal{M}(\mathcal{A}) \quad \text{continuous}$$

$$x \longmapsto |\ |_x \circ \varphi.$$

Ex

k valuation field $\Rightarrow |\mathcal{M}(k)| = 1$.

$x \in \mathcal{M}(k) \qquad |\ |_x \le |\ |$

if $|\lambda|_x < |\lambda| \Rightarrow \left|\frac{1}{\lambda}\right| < \left|\frac{1}{\lambda}\right|_x \le \left|\frac{1}{\lambda}\right|$

rmk

if x is a mult. seminorm on a ring $\mathcal{A}$ then

$P_x = \{a \in \mathcal{A} \mid |a|_x = 0\}$ is prime.

$\cdot \quad |b+a|_x = |b|_x \quad \forall b \in \mathcal{A},\ a \in P_x$

$\Rightarrow x$ induces a mult norm on $\mathcal{A}/P_x$ and its fractions $Q(\mathcal{A}/P_x)$

Def

$\mathcal{H}(x) =$ completion of $Q(\mathcal{A}/P_x)$

$\mathcal{A} \xrightarrow{\varphi} \mathcal{H}(x) \qquad x = |\ |_{\mathcal{H}(x)} \circ \varphi.$

~~x bounded~~

if $\mathcal{A}$ BR $\qquad x$ bounded $\Leftrightarrow$ φ bounded

Conversely if $\mathcal{A} \xrightarrow{\varphi} K$ is a map to a valuation field then $|\ |_\varphi = |\ |_K \circ \varphi$ is a mult seminorm on $\mathcal{A}$.

~~$\mathcal{M}(\mathcal{A})$ $\mathcal{A}$ BR $\to$ $|\ |_\varphi$~~

$$\mathcal{M}(\mathcal{A}) = \left\{ \frac{\text{bounded map from } \mathcal{A} \text{ to a valuation field}}{\sim} \right\}$$

$$\begin{array}{ccccc} & & K & & \\ & \overset{\varphi}{\nearrow} & & \searrow & \\ \mathcal{A} & & & & K'' \\ & \underset{\psi}{\searrow} & & \nearrow & \\ & & K' & & \end{array}$$

<u>THM</u> $\mathcal{A}$ BR

$\mathcal{M}(\mathcal{A})$ is a non empty, compact, Hausdorff space.

pf (idea)

Hausdorff (easy)

- <u>non empty</u>

  $m \subseteq \mathcal{A}$ max ideal is closed

  $\Rightarrow$ assume $\mathcal{A}$ field

  Zorn lemma $\Rightarrow \exists$ minimal bounded seminorm on $\mathcal{A}$.

  $\Rightarrow$ ~~p~~ show it is multiplicative.

- <u>Compact</u>

$$\mathcal{A} \longrightarrow \text{“}\widehat{\prod_{x \in \mathcal{M}(\mathcal{A})}}\text{”}\, \mathcal{H}(x) = \mathcal{B} \text{ is a BR}$$

$\mathcal{M}(\mathcal{B}) \longrightarrow \mathcal{M}(\mathcal{A})$ surjective continuous

Stone - Čech compactification of ~~$\mathcal{M}(\mathcal{A})$~~ the set $\mathcal{M}(\mathcal{A})$.

Def

$\mathcal{A}$ BR, non Archimedean

$r_1, \dots, r_n > 0 \in \mathbb{R}$

$$\mathcal{A}\{r^{-1}T\} = \mathcal{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$$

is the subset of $\mathcal{A}[[T_1, \dots, T_n]]$ of

$$f = \sum_{\nu \in \mathbb{N}^n} a_\nu T^\nu$$

s.t. $|a_\nu| r^\nu \longrightarrow 0$ as $|\nu| = \nu_1 + \dots + \nu_n \longrightarrow +\infty$

$$|f| \overset{\text{def}}{=} \max_{\nu \in \mathbb{N}^n} \left(|a_\nu| r^\nu \right)$$

$\mathcal{A}\{r^{-1}T\}$ is a BR.

- $\mathcal{A}[T_1, \dots, T_n] \subseteq \mathcal{A}\{r^{-1}T\}$ is dense.
- $|\,|$ mult. on $\mathcal{A}$ $\Rightarrow$ $|\,|$ mult on $\mathcal{A}\{r^{-1}T\}$

Def $\mathcal{A}$ BR.

$$\mathbb{A}^n_{\mathcal{A}} = \left\{ \begin{array}{l} \text{multiplicative seminorms on } \mathcal{A}[T_1, \dots, T_n] \\ \text{whose restriction to } \mathcal{A} \text{ is bounded.} \end{array} \right\}$$

with the weakest topology making

$$\begin{array}{ccc} \mathbb{A}^n_{\mathcal{A}} & \longrightarrow & \mathbb{R}_{\geq 0} \\ x & \longmapsto & |f|_x \end{array} \qquad f \in \mathcal{A}[T_1, \dots, T_n]$$

continuous.

