Seminar WS 16/17: Berkovich spaces, birational geometry and motivic zeta functions

TALK 1: INTRODUCTION

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1. Aim

In the seminar we are going through Nicaise-Xu proof of the following conjecture.

Conjecture 1 (Veys)**.** : Let k be a field of characteristic 0, let $f \in k[x-1,\dots,x_n]$ be a nonconstant polynomial such that $f(0)=0$. If α is a pole of order n of the motivic zeta function $Z_{mot}(f;s)$, then $\alpha = -\operatorname{lct}_0(f)$.

The aim of this talk is to introduce the objects appearing in the conjecture, namely, motivic zeta functions and the log canonical threshold, and to sketch its proof, that is the relation with Berkovich geometry.

The content of Section 2 is based on [Mus12], the content of Sections 3 and 4 is based on [Nic10], the content of Section 5 is based on [NX15].

2. Invariants of singularities

Let k be a field of characteristic 0, let $f \in k[x_1,\dots,x_n]$ be a nonconstant polynomial. Then $\{f=0\}$ defines a divisor Δ in $X := \mathbb{A}^n_k$. Let $x \in \Delta$ be a k-point.

Notation 2. For us, a *pair* is given by a smooth variety and a divisor, for example (X,Δ). Let $h: Y \to X$ be a *log resolution* of the pair (X,Δ), that is, h is a birational morphism, Y is a smooth k-variety and the union of the exceptional locus of h with the support of $h^*\Delta$ form a simple normal crossing divisor in Y. We denote by $\tilde{\Delta}$ the strict transform of Δ under h. We write

$$K_Y + \tilde{\Delta} = h^*(K_X+\Delta) + \sum_{E \text{ exc. prime}} a_E E,$$

$$h^*\Delta = \tilde{\Delta} + \sum_{E \text{ exc. prime}} N_E E,$$

$$K_Y = h^*(K_X) + \sum_{E \text{ exc. prime}} (\nu_E E - 1)E,$$

where the sums run over the set of prime exceptional divisors E of h. We recall that the pair (X,Δ) is *log canonical* if there is a log resolution $h: Y \to X$ such that all $a_E \geq -1$. We say that (X,Δ) is *log canonical around* $x \in \Delta$ if there exists an open neighborhood U of x in X such that $(U, \Delta|_U)$ is a log canonical pair.

Here are some invariants if the singularity $x \in \Delta$:

- *multiplicity*:
$$\operatorname{ord}_x f := \min\{\sum_{i+1}^{n} \alpha_i : \alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n_{\geq 0}, \frac{\partial^\alpha f}{\partial x^\alpha}(x) \neq 0\} \geq 1,$$
equality holds if and only if $x \in \Delta$ is smooth;
- *Milnor number* (for isolated singularities):
$$\mu_x f := \dim_k \left(\mathcal{O}_X, x/(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}) \right) \geq 0,$$
equality holds if and only if $x \in \Delta$ is smooth;

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- *log canonical threshold*:

  $\mathrm{lct}_x(f) := \sup\{\lambda \in \mathbb{R}_{>0} : (X, \lambda\Delta)$ is a log canonical pair around $x\}$,

  see Notation 2 for the definition of log canonical pair.

Example 3. For $f = x_1^{e_1} + \cdots + x_n^{e_n}$ and $x = 0 \in \mathbb{A}^n$, we have

$$\mathrm{ord}_0 f = \min_{1 \le i \le n} e_i, \qquad \mu_0 f = \prod_{i=1}^{n}(e_i - 1), \qquad \mathrm{lct}_0\, f = \min\{1, \sum_{i=1}^{n} \frac{1}{e_i}\}.$$

The log canonical threshold can be considered as "a refinement of the reciprocal of the multiplicity."

Properties of the log canonical threshold.

(1) The log canonical threshold is a rational number.
(2) If $x \in \Delta$ is smooth, then $\mathrm{lct}_x f = 1$. But the converse does not hold: take $n = 3$, $e_1 = e_2 = e_3 = 3$ in Example 3.
(3) The log canonical threshold can be computed in terms of a log resolution $h : Y \to X$ of (X, Δ): $\mathrm{lct}_x f = \min\{\frac{\nu_e}{N_E} : E$ exc. prime, $x \in h(E)\}$.
(4) The log canonical threshold can be computed in terms of jet schemes [Mus02]: $\min_{x \in \Delta} \mathrm{lct}_x f = \dim X - \sup_{m \ge 0} \frac{\dim_k \mathcal{L}_m(\Delta)}{m+1}$, where the m-th *jet scheme* of Δ is characterized by $\mathcal{L}_m(\Delta)(A) = \Delta(A[t]/(t^{m+1}))$ for all k-algebras A.
(5) There is a notion of *F-pure threshold* in positive characteristics. Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ and $x \in \{f = 0\}$. Denote by f_p the reduction $\mod p$ of f for every prime p, and by $\mathrm{fpt}_x f_p$ the F-pure threshold of f_p at the reduction mod p of x. Then $\mathrm{fpt}_x f_p \le \mathrm{lct}_x f$ for $p >> 0$, and $\lim_{p \to \infty} \mathrm{fpt}_x f_p = \mathrm{lct}_x f$. It is conjectured that $\mathrm{fpt}_x f_p = \mathrm{lct}_x f$ for infinitely many primes p.

