Preliminary for characteristic p Non-abelian Hodge Theory

Su

1. Introduction

First we recall a little bit of the main theorems in classical Non-abelian Hodge Theory. Let $X/\mathbb{C}$ be a smooth projective manifold over the complex numbers.

Definition 1.1. A **Higgs bundle** is defined to be a pair $(\mathcal{E}, \theta)$, where $\mathcal{E}$ is a holomorphic vector bundle of rank r on X and

$$\theta : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$$

(we call it Higgs field) is a morphism of $\mathcal{O}_X$-modules, satisfying $\theta \wedge \theta = 0$. Here by $\theta \wedge \theta$ we mean the composition

$$\mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\theta \otimes id_{\Omega_X}} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{id_{\mathcal{E}} \otimes \wedge} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_X.$$

Then we get a complex of locally free sheaves, the **holomorphic Dolbeault complex**:

$$\mathcal{E} \xrightarrow{\theta\wedge} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\theta\wedge} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_X \xrightarrow{\theta\wedge} \cdots$$

The condition $\theta \wedge \theta = 0$ insures that this is indeed a complex. Define the **Dolbeault cohomology** with coefficients in $\mathcal{E}$ to be the hyper cohomology

$$H^i_{Dol}(X, E) := R^i\Gamma(X, \mathcal{E} \xrightarrow{\theta\wedge} \mathcal{E} \otimes_{SO_X} \Omega^1_X \xrightarrow{\theta\wedge} \cdots)$$

Simpson in [Sim92] proved the following theorem

Theorem 1.2. *There is an equivalence of categories (We denote it by C) between the category of simple flat connections and the category of stable Higgs bundles with $c_1(\mathcal{E}) = c_2(\mathcal{E}) = 0$.*

Moreover, he gave a comparison between Dolbeault cohomology and de Rham cohomology.

Theorem 1.3. *There is a natural isomorphisms $H^i_{dR}(X, \mathcal{E}) \cong H^i_{Dol}(X, C(\mathcal{E}))$ for each $i \in \mathbb{N}$.*

Via some categorical arguments he further showed that the functor C extend to an equivalence between flat connections and semi-stable Higgs bundles with vanishing chern classes. Simpson called this correspondence **non-abelian Hodge theory**, and some other mathematicians called this **Simpson correspondence**.

During the following talks, we will introduce the Simpson correspondence in positive characteristic.

Let X be a smooth quasi-projective variety over a perfect field k, with $char(k) = p > 0$. We have commutative diagram of Frobenii:

$$\begin{array}{ccccc} X & \xrightarrow{F_{X/k}} & X^{(p)} & \xrightarrow{\widetilde{F_k}} & X \\ & \searrow & \downarrow & & \downarrow \\ & & spec(k) & \xrightarrow{F_k} & spec(k). \end{array}$$

In the diagram above, the composite of the relative Frobenius $F_{X/k}$ with $\widetilde{F_k}$ is the absolute Frobenius F_X of X, and F_k is the absolute Frobenius of $spec(k)$.

Theorem 1.4. *Suppose that X can be lift to $W_2(k)$, then there is an equivalence of categories from flat sheaves of nilpotent $\leq p-1$ on X to Higgs sheaves of nilpotent $\leq p-1$ on $X^{(p)}$. We call this equivalence* ***Cartier Transform****.*

Remark 1.5. There are 3 ways to give the correspondence in [OV07], [LSZ15], and [Oy17]respectively.

2. Flat connections and Higgs sheaves in characteristic p

We introduce our terminology. Throughout this section, we always assume that X is a smooth quasi-projective variety over a perfect field k with $char(k) > 0$.

2.1. Flat connections in characteristic p.

2.1.1. *The category of flat connections.*

Definition 2.1. Let X be a smooth quasi-projective variety over a perfect field k, $char(k) = p > 0$. $\mathcal{E}$ is a coherent sheaf on X. A **connection** on $\mathcal{E}$ is a k-linear morphism

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{X/k}$$

satisfying Leibniz's rule

$$\nabla(fe) = e \otimes df + f\nabla e,$$

where f and e are sections of $\mathcal{O}_U$ and $\mathcal{E}|_U$ respectively over an open subset $U \subset X$.

Example 2.2. The differential map $d : \mathcal{O}_X \to \Omega^1_X$ is a connection on $\mathcal{O}_X$. Therefore, if $\mathcal{F}$ is a locally free coherent sheaf of rank r on X with trivialization $\varphi_\alpha : \mathcal{F}|_{U_\alpha} \to \mathcal{O}_U^{\oplus r}$, the trivialization can induce a connection on each $\mathcal{F}|_{U_\alpha}$ by pulling back the canonical one.

If $X = \mathbb{A}^1_{\mathbb{F}_3} = spec(\mathbb{F}_3[x])$, for every connection ∇ on $\mathcal{O}_X$, we have $\nabla = d + f dx$, where $f \in \mathbb{F}_3[x]$.

The kernel of ∇ (as an abelian sheaf on the Zariski site of X), denoted $\mathcal{E}^{\nabla}$, is called the sheaf of germs of horizontal sections of $(\mathcal{E}, \nabla)$.

Remark 2.3. Remark that in complex analytic case, the kernel of a connection (as an abelian sheaf over the base space with manifold topology) is a finite rank local system which may not have so much local sections. But in positive characteristic case, if $e \in \mathcal{E}^{\nabla}$, every $f \in \mathcal{O}_U$, $f^p e$ is in $\mathcal{E}^{\nabla}$. Thus $\mathcal{E}^{\nabla}$ is an $\mathcal{O}_X^p$-module, or in other words $F_*(\mathcal{E}^{\nabla})$ is an $\mathcal{O}_X$-module, where $F : X \to X$ is the absolute Frobenius map.

One easily verifies that the difference between two connections is $\mathcal{O}_X$-linear and defines a section of $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \otimes_{\mathcal{O}_X} \Omega_X^1$. So, on free sheaf $\mathcal{E} = \mathcal{O}_X^{\oplus r}$ any connection ∇ is of the form $d + \omega$, where ω is a section of $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \otimes \Omega_X^1$.

A connection ∇ can be extended to a homomorphism of abelian sheaves(**by Leibniz rule**)

$$\nabla_i := \nabla \wedge id_{\Omega^i} + id_{\mathcal{E}} \otimes d : \mathcal{E} \otimes \Omega_X^i \to \mathcal{E} \otimes \Omega_X^{i+1},$$

by

$$\nabla_i(e \otimes \omega) = \nabla(e) \wedge \omega + e \otimes d(\omega),$$

where ω and e are local sections of $\Omega_{X/k}^i$ and $\mathcal{E}$ respectively, and $\nabla(e) \wedge \omega$ denotes the image of $\nabla(e) \otimes \omega$ under the canonical map

$$\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/k}^1 \otimes_{\mathcal{O}_X} \Omega_{X/k}^i \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/k}^{i+1}$$

which sends $e \otimes \tau \otimes \omega$ to $e \otimes \tau \wedge \omega$.