Rmk

$$\mathbb{A}^n_{\mathcal{A}} = \bigcup_{r_1, \dots, r_n > 0} \mathcal{M}(\mathcal{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\})$$

is a locally compact Hausdorff space.

Examples $k=k$ non Archimedean, non trivial field

$\mathbb{A}^1_k = ?$

$k \subseteq \mathbb{A}^1_k$

$a \in k \quad |f|_a = |f(a)| \qquad f \in k[T]$

points of type 1.

Given $a \in k$, $\rho \geq 0 \in \mathbb{R}$ $\quad E(a,\rho) = \{ c \in k \mid |a-c| \leq \rho \}$

$$\Big| \sum_n a_n (T-a)^n \Big|_{E(a,\rho)} = \max \{ |a_n| \rho^n \}$$

$a=0 \qquad k[T] \leq k\{ \rho^{-1} T \}$

type 2 pts: $\quad | \ |_{E(a,\rho)} \qquad \rho \in |k^*|$

type 3 pts $\quad | \ |_{E(a,\rho)} \qquad \rho \notin |k|$

such $\quad | \ |_{E(a,0)} = | \ |_a$

- Let $\mathcal{E} = \{ E_i = E(a_i, \rho_i) \}_{i \in I}$ s.t.

$$\rho_i \leq \rho_j \Rightarrow E_i \subseteq E_j \quad \left(\Rightarrow \ | \ |_{E_i} \leq | \ |_{E_j} \right)$$

$$| \ |_{\mathcal{E}} = \inf_{i \in I} | \ |_{E_i} \in \mathbb{A}^1_k$$

$E = \bigcap_i E_i$

$E \neq \emptyset \Rightarrow E$ disc $\quad | \ |_{\mathcal{E}} = | \ |_E$

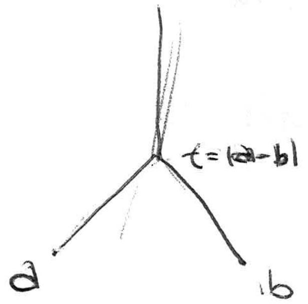
$E = \emptyset \quad | \ |_{\mathcal{E}}$ point of type 4.

$x \in \mathbb{A}^1_k$ $\qquad \mathcal{E} = \{ E(a, |T-a|_x) \}_{a \in A}$

$$| \ |_x = | \ |_{\mathcal{E}}.$$

$\boxed{4}$

rmk: if $|a-b| \leq \rho$ then $E(a,\rho) = E(b,\rho)$ and $| \ |_{E(a,\rho)} = | \ |_{E(b,\rho)}$



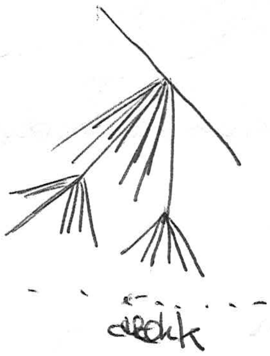
rmk $\forall c \in k$

$$\mathbb{R}_{\geq 0} \longrightarrow \mathbb{A}^1_k$$
$$\rho \longmapsto | \ |_{E(c,\rho)}$$

continuous.

$\tilde{k} = \dfrac{E(0,1)}{\{x \in k \mid |x| < 1\}}$ residue field

- points of type $\overset{2}{1}$ (~~resp 2~~) are dense in $\mathbb{A}^1_k$
- From all type 2 points departs $\tilde{k}$ branches.



leaves are pts of type 1 and 4

BERKOVICH SPACES. k non Arch. field

Def

A k-affinoid is a quotient of

$k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ $r_1, \dots, r_n > 0$

FACT

k-affinoids are Noetherian, are BR and all its ideals are closed.

Def $\mathcal{A}$ k-aff. $X = \mathcal{M}(\mathcal{A})$

. A domain in X is a closed subset $Y \subseteq X$ s.t.

$\exists\ \mathcal{A} \to \mathcal{A}_Y$ bounded map of k-affinoids

with $\mathcal{M}(\mathcal{A}_Y) \to \mathcal{M}(\mathcal{A})$ maps into Y, and

s.t. $\mathcal{A} \to \mathcal{A}_Y$ is universal for this property.

. A special set in X is a finite union of domains.

$$V = \bigcup_{i \in I} V_i$$

$$\mathcal{A}_V = \operatorname{Ker}\left(\prod \mathcal{A}_{V_i} \rightrightarrows \prod \mathcal{A}_{V_i} \hat{\otimes}_{\mathcal{A}} \mathcal{A}_{V_j}\right)$$

does not depend on the decomposition.