The log canonical threshold find numerous applications within the Minimal Model Program. This section is based on [Mus12], to which we refer the reader for further information.

3. Igusa's p-adic zeta functions

Assume that $f \in \mathbb{Z}[x_1, \ldots, x_n]$ is nonconstant and satisfy $f(0) = 0$. For every prime p we denote by f_p the polynomial f considered as an element of $\mathbb{Z}_p[x_1, \ldots, n]$. For every $r \ge 0$, let

$$N_{f_p}(r) := \#\{x \in (\mathbb{Z}/p^{r+1}\mathbb{Z})^n : f(x) = 0 \mod p\}.$$

The Igusa-Poincaré series associated to f_p is

$$Q_{f_p}(T) := \sum_{r \ge 0} N_{f_p}(r) T^{r+1} \quad \in \mathbb{Z}[[T]].$$

It is related to the Igusa zeta function of f_p at 0 by the following formula

$$Z(f_p; s) = 1 + \frac{(p^{-s} - 1) Q_{f_p}(p^{-s-n})}{p^{-s}},$$

where the Igusa zeta function of f_p at 0 is defined as the p-adic integral

$$Z(f_p; s) := \int_{\mathbb{Z}_p^n} p^{-v_p(f)s} \mathrm{d}\mu,$$

where s is a complex variable and $\mathrm{d}\mu$ is the Haar measure on $\mathbb{Q}_p$ that satisfy $\mu(\mathbb{Z}_p) = 1$.

Igusa proved that $Q_{f_p}(T)$ is a rational function by proving that $Z(f_p; s)$ is a rational function in p^{-s}. Moreover Igusa proves that the candidate poles for $Z(f_p; s)$ can be computed in terms of a log resolution $h : Y \to \mathbb{A}^n_{\mathbb{Q}_p}$ of $(\mathbb{A}^n_{\mathbb{Q}_p}, \{f_p = 0\})$.

Theorem 4 (Igusa).

$$Z(f_p; s) \in \mathbb{Z}\left[\frac{p^{-N_E s-\nu_E}}{1-p^{-N_E s-\nu_E}}\right],$$

where E run over the set of prime exceptional divisors of a log resolution of $(\mathbb{A}^n_{\mathbb{Q}_p}, \{f_p = 0\})$, N_E and ν_E are defined in Notation 2.

Denef gave an explicit expression of $Z(f_p; s)$ in terms of a log resolution of $(\mathbb{A}^N_{\mathbb{Q}_p}, \{f_p = 0\})$.

Conjecture 5 (Igusa monodromy conjecture)**.** Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$, then for almost all primes p, if α is a pole of $Z(f_p; s)$, then $e^{2\pi i \Re(\alpha)}$ is a monodromy eigenvalue of f for some $x \in \{f_p = 0\}$.

The conjecture is known to hold in dimension 2, in dimension 3 it is known for homogeneous polynomials. It is also proven for some families of hypersurfaces, but it is open in general.

4. Motivic zeta functions

We can think of the motivic zeta function of a polynomial as a generalization of its Igusa p-adic zeta functions. We start with a rough introduction to motivic integration. Let k be a field of characteristic 0.

Idea beyond motivic integration: replace $\mathbb{Q}_p$ by $k((t))$, and replace $p = \#\mathbb{F}_p = \#\mathbb{A}^1_{\mathbb{F}_p}(\mathbb{F}_p)$ by $\mathbb{L} := [\mathbb{A}^1_k] \in K_0(Var/k)$, where $K_0(Var/k)$ is the Grothendieck ring of k-varieties.

We recall that the m-th jet scheme $\mathcal{L}_m(X)$ of a k-variety X is characterized by $\mathcal{L}_m(X)(A) = X(A[t]/(t^{m+1}))$ for all k-algebras A. The *arc space* of X is $\mathcal{L}(X) := \varprojlim_m \mathcal{L}_m(X)$.

Motivic integration: integrate over $\mathcal{L}(\mathbb{A}^n_k)$ with value in $\mathcal{M}_k := K_0(Var/k)[\mathbb{L}^{-1}]$ using the motivic measure.

