Forschungsseminar
Supersingular K3 surfaces are unirational

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Introduction

A K3 surface X over an algebraically closed field k is a connected smooth proper surface which satisfies $\omega_X := \Omega^2_{X/k} \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. For the Hodge numbers we have $h^{0,2} = h^{2,0} = 1$ and $h^{1,1} = 20$. In characteristic zero the Hodge Chern class injects the Néron-Severi group into $H^1(X, \Omega^1_X)$ and its rank is therefore bounded by 20. In positive characteristic, the crystalline Chern class injects the Néron-Severi group into the slope 1 part of crystalline cohomology and it can happen its image generates the whole $W(k)$-module $H^2_{\mathrm{crys}}(X/W)$. In this case the rank of the Néron-Severi group of X is 22 and we say that it is (Shioda) supersingular. It was conjectured by Artin et al. that supersingular K3 surfaces are unirational (since all the examples they had were so). The aim of this seminar is to understand the proof of the following recent result of Liedtke:

Theorem ([Lie15b]). *Supersingular K3 surfaces in positive characteristic* ≥ 5 *are unirational.*

Lieblich also announced a proof of this result, see [Lie14], but we will follow the approach taken in [Lie15b]. Notice that for a smooth unirational variety in characteristic 0 one always has $H^i(X, \mathcal{O}_X) = 0$, for all $i \geq 1$. Therefore a K3 surface in charateristic 0 can never be unirational since we have $H^2(X, \mathcal{O}_X) \cong H^2(X, \omega_X) = H^0(X, \mathcal{O}_X)$. Whereas the unirationality in positive characteristic only implies $H^i(X, W\mathcal{O}_X)/\text{torsion} = 0$ for all $i \geq 1$. Actually we will see in the seminar that for a supersingular K3 surface, $H^2(X, W\mathcal{O}_X)$ is an infinitely generated $W(k)$-torsion module.

We will spend some time to understand the methods and results which are fundamental to understand K3 surfaces in positive characteristic. Tools that we need are a basic understanding of crystalline cohomology, formal groups, in particular the formal Brauer group of Artin-Mazur, their Dieudonné modules, F-crystals and their slope decompositions. Under the classical and fundamental results that we will discuss are the Rudakov-Shafarevich vanishing $H^0(X, \Omega^1_X) = 0$ (in characteristic 0 this is a direct consequence of Hodge symmetry, in positive characteristic it is some work), Illusie's refinement of the Igusa-Artin-Mazur inequality, maybe also the Theorem of Rudakov-Shafarevich that supersingular K3 surfaces always have potential good reduction and Ogus' theory of K3 crystals culminating in his crystalline Torelli Theorem for supersingular K3 surfaces, which says that such surfaces are up to isomorphism determined by their K3 crystals. After discussing this fundamental background material we will try to understand Liedtke's proof in detail.

The Talks

1.K3 surfaces-basics (15.10.)

Give the definition and first examples of K3 surfaces and compute first invariants, following [Huy, Ch 1, 1.,2.]. In particular prove Prop. 2.5 and the formula $c_2(X) = 24$ in [Huy, 2.4]. If time permits you can compute the Hodge diamond in characteristic 0, but we will do this over an arbitrary field later (after we proved the Rudakov-Shafarevich vanishing).

Fabio

2. Overview of crystalline cohomology and de Rham-Witt (22.10.)

Shortly describe the Witt vectors and the construction and the main properties of crystalline cohomology. Say a word on the de Rham-Witt complex and its relation with crystalline cohomology. You should cover the material from [Lie15a, 1.4, 1.5], don't forget to include Exercise 1.7 and Example 1.8 of *loc. cit.*. Some more details (for which we won't much time) can be found in [CL98] and [Ill94].