Consider the composite of ∇ and ∇_1. For any local sections $f \in \Gamma(U, \mathcal{O}_X)$, $e \in \Gamma(U, \mathcal{E})$,

$$\nabla_1\nabla(fe) = \nabla_1(e \otimes df + f\nabla e) = \nabla(e) \wedge df + e \otimes d(df) + \nabla(e) \wedge (-df) + f\nabla_1\nabla(e) = f\nabla_1\nabla(e).$$

i.e. $\nabla_1\nabla$ is an $\mathcal{O}_X$ linear map $K = \nabla_1\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/k}^2$. One easily verifies that

$$\nabla_{i+1}\nabla_i(e \otimes \omega) = K(e) \wedge \omega,$$

where ω and e are local sections of $\Omega_{X/k}^i$ and $\mathcal{E}$ respectively.

Definition 2.4. We call the linear map $K(\mathcal{E}, \nabla) : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}} \Omega_X^2$ to be the **curvature** with respect to ∇. And a connection ∇ is called **integrable** or **flat** if its curvature $K(\mathcal{E}, \nabla)$ is zero.

By abuse the notions, we also call the pair $(\mathcal{E}, \nabla)$ to be a **flat connection**.

Remark 2.5. Remark that if we define the connections as a map from $\mathcal{E} \to \Omega_X^1 \otimes_{\mathcal{O}} \mathcal{E}$, then there is a sign convention. We should change the definition of $\nabla_i : \Omega^i \otimes \mathcal{E} \to \Omega^{i+1} \otimes \mathcal{E}$ to be

$$d \otimes id_{\mathcal{E}} + (-1)^i id_{\Omega^i} \wedge \nabla$$

.

Example 2.6. Let $X = \mathbb{A}^1_{\mathbb{F}_3} = spec(\mathbb{F}_3[x])$, $\nabla = d + f dx$ is a connection on $\mathcal{O}_X$, then

$$K(\nabla, \mathcal{O}_X) = df.$$

Indeed, if $\mathcal{E}$ is a locally free coherent sheaf and ∇ is a connection on $\mathcal{E}$, we can choose a localization such that

$$\nabla|_{U_\alpha} = d_\alpha + \omega_\alpha,$$

then the curvature of ∇ can be represented locally as

$$\nabla_{\alpha,1}\nabla_\alpha = (d_\alpha + \omega_\alpha)(d_\alpha + \omega_\alpha) = d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha.$$

More over, even if $\mathcal{E}$ is not free, we still have a similar local formula for the curvature. Assume X is affine, $\mathcal{E}$ is generated by global sections s_1,...,s_q and ∇ be a connection on $\mathcal{E}$.

$$\nabla(s_1, \ldots, s_q) = (s_1, \ldots, s_q)(\omega_{ij}).$$

Then $D := d + (\omega_{ij})$ defines a connection on $\mathcal{O}_X^{\oplus q}$ so that

$$\mathcal{O}_X^{\oplus q} \to \mathcal{E},\ (\varphi_1, \ldots, \varphi_q) \mapsto \sum_i \varphi_i s_i$$

induces a morphism of connections, which means the following diagram commute.

$$\begin{array}{ccc} \mathcal{O}_X^{\oplus q} & \longrightarrow & \mathcal{E} \\ \downarrow{\scriptstyle d+(\omega_{ij})} & & \downarrow{\scriptstyle \nabla} \\ \mathcal{O}_X^{\oplus q} \otimes \Omega_X^1 & \longrightarrow & \mathcal{E} \otimes \Omega_X^1. \end{array}$$

Thus,

$$[K(\mathcal{O}_X^{\oplus q}, D)(\varphi_1, ..., \varphi_q)] = K(\mathcal{E}, \nabla)(\sum_i \varphi_i s_i).$$

and

$$K(\mathcal{E}, \nabla)(\sum_i \varphi_i s_i) = (\varphi_1, ..., \varphi_q)(d(\omega_{ij}) + (\omega_{ij}) \wedge (\omega_{ij}))(s_1, ..., s_q)^T.$$

For local sections D_1, D_2 of $\mathcal{T}X$,

$$(d\omega)(D_1, D_2) = D_1(\omega(D_2)) + D_2(\omega(D_1)) - \omega([D_1, D_2]).$$

So for $(\mathcal{O}_X^{\oplus q}, D) \twoheadrightarrow (\mathcal{E}, \nabla)$ we have

$$\begin{array}{rcl} D_{D_1}D_{D_2} - D_{D_2}D_{D_1} - D_{[D_1,D_2]} & = & [D_1, D_2] + D_1(\omega_{ij}(D_2)) + D_2(\omega_{ij}(D_1)) \\ & & +[(\omega_{ij}(D_1)), (\omega_{ij}(D_2))] - ([D_1, D_2] + (\omega_{ij}([D_1, D_2])) \\ & = & d[(\omega_{ij}) + (\omega_{ij}) \wedge (\omega_{ij})](D_1, D_2) \\ & = & K(\mathcal{O}_X^{\oplus q}, D)(D_1, D_2). \end{array}$$

Pass to $\mathcal{E}$, we get

$$\nabla_{D_1}\nabla_{D_2} - \nabla_{D_2}\nabla_{D_1} - \nabla_{[D_1,D_2]} = K(\mathcal{E}, \nabla)(D_1, D_2).$$

Therefore, for flat connection $(\mathcal{E}, \nabla)$, we have

$$[\nabla_{D_1}, \nabla_{D_2}] = \nabla_{[D_1, D_2]}.$$

In other words, ∇ defines a Lie algebra morphism

$$\nabla : \mathcal{T}X \to \mathcal{E}nd_k(\mathcal{E})$$

Remark 2.7. A flat connection over characteristic 0 is always locally free, because we can always derivate a non-trivial relation of local generators to a non-trivial relation in the fiber. On the contrary, in positive characteristic a flat connection is not always locally free. So we should consider connections on coherent sheaves which are not necessarily locally free.

