SEMINAR WS 16/17
BERKOVICH SPACES, BIRATIONAL GEOMETRY AND MOTIVIC ZETA FUNCTIONS

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1. Description

The aim of the seminar is to understand the paper *Poles of maximal order of motivic zeta functions* [NX15] by Nicaise-Xu.

As motivation, we report the abstract of a talk on this topic Nicaise gave in Hannover in July, 2016:

Igusas p-adic zeta function $Z(s)$ attached to an integer polynomial f in N variables is a meromorphic function on the complex plane that encodes the numbers of solutions of the equation $f = 0$ modulo powers of a prime p. It is expressed as a p-adic integral, and Igusa proved that it is rational in p^{-s} using resolution of singularities and the change of variables formula. From this computation it is immediately clear that the order of a pole of $Z(s)$ is at most N, the number of variables in f. In 1999, Wim Veys conjectured that the real part of every pole of order N equals minus the log canonical threshold of f (which is an invariant of the singularities of f).

Nicaise and Xu prove Veys' conjecture by means of Berkovich skeletons and birational geometry. This seminar roughly covers the papers [MN13, NX15] and includes short introductions to Berkovich spaces, the minimal model program, p-adic and motivic zeta functions.

2. The talks

2.1. Introduction. (27.10. Marta)

Introduce Veys' conjecture both in the p-adic and in the motivic setting. Explain the connection with Berkovich spaces, state the results [NX15, Theorem 2.4] and [NX15, Theorem 4.10], and explain their role in the proof of the conjecture.

If time permits mention also the results in [NX15, §5].

2.2. Birational geometry. (3.11. Efstathia)

Use [CKL11], [Fuj11], [KM98] and [BCHM10] as main references, and [Deb01, §7] as a gentle introduction to the subject.

Singularities Introduce the notion of log resolution [KM98, (10) at p. 10] and of discrepancy [Fuj11, §4.4] Introduce terminal, canonical, log terminal and log canonical pairs in terms of a log resolution [KM98, Definition 2.34], [BCHM10, p. 25].

Introduce dlt pairs [KM98, Theorem 2.44], [BCHM10, p. 25].

Introduce log canonical centers [Fuj11, Definitions 4.5–4.6].

Introduce the log canonical threshold [CKL11, Definition 2.21], [Kol97, §8], [NX15, (2.1)] and explain why the definitions are equivalent.

Minimal model program Introduce the cone of curves and of relative curves [Deb01, §1.11-1.12].

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Introduce ample, semiample, nef, big, effective and pseudoeffective divisors and the relations between their cones and with the cone of curves (see for example [Cos10, §1] or [Laz04]).

State the Negativity Lemma [KM98, 3.39].

State the Base-point-free theorem [CKL11, Theorem 3.4], [Fuj11, Theorem 13.1] (cf. [Deb01, Theorem 7.32]).

Recall the notion of extremal ray, see [Fuj11, Definition 16.2] for example. State the Cone theorem [CKL11, Theorem 3.1], [Fuj11, Theorem 16.6] (cf. [Deb01, Theorem 7.38]).

State the Contraction theorem [CKL11, Theorem 3.2], [Fuj11, Theorem 16.4] (cf. [Deb01, Theorem 7.39]) and sketch briefly its proof.

Introduce the three types of contractions that can be obtained [CKL11, 3.7–3.8] (cf. [Deb01, 7.42]), explain that divisorial contractions and fibrations produce a $\mathbb{Q}$-factorial variety with mild singularities but small contractions don't.

Introduce the notion of flip [CKL11, 4.8–4.9]. Explain the problem of termination of flips (cf. sentence after proof of [Deb01, Proposition 7.44]).

Introduce the notion of relative MMP [CKL11, §3 at p. 9] and explain [CKL11, Figure 1 p.19].

Introduce the notion of relative $(K_X + \Delta)$-MMP with scaling [CKL11, Definition 4.18] (cf. [BCHM10, Lemma 3.10.9] and the sentence thereafter).