Pedro

3. Formal groups and Dieudonné modules (29.10.)

Give an overview of formal groups and their Dieudonné modules following the presentation in [Fon77]. The numbering in the following refers to [Fon77]; the case of an algebraically closed ground field suffices, so please restrict to this situation if it simplifies the presentation. Here are some details what should be covered: Define a formal k-group (with k a field) as in I, 5.1. Give the following example: if G is a commutative k-group scheme then $\hat{G}(R) := \mathrm{Ker}(G(R) \to G(R_{\mathrm{red}}))$, R a finite k-algebra, defines a formal k-group which is called the completion of G along the zero section. Then give I, Prop. 6.6. Following I, §7, explain that over a perfect field k a formal k-group is a product of its étale and its connected part (see top of p. 47). Say a word on Frobenius and Verschiebung and how to use them to define connected and unipotent formal k-groups, see I, 7.4-7.6. Then I, §9 Thm 1 and 9.6 and 9.7. Introduce the functor $\underline{M}$ (and if time allows also $\underline{G}$) and all the necessary notation from III, §1, 1.1-1.3. Then go through III, 6.1, in particular define the notion of p-divisible group (= Barsotti-Tate group) and the notion of height. State the main theorem of Dieudonné theory for smooth p-groups: III, Prop. 6.1 (one also gets that the unipotent smooth formal groups correspond to those Dieudonné modules on which the Verschiebung acts topologically nilpotent). Conclude by saying that a 1-dimensional smooth connected formal k-group ($k = \bar{k}$) is either equal to $\hat{\mathbb{G}}_a$ or is a p-divisible group which is uniquely determined by its height (see e.g. [Zin84, V, Thm 5.33], but the point of view is dual to Fontaine). As a final remark, say that the Dieudonné module of the p-divisible group $\varinjlim_n A[p^n]$ associated to an abelian variety is equal to $H^1_{\mathrm{crys}}(A/W)$, see e.g. [Ill79, II, Rem. 3.11.2].

Lei

4. The formal Brauer group (5.11.)

Define the formal Brauer group following Artin-Mazur [AM77] (the numbering in the following refers to this article). Define the deformation sheaf $\Phi^q(X/S, E)$ as in II, (1.4), give the *Main examples* and Prop 1.7. Then jump to II, 4., explain Cor. (4.1) and (4.2) and give the definition of the formal Brauer group. Note if X is a K3 surface over $k = \bar{k}$, then the formal Brauer group of X/k is a smooth connected formal k-group. Explain the sheaf of p-typical curves on E, $TC(E)$ as in I, 3. (see also [Fon77, V, 3.3], where $TC = CT$). Artin-Mazur then write $\mathbf{D}$ instead of TC but please stick to the TC notation. Then explain that in the case $X = \operatorname{Spec} k$, then $TC(E)$ is kind of dual to the Dieudonné module $\underline{M}(E)$ from the previous talk, see [Fon77, V, Prop 3.2] and that it is really dual if E is a p-divisible group, see [Fon77, Prop 3.4]. Give II, Prop 2.13 and compute $TC(\mathbb{G}_{m,X}) = W\mathcal{O}_X$. We get $TC(\widehat{\mathrm{Pic}_{X/k}}) = H^1(X, W\mathcal{O}_X)$ and $TC(\widehat{\mathrm{Br}_{X/k}}) = H^2(X, W\mathcal{O}_X)$. In the case where X is a K3 surface the formal Brauer group is 1-dimensional and hence by the previous talk is either a p-divisible group of finite height h or $\hat{\mathbb{G}}_a$ ($h = \infty$). Conclude that $H^2(X, W\mathcal{O}_X)$ is a torsion group (which is infinitely generated) iff $h = \infty$; and else $H^2(X, W\mathcal{O}_X)$ is a free $W(k)$-module of rank h.

Elena

5. F-cystals and slopes (12.11.)

Define F-crystals, their slope decomposition and their associated Hodge and Newton polygons and the connection with geometry, following [Lie15a, 3.1-3.3] (see also [Kat79]). Also do Exercise 3.9 and explain the case of abelian varieties 3.4. The case of K3 surfaces will be discussed in detail in talk 7.

Efstathia

6. The Rudakov-Shafarevich vanishing theorem (19.11.)

Sketch the proof of the Rudakov-Shafarevich vanishing theorem which says that a K3 surface over a field k of positive characteristic has no global vector fields (equivalently $H^0(X, \Omega^1_{X/k}) = 0$; notice that this holds in characteristic 0 by Hodge symmetry). To this end proceed as follows: Assume that $H^0(X, \Omega^1_{X/k}) \neq 0$. Then the the tangent bundle $\mathcal{T}_X$ is not μ-stable. Prove that this implies that X is unirational, see [Huy, 9, Prop. 4.6]. Sketch the proof of [Nyg79, lem 3.3] which uses the unirationality of X. Then follow the argument from [LN80].