Definition 2.8. Let $(\mathcal{E}, \nabla)$, $(\mathcal{F}, \nabla')$ be two flat connections. We define a **morphism** between $(\mathcal{E}, \nabla)$ and $(\mathcal{F}, \nabla')$ to be a morphism $\varphi : \mathcal{E} \to \mathcal{F}$ of $\mathcal{O}_X$-modules such that it commutes with ∇, i.e., $\varphi \otimes id_\Omega \circ \nabla = \nabla' \circ \varphi$.

We see that flat connections, or in other words, modules with integrable connections forms an abelian category, and we denote it by $\mathbf{MIC}(X)$. The category has an internal Hom functor and a tensor functor as follows:

$$(\mathcal{E}, \nabla) \otimes (\mathcal{F}, \nabla') := (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}, \nabla \otimes id_{\mathcal{F}} + id_{\mathcal{E}} \otimes \nabla').$$

$$\mathbf{Hom}((\mathcal{E}, \nabla), (\mathcal{F}, \nabla')) := (\mathcal{H}om(\mathcal{E}, \mathcal{F}), \nabla'').$$

Here ∇'' is defined by the formula:

$$\nabla''_D(\varphi)(e) = \nabla'_D(e) - \varphi(\nabla_D(e)),$$

where D, φ and e are local sections of $\mathcal{T}X$, $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ and $\mathcal{E}$ respectively. The horizontal sections of internal Hom turn out to be the Hom set in $\mathbf{MIC}(X)$

If $f : X \to Y$ be a smooth k-morphism, we can define a pull back functor $f^* : \mathbf{MIC}(Y) \to \mathbf{MIC}(X)$,

$$f^*(\mathcal{E}, \nabla) = (f^*\mathcal{E}, f^*\nabla),$$

$f^*\nabla$ defined by the formula:

$$f^*\nabla := id_{\mathcal{O}_X} \otimes f^{-1}\nabla + d_{\mathcal{O}_X} \otimes id_{f^{-1}\mathcal{E}}$$

Definition 2.9. Let $(\mathcal{E}, \nabla)$ be a flat connection. Then ∇_i induce a complex

$$\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_X := \cdots \to 0 \to \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\nabla_1} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_X \xrightarrow{\nabla_2} \cdots$$

which is called the **de Rham complex** of $(\mathcal{E}, \nabla)$. Define the **de Rham cohomology** associated to $(\mathcal{E}, \nabla)$ to be the hyper-cohomology

$$H^i_{dR}(X, \mathcal{E}, \nabla) := R^i\Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_X).$$

In particular, $H^0_{dR}(X, \mathcal{E}, \nabla) = \Gamma(X, \mathcal{E}^\nabla)$.

As katz announced in [Ka70] Grothendieck in [Gro69] proved that the functors $H^i_{dR}(X,\cdot) : \mathbf{MIC}(X) \to k$-mod are the right derived functor of $H^0_{dR}(X,\cdot)$.

In particular if $\mathcal{E}$ is locally free, we have

$$Ext^1((\mathcal{E},\nabla),(\mathcal{F},\nabla')) = H^1_{dR}(X,\mathbf{Hom}(E,F)).$$

Remark 2.10. Over the complex numbers, a de Rham complex associated to a flat connection $(\mathcal{E},\nabla)$ is always a resolution of $\mathcal{E}^\nabla$ (because Cauchy's theorem tells us that we can always find flat local bases and d is exact [Con, Corollary 2.7]).

On the contrary, in characteristic p cases, we have

Proposition 2.11. *Let $(\mathcal{E},\nabla)$ be a flat connection on X, $\mathcal{E}\otimes_{\mathcal{O}_X}\Omega^\bullet_X$ be the de Rham complex associated to $\mathcal{E}$ and $F_{X/k}$ be the relative Frobenius map. Then $F_{X/k*}\mathcal{E}\otimes_{\mathcal{O}_X}\Omega^\bullet_X$ is a complex of $\mathcal{O}_{X^{(p)}}$-modules.*

And in particular for $(\mathcal{O}_X, d)$, we have the following theorem:

Theorem 2.12 (Cartier isomorphism)**.** *There is a unique isomorphism of $\mathcal{O}_{X^{(p)}}$-modules*

$$C^{-1} : \Omega^i_{X^{(p)}} \to \mathcal{H}^i(F_{X/k*}\Omega^\bullet_X)$$

which verifies

- $C^{-1}(1) = 1$
- $C^{-1}(\omega\wedge\tau) = C^{-1}(\omega)\wedge C^{-1}(\tau)$
- $C^{-1}(d\widetilde{F}^*_k(f)) = [f^{p-1}df] \in \mathcal{H}^1(F_{X/k*}\Omega^\bullet_X)$

Proof. Omitted, see [Ka70, Theorem 7.2]. □

Example 2.13. Let $X = \mathbb{A}^1_{\mathbb{F}_3} = spec(\mathbb{F}_3[x])$, then we have $\Omega^1_X = \mathbb{F}_3[x]dx$,

$$d : \mathbb{F}_3[x] \to \mathbb{F}_3[x],\ \sum_{i\geq 0} a_i x^i \mapsto \sum_{i\geq 1} i a_i x^{i-1}.$$

So, $d(\sum_{i\geq 0} a_i x^{3i}) = 0$, $(x^{3i+1})' = x^{3i}$, $(\frac{1}{2}x^{3i+2})' = x^{3i+1}$ and x^{3i+2} has no pre-image. One conclude that $ker(d) = \mathbb{F}_3[x^3] = (\mathcal{O}_X)^p$, $coker(d) = \mathbb{F}_3[x^3]x^2dx$ and there is an isomorphism

$$C^{-1} : \Omega^1_{(\mathcal{O}_X)^p} = \mathbb{F}_3[x^3]dx^3 \longrightarrow coker(d),$$

given by

$$gdf^p \mapsto gf^{p-1}df.$$

where $g \in (\mathcal{O}_X)^p$, and $f \in \mathcal{O}_X$.

2.1.2. *p-curvatures.* In characteristic p cases, the p-th power map will always induces something very important (for example the Frobenius map). Recall the Leibniz rule

$$D^n(gh) = \sum_{i=0}^{n} \binom{n}{i} D^i(g) D^{n-i}(h),$$

where D, g, h are sections of $\mathcal{D}er(X/k) = \mathcal{T}X$, $\mathcal{O}_X$ and $\mathcal{O}_X$ respectively over an open subset of X. Putting $n = p$, one find that

$$D^p(gh) = D^p(g) \cdot h + g \cdot D^p(h),$$

i.e. , that the p-th iterate of a derivation is a derivation.