State [BCHM10, Theorem 1.1] in the relative setting (see [CKL11, Theorem 4.19] and [BCHM10, Theorems C and F(1) in §2]).

2.3. **Berkovich spaces.** (3.11. Fabio, and 10.11. Wouter)

- **Berkovich spaces.** Introduce Banach rings and the spectrum of a Banach ring and bounded morphisms [Ber09, 1.2.1–1.2.5] (cf. [Ber90, §1]). Mention some examples from [Ber09, 1.2.2] and present [Ber09, 1.2.2 (v)] in detail. Explain [Ber90, Theorem 1.2.1].

  Introduce the analytic affine space [Ber09, 1.3.1–1.3.2], [Ber90, §1.5].

  Describe the points of the affine line [Ber09, 1.3.6] (cf. [Ber90, §1.4.4] and [Wer13, p. 7–10]).

  Introduce (strictly) affinoid algebras [Ber90, §2.1] (cf. [Ber09, 2.1]), explain [Ber90, Proposition 2.1.3].

  Introduce affinoid domains [Ber90, §2.2] and special subsets [Ber90, p. 30].

  Introduce affinoid spaces [Ber90, §2.3].

  Introduce Berkovich analytic spaces [Ber90, §3.1], including [Ber90, Theorem 3.2.1].

  Berkovich analytification. Introduce Berkovich analytification [Ber90, §3.4] with proof of [Ber90, Theorem 3.4.1]. In particular show that the analytic affine space previously introduced is the analytification of the affine space. State the results [Ber90, 3.4.2–3.4.14] and give an idea of some proofs.

  Justify the equivalent description of the topological space of the analytification given in [Nic14, (2.1.1)].

  Present example of Tate curve with description of points and skeleton [Bak08, 5.2] (cf. [Ber90, p. 84]), use [Ber90, 4.3.2, 4.3.3].

2.4. **Models and monomial points.** (24.11. + 30 min. on 1.12. Shane)

Use the notation from [NX15].

Notation: We work over a complete DVR [MN13, §1.9], with case of special interest given by the ring of formal power series $k[[t]]$. For the sets of birational, divisorial and monomial points use the notation $X^{\mathrm{bir}}, X^{\mathrm{div}}, X^{\mathrm{mon}}$ of [Nic14] and [NX15].

Introduce birational points: see [MN13, (2.1.2)] and [Nic14, (2.1.3)]. Write X^{bir} for the set of birational points.

Introduce (formal) R-models and the specialization map (also called reduction map) [MN13, (2.2.1)–(2.2.2)], [NX16, (2.1.2)], [Nic14, (2.2.2)–(2.2.4)] (cf. [Ber09, 4.3.6]). Denote the generic fiber by $\widehat{\mathscr{X}_K}$ and the specialization map by $\mathrm{sp}_{\mathscr{X}}$.

Introduce *snc* models over R [MN13, (2.2.3)–(2.2.5)], [Nic14, (2.2.1)] and over a curve [NX16, §2.2] and [Nic14, (2.2.1)].

Introduce divisorial points [MN13, (2.4.1)] and [Nic14, (2.3.1)].

Introduce monomial points [MN13, (2.4.2)–(2.4.9)], [Nic14, (2.3.2)–(2.3.10)]. It suffices to sketch the definition of v_α in [MN13, Proposition 2.4.6]. Present [MN13, (2.4.10)]. Prove the equivalence of (1) and (2) in [MN13, Proposition 2.4.11].

Prove [MN13, Proposition 2.4.12] and introduce from [MN13, §2.3] the Zariski-Riemann space and the properties that are needed for the proof.

2.5. **Skeleton.** (1.12. 60 min. Pedro)

Introduce the Berkovich skeleton $Sk(\mathscr{X})$ associated to a *snc* model and describe its relation to the dual intersection complex of the special fiber $\mathscr{X}_k$ [MN13, (3.1.2)–(3.1.3], prove [MN13, Proposition 3.1.4] (cf. [Nic14, (2.4.1)–(2.4.6)]).