Let $f \in k[x_1, \ldots, x_n]$ be a nonconstant polynomial such that $f(0) = 0$. Let Δ be the hypersurface of $\mathbb{A}^n_k$ defined by $f = 0$.

The "naive" *motivic zeta function* of f at 0 is defined as the motivic integral

$$Z_{mot}(f; s) := \int_{\mathcal{L}(\mathbb{A}^n_k)} \mathbb{L}^{-\mathrm{ord}_t(f)s} \quad \in \mathcal{M}_k[[\mathbb{L}^{-s}]],$$

where s is a complex variable. It is related to the series

$$Q_{mot}(f; T) := \sum_{m \geq 0} [\mathcal{L}_m(\Delta)] T^{m+1} \quad \in K_0(Var/k)[[T]],$$

analogous to the Igusa-Poincaré series. The motivic zeta function of f is rational in $\mathbb{L}^{-s}$ as the following theorem shows.

Theorem 6 (Denef-Loeser)**.**

$$Z_{mot}(f; s) \in \mathcal{M}_k\left[\frac{\mathbb{L}^{-N_E s-\nu_E}}{1-\mathbb{L}^{-N_E s-\nu_E}}\right],$$

where E run over the set of prime exceptional divisors of a log resolution of $(\mathbb{A}^n_k, \Delta)$, N_E and ν_E are defined in Notation 2.

Denef and Loeser gave an explicit formula for $Z_{mot}(f; s)$ in terms of a log resolution of $(\mathbb{A}^n_k, \Delta)$ which is analogous to the one for the p-adic zeta functions of Igusa. Moreover, they prove that if $f \in \mathbb{Z}[x_1, \ldots, x_n]$, then $Z_{mot}(f; s)$ specializes to $Z(f_p; s)$ for almost all primes p.

Conjecture 7 (Motivic monodromy conjecture by Denef-Loeser)**.** Let $f \in k[x_1, \dots, x_n]$, if $\alpha \in \mathbb{Q}$ is a pole of $Z_{mot}(f; s)$, then $e^{2\pi i \alpha}$ is a monodromy eigenvalue of f for some $x \in \{f = 0\}$.

The motivic monodromy conjecture implies Igusa's monodromy conjecture, via a specialization argument. Both conjectures are open.

5. Veys' conjecture

Let k be a field of characteristic 0, $f \in k[x_1, \dots, x_n]$ a noncostant polynomial such that $f(0) = 0$.

Conjecture 8 (Veys)**.** If α is a pole of order n of the motivic zeta function $Z_{mot}(f; s)$, then $\alpha = -\operatorname{lct}_0(f)$.

Vey's conjecture has been proven by Nicaise-Xu as follows.

Setting: Let $R := k[[t]]$ and $K := k((t))$. The polynomial f induces a morphism $f : \mathbb{A}^n_k \to \mathbb{A}^1_k$, let $\Delta := f^{-1}(0) = \{f = 0\} \subseteq \mathbb{A}^n_k =: X$ and $x := 0 \in \mathbb{A}^n_k$. Let $h : Y \to X$ be a log resolution of (X, Δ). We define the R-schemes $\mathscr{X}$ and $\mathscr{Y}$ via the following square diagrams

$$\begin{tikzcd} \mathscr{Y} \arrow[r] \arrow[d] & \mathscr{X} \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ Y \arrow[r, "h"] & X \arrow[r, "f"] & \operatorname{Spec} k[t]. \end{tikzcd}$$

Then the special fiber of $\mathscr{X} \to \operatorname{Spec} R$ is $\mathscr{X}_k = \Delta$, the generic fiber $\mathscr{X}_K$ is smooth, and hence, $\mathscr{X}_K = \mathscr{Y}_K$.

Let $\widehat{\mathscr{X}}$ be the t-adic completion of $\mathscr{X}$, and $(\mathscr{X}_K)^{\mathrm{an}}$ the Berkovich analytification of $\mathscr{X}_K$. Then, the generic fiber of the completion $\widehat{\mathscr{X}_K}$ satisfies $\widehat{\mathscr{X}_K} \subseteq (\mathscr{X}_K)^{\mathrm{an}}$ with equality if $\mathscr{X}$ is proper over R.

Skech of proof: Up to shirinking X around the point x we can assume that $\mathscr{X}_K$ admits a global pluricanonical form. Nicaise-Xu use such a pluricanonical form of $\mathscr{X}_K$ to define a weight function $\mathrm{wt}_\Delta : \widehat{\mathscr{X}_K} \to \mathbb{R} \cup \{+\infty\}$, that depends on the special fiber $\mathscr{X}_k = \Delta$.