Let $(\mathcal{E}, \nabla)$ be a flat connection over X, then

$$D \in \mathcal{T}X \mapsto (\nabla_D)^p - \nabla_{(D^p)} \in \mathcal{E}nd_{\underline{k}}(\mathcal{E})$$

defines a map

$$\psi^0_\nabla : \mathcal{T}X \to \mathcal{E}nd_{\underline{k}}(\mathcal{E}),$$

Theorem 2.14 (Basic properties of ψ^0_∇)**.** *Let $(\mathcal{E}, \nabla)$ be a flat connection, $\psi^0_\nabla : \mathcal{T}X \to \mathcal{E}nd_k(\mathcal{E})$ be the map defined as above, then*

1 *For any local section $D \in \mathcal{T}X$, $\psi_\nabla(D)^0$ is $\mathcal{O}_X$ linear, i.e., ψ^0_∇ is a morphism from $\mathcal{T}X$ to $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$.*

2 *$\psi^0_\nabla : \mathcal{T}X \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ is p-linear, i.e., it is additive, and $\psi^0_\nabla(gD) = g^p \psi^0_\nabla(D)$ for any local section g of $\mathcal{O}_X$.*

3 *For any local sections D, D' of $\mathcal{T}X$, $\psi_\nabla(D)$ and $\nabla_{D'}$ commute as element in $\mathcal{E}nd_k(\mathcal{E})$.*

Proof. It is not straight forward to check that the map ψ^0_∇ is additive and p-linear. The readers could consult to [Ka70] to find proofs. A much better treatment is in [BMR08] or in Michael's talk. We just prove 1 and 3

(1) For local section f, e in $\mathcal{O}_X$ and $\mathcal{E}$ respectively,

$$\begin{aligned} \psi^0_\nabla(D)(fe) &= (\nabla_D)^p(fe) - \nabla_{(D^p)}(fe) \\ &= D^p(f)e + f(\nabla_D)^p(e) - (D^p(f)e + f\nabla_{D^p}(e)) \\ &= f\psi^0_\nabla(D)(e) \end{aligned}$$

(2) Not obviously, see [BMR08, Lemma 1.3.1] and the following example.

(3) The question is local on X, so we may assume that X is affine and étale over $\mathbb{A}^d_k$, so that $\mathcal{T}X$ is freely generated by $\partial_1, \cdots, \partial_d$. Let

$$D = \sum_i a_i \partial_i \quad D' = \sum_i b_i \partial_i,$$

we have

$$\begin{aligned}
[\psi^0_\nabla(D), \nabla_{D'}] &= [\psi^0_\nabla(\textstyle\sum_i a_i\partial_i), \nabla_{\sum_j b_j\partial_j}] \\
\text{(by } p\text{-linear property)} &= \textstyle\sum_{i,j}[a_i^p\psi^0_\nabla(\partial_i), b_j\nabla_{\partial_j}] \\
\text{(Leibniz rule for Lie bracket)} &= \textstyle\sum_{i,j} a_i^p b_j[\psi^0_\nabla(\partial_i), \nabla_{\partial_j}] \\
&= \textstyle\sum_{i,j} a_i^p b_j[(\nabla_{\partial_i})^p, \nabla_{\partial_j}] \\
\text{(Leibniz rule for Lie bracket)} &= 0
\end{aligned}$$

□

Remark 2.15. For connection $(\mathcal{E}, \nabla)$, we can choose a finite covering of X consisting of affine open sets $\{U_\alpha\}$ such that each U_α is étale over $\mathbb{A}^d_k$, and on U_α, the sheaf Ω^1_X is free $\mathcal{O}_X$-module, with bases $(dx_1^\alpha, \cdots, dx_d^\alpha)$. On the other hand, the tangent sheaf $\mathcal{T}X = Der(X/k)$ is locally free and generated by $(\frac{\partial}{\partial x_1^\alpha}, \cdots, \frac{\partial}{\partial x_d^\alpha})$. (And we shortly denote $\frac{\partial}{\partial x_i^\alpha}$ by ∂_i) In this case, the p-th iterate of $\frac{\partial}{\partial x_i^\alpha}$ is zero and

$$\psi^0_\nabla\left(\frac{\partial}{\partial x_i^\alpha}\right) = (\nabla_{\frac{\partial}{\partial x_i^\alpha}})^p$$

Thus ψ^0_∇ induce a $\mathcal{O}_X$-linear morphism $\psi_\nabla : F_X^*\mathcal{T}X \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$.

Definition 2.16. Since $F_X^*\mathcal{T}X = F_{X/k}^* \circ \widetilde{F_k}^*\mathcal{T}X = F_{X/k}^*\mathcal{T}X^{(p)}$, we get

$$\psi_\nabla : F_{X/k}^*\mathcal{T}X^{(p)} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}),$$

or equivalently a $\mathcal{O}_X$-linear morphism

$$\psi_\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} F_{X/k}^*\Omega^1_{X^{(p)}},$$

or

$$\psi_\nabla \in \Gamma(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \otimes_{\mathcal{O}_X} F_{X/k}^*\Omega^1_{X^{(p)}}).$$

We call this map to be the p-curvature of ∇.

Remark 2.17. Here Hélène suggest an important remark that the p-curvature map is flat, which is a key step to show Hitchin's map is proper.

More precisely, Flat means that the p-curvature map is a morphism of flat connections from $(\mathcal{E}, \nabla)$ to $(\mathcal{E} \otimes_{\mathcal{O}_X} F_{X/k}^*\Omega^1_{X^{(p)}}, \nabla \otimes id + id \otimes (d \otimes id_{\Omega^1_{X^{(p)}}}))$. This is a corollary of Theorem 2.14 (3).

Example 2.18. Let $X = \mathbb{A}^1_{\mathbb{F}_3} = spec(\mathbb{F}_3[x])$, $\nabla = d + f dx$ is a connection on $\mathcal{O}_X$. Then

$$\begin{aligned}
\psi^0_\nabla(\tfrac{d}{dx}) &= (\tfrac{d}{dx} + f)(\tfrac{d}{dx} + f)(\tfrac{d}{dx} + f) \\
&= (\tfrac{d}{dx})^3 + f(\tfrac{d}{dx})^2 + \tfrac{d}{dx}(f\tfrac{d}{dx}) + \tfrac{d}{dx}(\tfrac{d}{dx}(f\cdot)) \\
&\quad f^2\tfrac{d}{dx} + f\tfrac{d}{dx}(f\cdot) + \tfrac{d}{dx}(f^2\cdot) + f^3 \\
&= 0 + f(\tfrac{d}{dx})^2 + f'\tfrac{d}{dx} + f(\tfrac{d}{dx})^2 + f'' + 2f'\tfrac{d}{dx} + f(\tfrac{d}{dx})^2 \\
&\quad f^2\tfrac{d}{dx} + ff' + f^2\tfrac{d}{dx} + 2ff' + f^2\tfrac{d}{dx} + f^3 \\
&= f'' + f^3
\end{aligned}$$

Remark that since the derivations are not commutative, we don't have freshman's dream for derivations.