Example $f_i, g_j \in \mathcal{A}$, $p_i, q_j > 0$

$$\{ x \in \mathcal{M}(\mathcal{A}) \mid |f_i|_x \le p_i, \ |g_j|_x \ge q_j \}$$

$$\mathcal{A}_Y = \mathcal{A}\{ p^{-1}T, \ q \cdot S \} / (T_i - f_i, \ g_j S_j - 1)$$

FACT

$\forall x \in X$ $\exists$ a basis of neighborhoods of x made of domains

- $Y \subseteq X$ domain $\Rightarrow$ $\mathcal{M}(\mathcal{A}_Y) = Y$

$$\mathcal{A} \to \mathcal{A}_Y \quad \text{flat}$$

Def $X = \mathcal{M}(\mathcal{A})$

$$\mathcal{O}_X(U) = \varprojlim_{\substack{Y \subseteq U \\ \text{special} \\ \text{set}}} \mathcal{A}_Y$$

FACT

$(X, \mathcal{O}_X)$ is a locally ringed space called k-affinoid space.

Rmk $\mathcal{O}_X(X) = \mathcal{A}_X = \mathcal{A}.$

Def

A Berkovich space ~~$(X, \mathcal{O}_X)$~~ is a locally ringed space $(X, \mathcal{O}_X)$ s.t. $\exists\ X = \bigcup_i U_i$ open covering s.t. $(U_i, \mathcal{O}_X|_{U_i})$ is an open of a k-affinoid space.

WRONG!

PROBLEM

$$\left\{ \begin{array}{c} k\text{-affinoid} \\ \text{algebras} \end{array} \right\} \xrightarrow{\mathcal{M}} \left\{ \begin{array}{c} \text{locally ringed} \\ \text{spaces} \end{array} \right\}$$

faithful but not full

Def

A k-quasi affinoid space is a k-~~quasi~~ affinoid algebra $\mathcal{A}$ with an open immersion of locally ringed spaces.

$$U \to \mathcal{M}(\mathcal{A})$$

A map of k-quasi affinoids is

$$\begin{tikzcd} \mathcal{M}(\mathcal{A}_D) = D \arrow[r, "\text{subsp. of l.r.s.}"] & U \arrow[r] \arrow[d, "f"] & \mathcal{M}(\mathcal{A}) \\ & V \arrow[r] & \mathcal{M}(\mathcal{B}) \end{tikzcd}$$

a map of locally ringed spaces s.t. $\forall\ D \subseteq U$ domain in $\mathcal{M}(\mathcal{A})$

$\mathcal{M}(\mathcal{A}_D) \to \mathcal{M}(\mathcal{B})$ comes from a bounded map $\mathcal{B} \to \mathcal{A}_D$.

Let $(X, \mathcal{O}_X)$ be a locally ringed space

A k-analytic chart on X is

$$\begin{array}{ccc} U & \longrightarrow & \mathcal{M}(\mathcal{A}) \leftarrow \text{k-quasi affinoid} \\ \text{open immersion} \downarrow & & \\ X & & \end{array}$$

Two k-analytic charts are compatible if

$$\begin{tikzcd} & U \arrow[r] \arrow[dr] & \mathcal{M}(\mathcal{A}) \\ U \times_X V \arrow[ur] \arrow[d, "id"'] & & X \\ U \times_X V \arrow[r] & V \arrow[r] \arrow[ur] & \mathcal{M}(\mathcal{B}) \end{tikzcd}$$

is $id_{U \times_X U}$ is an isomorphism of k-quasi affinoids.

- A k-analytic atlas is a collection of mutually compatible k-analytic charts $\begin{array}{c} U_i \to \mathcal{M}(\mathcal{A}_i) \\ \downarrow \\ X \end{array}$ s.t. $\coprod_i U_i \to X$ surjective
- two k-analytic atlases are equivalent if any chart of one atlas is compatible with any chart of the other atlas.

Def A structure of Berkovich space on $(X, \mathcal{O}_X)$ is an equivalence class of k-analytic atlases.

A map of Berkovich spaces $f: X \to Y$
is a locally ringed map. s.t. ~~there exists~~
$\forall x \in X$

$\exists$ $\begin{array}{c} U \to \mathcal{M}(\mathcal{A}) \\ \downarrow \\ X \end{array}$, $\begin{array}{c} V \to \mathcal{M}(\mathcal{B}) \\ \downarrow \\ Y \end{array}$ compatible

$x \in U$, $y \in f(U) \subseteq V$ s.t.

$$\begin{array}{ccc} U & \longrightarrow & \mathcal{M}(\mathcal{A}) \\ {\scriptstyle f|_U}\downarrow & & \\ V & \longrightarrow & \mathcal{M}(\mathcal{B}) \end{array}$$

is a map of k-quasi affinoids.

THM
A connected Berkovich space is arc wise connected.