They define the *Berkovich skeleton* $\mathrm{Sk}(\mathscr{Y}) \subseteq \widehat{\mathscr{Y}_K} = \widehat{\mathscr{X}_K}$ of $\mathscr{Y}$, and show that $\mathrm{Sk}(\mathscr{Y})$ is isomorphic to the dual complex of the strict normal crossing divisor $\mathrm{Supp}(h^*(\Delta))$ (the vertices of the dual complex correspond to the prime divisors in the support of $h^*(\Delta)$, the m-dimensional faces correspond to connected components of intersections of m prime divisors in the support of $h^*(\Delta)$, see example an page 6). They prove that the weight function wt_Δ restricts to a piecewise affine function on the skeleton $\mathrm{Sk}(\mathscr{Y})$ which is computable in terms of h: $\mathrm{wt}_\Delta(v_E) = \frac{\nu_E}{N_E}$ if v_E is the valuation point of $\mathrm{Sk}(\mathscr{Y})$ associated to a prime exceptional divisor E of h. We observe that those points are vertices of $\mathrm{Sk}(\mathscr{Y})$.

Nicaise-Xu expect that the weight function wt_Δ induces a flow on $\mathrm{Sk}(\mathscr{Y})$. Some evidence is provided by the following result. For $w \in \mathbb{R}$, let $\mathrm{Sk}(\mathscr{Y})^{\leq w} := \{\xi \in \mathrm{Sk}(\mathscr{Y}) : \mathrm{wt}_\Delta(\xi) \leq w\}$.

Theorem 9. *There exists a collapse of* $\mathrm{Sk}(\mathscr{Y})$ *to the essential skeleton* $\mathrm{Sk}^{ess}(\mathscr{Y})$*, which collapses each* $\mathrm{Sk}(\mathscr{Y})^{\leq w}$ *to* $\mathrm{Sk}^{ess}(\mathscr{Y})$*.*

Here by *collapse* we mean a sequence of elementary collapses, where an elementary collapse is the following operation: If τ is a maximal face of $\mathrm{Sk}(\mathscr{Y})$ and σ is a maximal face of τ which is not a face of any other face of $\mathrm{Sk}(\mathscr{Y})$, and elementary collapse is obtained by deleting the interiors of τ and σ from $\mathrm{Sk}(\mathscr{Y})$. See example at page 6.

Let $\mathrm{Sk}(\mathscr{Y}, x)$ be the union of faces of $\mathrm{Sk}(\mathscr{Y})$ that correspond to intersections I of exceptional divisors for h such that $x \in h(I)$. Up to shrinking X around x we can assume that $\mathrm{Sk}(\mathscr{Y}, x) = \mathrm{Sk}(\mathscr{Y})$.

Nicaise-Xu prove Veys' conjecture by combining the following theorem with Denef-Loeser's explicit formula for $Z_{mot}(f; s)$ in terms of the log resolution h.

Theorem 10. *If τ is a maximal face of $\mathrm{Sk}(\mathscr{Y}, x)$ and wt_Δ is constant on τ with value w, then w is minimal and $w = -\operatorname{lct}_0 f$.*

References

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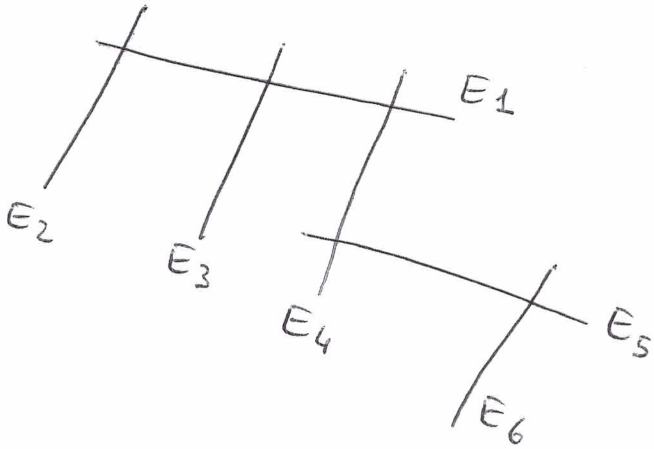
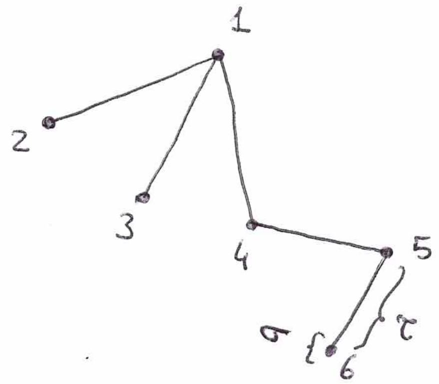
Example of dual complex for the snc divisor



dual complex



Example of elementary collapse:

the collapse of τ, σ from the complex gives the complex