Exercise In non commutative ring $\mathbb{Z}\{x, y, z\}$, one can calculate that

$$(x+y)^3 - (x^3 + 3x^2y + 3xy^2 + y^3) = [[y,x],x] + [y,[y,x]] + 3x[y,x] + 3[y,x]y,$$

i.e.,

$$(x+y)^3 = x^3 + y^3 + \text{Lie bracket} + 3(\dots).$$

Then, one have:

$$\begin{aligned}(\nabla_{D_1} + \nabla_{D_2})^3 &= \nabla_{D_1}^3 + \nabla_{D_2}^3 + [[\nabla_{D_2}, \nabla_{D_1}], \nabla_{D_1}] + [\nabla_{D_2}, [\nabla_{D_2}, \nabla_{D_1}]], \\ \nabla_{(D_1+D_2)^3} &= \nabla_{D_1}^3 + \nabla_{D_2}^3 + \nabla[[D_2, D_1], D_1] + \nabla[D_2, [D_2, D_1]],\end{aligned}$$

which imply

$$\psi_\nabla^0(D_1 + D_2) = \psi_\nabla^0(D_1) + \psi_\nabla^0(D_2).$$

On the other hand,

$$\begin{aligned}\nabla_{\lambda\frac{d}{dx}}\nabla_{\lambda\frac{d}{dx}}\nabla_{\lambda\frac{d}{dx}} &= \lambda\nabla_{\frac{d}{dx}}(\lambda\nabla_{\frac{d}{dx}}(\lambda\nabla_{\frac{d}{dx}})) \\ &= \lambda\nabla_{\frac{d}{dx}}(\lambda\lambda'\nabla_{\frac{d}{dx}} + \lambda^2\nabla^2_{\frac{d}{dx}}) \\ &= \lambda(\lambda\lambda')'\nabla_{\frac{d}{dx}} + \lambda^2\lambda'\nabla_{\frac{d}{dx}} + 2\lambda^2\lambda'\nabla^2_{\frac{d}{dx}} + \lambda^3\nabla^3_{\frac{d}{dx}} \\ &= (\lambda\lambda'^2 + \lambda^2\lambda'')\nabla_{\frac{d}{dx}} + \lambda^3\nabla^3_{\frac{d}{dx}}\end{aligned}$$

$$\lambda\frac{d}{dx}(\lambda\frac{d}{dx})(\lambda\frac{d}{dx}) = (\lambda\lambda'^2 + \lambda^2\lambda'')\frac{d}{dx} + \lambda^3\frac{d}{dx}^3$$

So, we have

$$\psi_\nabla^0(\lambda\frac{d}{dx}) = \lambda^p\psi_\nabla^0(\frac{d}{dx}).$$

Definition 2.19. We say a flat connection $(\mathcal{E}, \nabla)$ is **nilpotent** of exponent $\leq n$, if for any local sections $D_1, \cdots, D_n$ of $\mathcal{T}X$,

$$\psi_\nabla(D_1)\cdots\psi_\nabla(D_n) = 0.$$

Flat connections of nilpotent $\leq n$ form a full subcategory of $\mathbf{MIC}(X)$, denoted by $\mathbf{MIC}_{\leq n}(X)$. We further define

$$\mathbf{MICN}(X) := \bigcup_{n\geq 1}\mathbf{MIC}_{\leq n}(X).$$

Proposition 2.20. *The following statements are equivalent:*

(1) $(\mathcal{E}, \nabla) \in \boldsymbol{MIC}_{\leq n}(X)$.
(2) *There exists a covering of X by affine open subsets $\{U_\alpha\}$, each U_α is étale over $\mathbb{A}_k^d$ with coordinates $x_1, \cdots, x_d$ such that for every r-tuple $(w_1, \cdots, w_r)$ of integers with $\sum_i w_i = d$,*

$$\nabla_{\partial_1}^{pw_1}\cdots\nabla_{\partial_d}^{pw_r} = 0$$

(3) *There exists a filtration in* $\in \boldsymbol{MIC}(X)$ *of* $(\mathcal{E}, \nabla)$ *of length* $\leq n$ *whose associated graded objects have* p*-curvature zero.*

Therefore for any flat connection $(\mathcal{E}, \nabla)$ *of exponent* $\leq n$*, there is an exact sequence*

$$0 \to (\mathcal{E}', \nabla') \to (\mathcal{E}, \nabla) \to (\mathcal{E}'', \nabla'') \to 0$$

in $\boldsymbol{MIC}(X)$*, so that* $\mathcal{E}' \in \boldsymbol{MIC}_{\leq 1}(X)$ *and* $\mathcal{E}'' \in \boldsymbol{MIC}_{\leq n-1}(X)$

Proof. p-linearity implies (1) iff (2).

(1)imply(3): $\psi_\nabla(\mathcal{T}X)(\mathcal{E}) : \mathcal{F}^1$ is a sub $\mathcal{O}_X$-module of $\mathcal{E}$. Since p-curvature commutes with ∇, $(\mathcal{F}^1, \nabla|_{\mathcal{F}^1})$ is well defined and nilpotent of exponent $\leq n-1$. Do induction, we get the filtration.

(3)imply(1): If we have exact sequence of flat connections

$$0 \to (\mathcal{E}', \nabla') \to (\mathcal{E}, \nabla) \to (\mathcal{E}'', \nabla'') \to 0$$

then,

$$[\psi_\nabla(D)(e)] = \psi_{\nabla''}(D)([e]) \in \mathcal{E}''.$$

Therefore, for any local sections $D_1, \cdots, D_n$ of $\mathcal{T}X$ and e of $\mathcal{E}$, $[\psi_\nabla(D_1) \cdots \psi_\nabla(D_n)(e)] = 0$ in each graded object. So, we have $\psi_\nabla(D_1) \cdots \psi_\nabla(D_n) = 0$

□

Proposition 2.21.