Wouter

7. The inequality of Igusa-Artin-Mazur and K3 surfaces in positive characteristic (26.11.)

First discuss Illusie's refinement of the inequality of Igusa-Artin-Mazur, which describes the Picard rank of a smooth projective scheme over an algebraically closed field in terms of its second Betti number, the rank of the slope less than 1 part in the second crystalline cohomology modulo torsion of X and the rank of the p-adic Tate module of its Brauer group. See [Ill79, II, 5.7], especially explain the exact sequence [Ill79, (5.8.5)] and Prop 5.12. Now assume X is a K3 surface over an algebraically closed field. Compute the Hodge diamond of X and its crystalline cohomology as in [Lie15a, Prop 2.4, Prop. 2.5] and [Lie15a, 3.5] especially Exercise 3.10, see also [Ill79, II, 7.2]. On the way introduce supersingular K3 surfaces as those K3 surfaces which have Picard rank 22 and use the Igusa-Artin-Mazur inequality to show that this implies that $H^2_{\mathrm{crys}}(X)$ is isomorphic to $NS(X) \otimes_{\mathbb{Z}} W(k)$ which implies that the height of $\hat{\mathrm{Br}}_{X/k}$ is ∞. It follows from the Tate conjecture for K3 (proved in characteristic ≥ 3) that the reverse implication holds as well. Also introduce the Artin invariant of a supersingular K3 surface σ_0, cf. [Lie15a, Prop 4.7, (1)]. Give examples of supersingular K3 surfaces, see .e.g. [Lie15a, Ex 4.10].

Enlin

8./9. Continuous families of torsors (3.12.)/(10.12.)

Discuss section 3 of [Lie15b] in detail! If time permits, it would be nice to see the idea of the proof of the Theorem of Rudakov-Shafarevich-Zink stating that supersingular K3 surfaces in characteristic $p \geq 5$ have potential good reduction. For this first explain [RTS82, §1, Prop 3] (behavior of the height of the formal Brauer group of a K3 under specialization), then [RTS82, §6, Thm 3].

Marta/ NN

10. /11. Supersingular K3 crystals (17.12.)/(7.1.)

Discuss [Lie15a, 4.] in detail! (Note that Example 4.10 was already given in talk 7.)

Tanya/ NN

12. The moduli space of N-marked K3 surfaces (14.1.)

Show that the functor which associates to an algebraic space T over a field the set of isomorphism classes of N-marked K3 surfaces over T is representable by a nice smooth algebraic space. To this end go through [Ogu83, §2] and prove [Ogu83, Thm 2.7].

Valentina

13. Ogus' crystalline Torelli theorem (21.1.)

Discuss [Lie15a, 5.] in detail.

NN

14. N-rigidified supersingular K3 crystals and moving torsors (28.1.)

The aim of this talk is to prove [Lie15b, Cor 4.6], which says that given a supersingular K3 surface of a given Artin invariant $\sigma_0 \geq 2$ one finds a purely inseparable isogeny of height 2 to a supersingular K3 surface of Artin invariant $\sigma_0 - 1$. To this end introduce rigidified $K3$ crystals as in[Lie15b, Def. 4.2] and explain the description of their moduli space as in [Lie15b, Thm 4.3]. Then give the relation with the moving torsors from talks 8./9. and prove [Lie15b, Cor 4.6].

Kay

15. Supersingular K3 surfaces are unirational (4.2.)

First explain Shioda's Theorem [Shi77], which says that in positive characteristic > 2 a Kummer surface is unirational if and only if it is supersingular, see [Shi77, Thm 1.1]. Then give Liedtke's Theorem [Lie15b, Thm 5.1], which in particular says that in positive characteristic > 5 any supersingular K3 surface is inseparably isogeneous to an abelian Kummer surface and conclude that any $K3$ surface is unirational [Lie15b, Thm 5.3]. Finish with [Lie15b, Cor 5.4, Thm 5.5].

Kay

16. Free slot (11.2.)

NN

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