- $\boldsymbol{MIC}_{\leq n}(X)$ *is an abelian category embed in* $\boldsymbol{MIC}(X)$*,*
- $\boldsymbol{MICN}(X)$ *is stable under internal* $\boldsymbol{Hom}$ *and internal tensor. If* A*,* B *are objects of* $\boldsymbol{MIC}_{\leq n}(X)$ *and* $\boldsymbol{MIC}_{\leq m}(X)$ *respectively, then* $A \otimes B$ *and* $\boldsymbol{Hom}(A, B)$ *are both in* $\boldsymbol{MIC}_{\leq m+n-1}(X)$*.*

In particular, for $\mathbf{MIC}_{\leq 1}$, we have

Theorem 2.22 (Cartier descent)**.** *There is an equivalence of categories between* $\boldsymbol{MIC}_{\leq 1}(X)$ *an* $Coh(X^{(p)})$ *given as follows:*

$$C : (\mathcal{E}, \nabla) \in \boldsymbol{MIC}_{\leq 1}(X) \mapsto F_{X/k*}(\mathcal{E}^\nabla) \in Coh(X^{(p)}),$$

$$C^{-1} : \mathcal{F} \in Coh(X^{(p)}) \mapsto (F^*_{X/k}\mathcal{F}, \nabla^{can}) \in \boldsymbol{MIC}_{\leq 1}(X)$$

the canonical connection associated to $\mathcal{F}$ *is defined to be* $\nabla^{can} := d_{\mathcal{O}_X} \otimes id_{F^{-1}_{X/k}\mathcal{F}}$*.*

Proof. We can directly check that $(\mathcal{O}_X, d)$ is a flat connection with zero p-curvature, so the canonical connection is integrable and of p-curvature zero on $F^*_{X/k}\mathcal{F}$, so C, C^{-1} are well defined.

$\mathbf{C} \circ \mathbf{C^{-1}} = \mathbf{id}$. Tensor $F^{-1}_{X/k}(\mathcal{F})$ with the exact sequence

$$0 \to (\mathcal{O}_X)^p \to \mathcal{O}_X \to \Omega^1_X,$$

we get

$$0 \to \mathcal{F} \otimes_{(\mathcal{O}_X)^p} (\mathcal{O}_X)^p \to \mathcal{F} \otimes_{(\mathcal{O}_X)^p} \mathcal{O}_X \to \Omega^1_X \otimes_{(\mathcal{O}_X)^p} \mathcal{F}$$

push forward by $F_{X/k*}$, we have

$$\mathcal{F} \cong F_{X/k*}((F_{X/k}^{*}\mathcal{F})^{\nabla_{can}}).$$

$\mathbf{C} \circ \mathbf{C^{-1}} = \mathbf{id}$. The following diagram is commutative

$$\begin{array}{ccc} F_{X/k}^{*}(\mathcal{E}^{\nabla}) & \xrightarrow{m} & \mathcal{E} \\ \downarrow & & \downarrow \varphi \\ F_{X/k}^{*}(\mathcal{E}^{\nabla}) \otimes_{\mathcal{O}_X} \Omega_X^1 & \longrightarrow & \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^1. \end{array} \qquad \begin{array}{ccc} f \otimes e & \longmapsto & fe \\ \downarrow & & \downarrow \\ df \otimes e & \longmapsto & dfe. \end{array}$$

So to show $(\mathcal{E}, \nabla) \cong (F_{X/k}^{*}(\mathcal{E}^{\nabla}), \nabla_{can})$, it is equivalent to show $m : F_{X/k}^{*}(\mathcal{E}^{\nabla}) \to \mathcal{E}$ is an isomorphism. The question is local on X, so we may suppose X is affine and étale over $\mathbb{A}_k^d$, with $\mathcal{T}X$ freely generated by $\partial_1, \cdots, \partial_d$. We can check the isomorphism stalk-wise, and by fully faithfulness of completion, the problem reduce to the case $X = spec(k[|x_1, \cdots, x_d|])$. In this case, one can construct the inverse map T of m. We show $d = 1$ case and the general cases are similar.

First, let us consider $F_{X/k}^{-1}(\mathcal{O}_{X^{(p)}})$-linear morphism P on $\mathcal{E}$, given by

$$P := \sum_{0 \leq w \leq p-1} \frac{(-x)^w}{w!} (\nabla_{\partial_x})^w$$

then, one have

$$\begin{aligned} \nabla_{\partial_x} P(e) &= \nabla_{\partial_x} \textstyle\sum_{0 \leq w \leq p-1} \frac{(-x)^w}{w!} (\nabla_{\partial_x})^w \\ &= \textstyle\sum_{1 \leq w \leq p-1} -\frac{(-x)^{(w-1)}}{(w-1)!} (\nabla_{\partial_x})^w + \sum_{0 \leq w \leq p-2} \frac{(-x)^w}{w!} (\nabla_{\partial_x})^{(w+1)} \\ &= 0 \end{aligned}$$

and if $e \in \mathcal{E}^{\nabla}$,

$$P(e) = \sum_{0 \leq w \leq p-1} \frac{(-x)^w}{w!} (\nabla_{\partial_x})^w(e) = 1 \cdot e = e$$

Then, we can define

$$T := \sum_{0 \leq \ell \leq p-1} \frac{x^\ell}{\ell!} \otimes P \circ (\nabla_{\partial_x})^\ell.$$

$\mathbf{m} \circ \mathbf{T} = \mathbf{id}$. Degree n term of the equality $e^x e^y = e^{x+y}$ give us

$$\sum_{\ell+w=n} \frac{x^\ell}{\ell!} \frac{y^w}{w!} = \frac{(x+y)^n}{n!}.$$

Then with this equality, one easily checks $m \circ T = id$.
$\mathbf{T} \circ \mathbf{m} = \mathbf{id}$. In our case since $X = spec(k[|x|])$, $F_{X/k}^{*}(\mathcal{E}^{\nabla})$ is generated as k-vector space by

$$\{x^i \otimes e | 0 \leq i \leq (p-1),\ e \in \mathcal{E}^{\nabla}\}.$$

We just have to show $T(x^i \cdot e) = x^i \otimes e$.

$$\begin{aligned}
T(x^i \cdot e) &= \textstyle\sum_{0\le \ell \le p-1} \frac{x^\ell}{\ell!} \otimes P \circ (\nabla_{\partial_x})^\ell (x^i \cdot e) \\
&= \textstyle\sum_{0\le \ell \le p-1} \frac{x^\ell}{\ell!} \otimes \sum_w \frac{(-x)^w}{w!} (\nabla_{\partial_x})^w \circ (\nabla_{\partial_x})^\ell (x^i \cdot e) \\
&= \textstyle\sum_{0\le \ell, w \le p-1} \frac{x^\ell}{\ell!} \otimes \frac{(-x)^w}{w!} (\nabla_{\partial_x})^{(w+\ell)} (x^i \cdot e) \\
&= \textstyle\sum_{0\le n \le p-1} \sum_{\ell+w=n} \frac{x^\ell}{\ell!} \otimes \frac{(-x)^w}{w!} (\nabla_{\partial_x})^n (x^i \cdot e) \\
&= \textstyle\sum_{0\le n \le p-1} \frac{(x\otimes 1 - 1\otimes x)^n}{n!} (\nabla_{\partial_x})^n (x^i \cdot e) \\
&= \textstyle\sum_{0\le n \le p-1} \frac{(x\otimes 1 - 1\otimes x)^n}{n!} (\partial_x^n (x^i) \cdot e) \\
&= \textstyle\sum_{0\le n \le i} \frac{(x\otimes 1 - 1\otimes x)^n}{n!} (\frac{i!}{(i-n)!} (1\otimes x)^{n-i} \cdot e) \\
&= (x\otimes 1 - 1\otimes x + 1 \otimes x)^i e \\
&= x^i \otimes e.
\end{aligned}$$

For dimension $d > 1$ case, we define

$$P := \sum_{0\le w_1,\cdots,w_d \le p-1} \Pi_{i=1}^d \frac{(-x)^{w_i}}{w_i!} \Pi_{i=1}^d (\nabla_{\partial_i})^{w_i},$$

and

$$T := \sum_{0\le \ell_1,\cdots,\ell_d \le p-1} \Pi_{i=1}^d \frac{x_i^{\ell_i}}{\ell_i!} \otimes P \circ \Pi_{i=1}^d (\nabla_{\partial_i})^{\ell_i}.$$

□

2.2. **Higgs sheaves in characteristic p.**

Definition 2.23. A **Higgs sheaf** is defined to be a pair $(\mathcal{E}, \theta)$, where $\mathcal{E}$ is a coherent sheaf on X and

$$\theta : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^1$$

(we call it a Higgs field) is a morphism of $\mathcal{O}_X$-modules, satisfying $\theta \wedge \theta = 0$. Here $\theta \wedge \theta$ is the composition of

$$\mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^1 \xrightarrow{\theta \otimes id_{\Omega_X}} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^1 \otimes_{\mathcal{O}_X} \Omega_X^1 \xrightarrow{id_{\mathcal{E}} \otimes \wedge} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^2.$$

Then we get a complex of coherent sheaves, the **Dolbeault complex**:

$$\mathcal{E} \xrightarrow{\theta\wedge} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^1 \xrightarrow{\theta\wedge} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^2 \xrightarrow{\theta\wedge} \cdots$$

The condition $\theta \wedge \theta = 0$ insures that this is a complex. Define the **Dolbeault cohomology** with coefficients in $\mathcal{E}$ to be the hyper cohomology

$$H^i_{Dol}(X, E) := R^i\Gamma(X, \mathcal{E} \xrightarrow{\theta\wedge} \mathcal{E} \otimes_{SO_X} \Omega_X^1 \xrightarrow{\theta\wedge} \cdots)$$

Let $(\mathcal{E}, \theta)$ and $(\mathcal{F}, \eta)$ be two Higgs sheaves, a morphism between $(\mathcal{E}, \theta)$ and $(\mathcal{F}, \eta)$ is a morphism $\varphi : \mathcal{E} \to \mathcal{F}$ of $\mathcal{O}_X$-modules so that it preserve the Higgs fields. i.e. $\varphi \otimes id_{\Omega_X^1} \circ \theta = \eta \circ \varphi$. Therefore Higgs sheaves form an abelian category denoted by $\mathbf{HIG}(X)$.

The category has an internal Hom functor and a tensor functor as follows:

$$(\mathcal{E}, \theta) \otimes (\mathcal{F}, \theta') := (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}, \theta \otimes id_{\mathcal{F}} + id_{\mathcal{E}} \otimes \theta').$$

$$\mathbf{Hom}((\mathcal{E},\theta),(\mathcal{F},\theta')) := (\mathcal{H}om(\mathcal{E},\mathcal{F}),\theta'').$$

Here θ'' is defined by the formula:

$$\theta''_D(\varphi)(e) = \theta'_D(e) - \varphi(\theta_D(e)),$$

where D, φ and e are local sections of $\mathcal{T}X$, $\mathcal{H}om(\mathcal{E},\mathcal{F})$ and $\mathcal{E}$ respectively.

A Higgs sheaf $(\mathcal{E},\theta)$ is said to be **nilpotent** of exponent $\leq n$ if for all local sections $D_1, \ldots, D_n$ of $\mathcal{T}X$,

$$\theta(D_1)\cdots\theta(D_n) = 0.$$

Denote $\mathbf{HIG}_{\leq n}$ the full subcategory of nilpotent Higgs sheaves of exponent $\leq n$. Note that $\mathbf{HIG}_1$ is just the category of coherent $\mathcal{O}_X$-modules. We further define

$$\mathbf{HIGN}(X) := \bigcup_{n\geq 1} \mathbf{HIG}_{\leq n}(X).$$

Proposition 2.24.

- *$\boldsymbol{HIG}_{\leq n}(X)$ is an abelian category embed in $\boldsymbol{HIG}(X)$,*
- *$\boldsymbol{HIGN}(X)$ is closed under internal **Hom** and internal tensor. If A, B are objects of $\boldsymbol{HIG}_{\leq n}(X)$ and $\boldsymbol{MIC}_{\leq m}(X)$ respectively, then $A\otimes B$ and $\boldsymbol{Hom}(A,B)$ are both in $\boldsymbol{MIC}_{\leq m+n-1}(X)$.*

Proposition 2.25. *If $(\mathcal{E},\theta) \in \boldsymbol{HIG}_{\leq n}$, then there exists a filtration in $\boldsymbol{HIG}(X)$ of length $\leq n$ whose associated graded objects have Higgs fields zero.*

Therefore for any Higgs sheaf $(\mathcal{E},\theta)$ of exponent $\leq n$, there is an exact sequence

$$0 \to (\mathcal{E}',\theta') \to (\mathcal{E},\theta) \to (\mathcal{E}'',\theta'') \to 0$$

in $\mathbf{HIG}(X)$, so that $\mathcal{E}' \in \mathbf{HIG}_{\leq 1}(X)$ and $\mathcal{E}'' \in \mathbf{HIG}_{\leq n-1}(X)$

In particular, if a Higgs sheaf $(\mathcal{E},\theta)$ of exponent ≤ 2, there is an exact sequence

$$0 \to (\mathcal{E}',\theta') \to (\mathcal{E},\theta) \to (\mathcal{E}'',\theta'') \to 0$$

in $\mathbf{HIG}(X)$, so that $\mathcal{E}',\mathcal{E}'' \in \mathbf{HIG}_{\leq 1}(X)$. Then the Higgs field of $\mathcal{E}$ uniquely determines a morphism $\varphi : \mathcal{E}'' \to \mathcal{E}$, and vice versa.

Proposition 2.26. *$\mathcal{F},\mathcal{Q} \in \boldsymbol{HIG}_{\leq 1}(X)$. Then their extensions of Higgs sheaves can be characterised by*

$$Ext^1((\mathcal{Q},0),(\mathcal{F},0)) \cong Ext^1_{\mathcal{O}_X}(\mathcal{Q},\mathcal{F}) \oplus Hom(\mathcal{Q},\mathcal{F}\otimes_{\mathcal{O}_X}\Omega^1_X).$$

3. Deligne-Illusie's (decomposition) splitting theorem

Definition 3.1. Let $\mathcal{A}$ be an abelian category and $D^b(\mathcal{A})$ be the bounded derived category associate to $\mathcal{A}$. $L \in D^b(\mathcal{A})$ is said to be **split** or **decomposable** if L is quasi-isomorphic to a complex with zero differential. In other words, there is an quasi isomorphism:

$$L \cong \bigoplus H^i(L)[-i]$$

Let X be a smooth quasi-projective variety over a perfect field k, $char(k) = p > 0$. We say that X can be lifted to $W_2(k)$, if there is a scheme $\widetilde{X}$ smooth over $spec(W(2))$ lying in the following Cartesian diagram:

$$\begin{array}{ccc} X & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ spec(k) & \xrightarrow{i} & spec(W_2(k)). \end{array}$$

Theorem 3.2 (Deligne-Illusie [DI87])**.** *If a smooth quasi-projective variety X can be lifted to $W_2(k)$, then $\tau_{\leq p-1}F_{X/k*}\Omega^\bullet_X$ is decomposable in $D^b(X^{(p)})$. i.e. there is an quasi-isomorphism*

$$\bigoplus_{i\leq p-1} \mathcal{H}^i(F_{X/k*}\Omega^\bullet_X)[-i] \cong \tau_{\leq p-1}F_{X/k*}\Omega^\bullet_X,$$

compose it with the Cartier isomorphism, we have quasi-isomorphic

$$\bigoplus_{i\leq p-1} \Omega^i_{X^{(p)}}[-i] \cong \tau_{\leq p-1}F_{X/k*}\Omega^\bullet_X,$$

Corollary 3.3. *If $dim(X) < p$, then the Frobenius push-forward of de Rham complex on X is decomposable.*

4. An observations from Hélène

Theorem 4.1 (Hélène)**.** *Let X be a smooth quasi-projective variety which can be lifted to $W_2(k)$, then there is an equivalence of categories from flat sheaves of nilpotent ≤ 2 on X to Higgs sheaves of nilpotent ≤ 2 on $X^{(p)}$.*

Proof. For convenient, we assume $p > dim(X)$ and flat sheaf $(\mathcal{E}, \nabla) \in \mathbf{MIC}_{\leq 2}(X)$ has a locally free quotient $\mathcal{Q}$. i.e. there is an exact sequence

$$0 \to (\mathcal{F}, \nabla') \to (\mathcal{E}, \nabla) \to (\mathcal{Q}, \nabla'') \to 0$$

with $(\mathcal{F}, \nabla'), (\mathcal{Q}, \nabla'') \in \mathbf{MIC}_{\leq 1}(X)$. By Cartier descent, there exist $\mathcal{F}^0, \mathcal{Q}^0 \in \mathbf{HIG}_{\leq 1}(X^{(p)})$, such that

$$(\mathcal{F}, \nabla') = (F^*_{X/k}\mathcal{F}^0, \nabla'_{can}), (\mathcal{Q}, \nabla'') = (F^*_{X/k}\mathcal{Q}^0, \nabla''_{can}).$$

Thus, $\mathbf{Hom}((\mathcal{Q},\nabla''),(\mathcal{F},\nabla')) = (F^*_{X/k}\mathcal{H}om(\mathcal{Q}^0,\mathcal{F}^0),\nabla^h_{can})$. And we have the following computation:

$$\begin{aligned}
Ext^1((\mathcal{Q},\nabla''),(\mathcal{F},\nabla')) &= H^1_{dR}(X,\mathcal{H}om(\mathcal{Q},\mathcal{F}),\nabla^h)\\
&= H^1_{dR}(X,F^*_{X/k}\mathcal{H}om(\mathcal{Q}^0,\mathcal{F}^0),\nabla^h_{can})\\
&= R^1\Gamma(X,F^*_{X/k}\mathcal{H}om(\mathcal{Q}^0,\mathcal{F}^0)\otimes_{\mathcal{O}_X}\Omega^\bullet_X,\nabla^h_{can})\\
&= R^1\Gamma(X^{(p)},F_{X/k*}(F^*_{X/k}\mathcal{H}om(\mathcal{Q}^0,\mathcal{F}^0)\otimes_{\mathcal{O}_X}\Omega^\bullet_X))\\
&= R^1\Gamma(X^{(p)},\mathcal{H}om(\mathcal{Q}^0,\mathcal{F}^0)\otimes_{\mathcal{O}_{X^{(p)}}} F_{X/k*}\Omega^\bullet_X, id\otimes F_{X/k*}d)\\
&\quad\text{(the de Rham complex split)}\\
&= R^1\Gamma(X^{(p)},\mathcal{H}om(\mathcal{Q}^0,\mathcal{F}^0)\otimes(\textstyle\bigoplus_i\Omega^i_{X^{(p)}}[-i]))\\
&= Ext^1_{\mathcal{O}_{X^{(p)}}}(\mathcal{Q}^0,\mathcal{F}^0)\oplus Hom_{\mathcal{O}_{X^{(p)}}}(\mathcal{Q}^0,\mathcal{F}^0\otimes_{\mathcal{O}_X^{(p)}}\Omega^1_{X^{(p)}}).
\end{aligned}$$

This gives an Higgs sheaf $(\mathcal{H},\theta)$ and vice versa. □

Remark 4.2. This proof uses Cartier descent and Deligne-Illusie's splitting theorem.

The assumption of $dim(X) > p$ is not necessary, because we can use truncated de Rham complex to compute H^1.

Open problems

1 From this computation, can we get the Lan-Sheng-Zuo's explicit construction with Cech cocycle classes?
2 Can we drop off the assumption that $\mathcal{Q}$ is locally free?
3 Can we prove the characteristic p Simpson correspondence following Simpson's way. Simpson showed his correspondence by first give the correspondence on simple objects, then compare the cohomology groups, and finally give the correspondence for all extensions. In our case, the first step is Cartier descent, What about the next two?

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