Revisit the example of the Tate curves and explain the relation between divisorial/monomial points and the points of type (1)–(4) on Berkovich spaces.

2.6. **Retraction.** (8.12. Simon)

Construct the retraction map $\rho_{\mathscr{X}}$ [MN13, (3.1.5)] [NX16, (3.1.1)–(3.1.2)]. Present [NX16, Theorem 3.1.3], explaining the proof, and [NX16, (3.1.4)]. See also [Nic14, (2.4.8)–(2.4.13)].

Present [MN13, (3.1.6)–(3.2.4)]. We are mostly interested in the piecewise affine structure.

2.7. **Weight function.** (15.12. Lei, and 5.1. Fei)

Present [MN13, §4.2–4.4].

2.8. **Essential skeleton and relation to birational geometry.** (12.1. Marcin)

Present [MN13, §4.5–4.6] and [MN13, §6.1, §6.3–6.4].

2.9. **The weight function computes the log canonical threshold.** (19.1.)

Local: Recall from [NX15, (4.1)–(4.4)] what needed to state [NX15, Theorem 2.4 at p. 11], prove [NX15, Theorem 2.4].

2.10. **Evidence towards existence of flow.** (26.1.)

Local: Recall what needed from [NX15, (4.1)–(4.4)]. Present [NX15, (4.5)–(4.9)], prove [NX15, Theorem 4.10]. Use [dFKX14] for the definitions and the results mentioned in the proof. Deduce [NX15, Corollary 4.13] and present [NX15, Remark 4.14].

If time permits, mention the examples: [NX15, (4.15)–(4.16)].

2.11. ***p*-adic integration and the Igusa zeta function.** (2.2. Tanya)

The references are Igusa's original articles [Igu74, Igu75], Igusa's book [Igu00], Denef's Bourbaki survey [Den91], Nicaise's survey [Nic10].

Introduce the basics of p-adic integration: definition of Haar measure on a p-adic field and of p-adic integral, change of variables formula [Igu00, §7.4] (cf. [Nic10, §2.2], [Igu74, §1]).

Define the the Igusa zeta function $Z_f(s)$ associated to a polynomial f [Nic10, §2.1,§2.3], [Igu75, (68)], [Den91, §1.1-1.2]. We are mainly interested in the case where $\chi = \chi_{\mathrm{triv}}$ and Φ is the characteristic function of R^n (because of [Nic10, §2.3] and [Den91, §1.2]).

Explain the proof of rationality of $Z_f(s)$ and the explicit formula via resolution of singularities. References: [Nic10, §2.4], [Den91, §1.3], [Igu75, Theorem 1], [Igu00, §11.5].

2.12. **Motivic integration and motivic zeta functions.** (9.2. Yun)

The main reference is [DL00]. See [Nic10, §4–5] for a short survey and [Loo02] for a long survey.

Introduce the $\hat{\mu}$-equivariant Grothendieck ring of varieties [DL00, §2.4], [Loo02, §5] (see [DL00, §2.3], [Nic10, §4.2] and [Loo02, §2] for the usual Grothendieck ring of varieties).

Introduce arc spaces [DL00, §2.1], [Nic10, §4.3] and [Loo02, §2] (we need only the X smooth case).

Introduce the basics of motivic integration (definition of motivic measure and of motivic integral, transformation formula, whatever else is needed to explain the proof of [Loo02, Theorem 5.4]). References: [Loo02, §2–3], [Nic10, §4.4–4.5].

Define the motivic zeta function associated to a hypersurface. References: [DL00, §3.2], [Nic10, §5.1], [Loo02, §5]. Define the naive zeta function associated to a hypersurface [DL00, §3.2].

Explain the proof of the explicit formula for the motivic zeta function [Loo02, Theorem 5.4]. Deduce the explicit formula for the naive zeta function [DL00, Corollary 3.3.2].

Define the topological zeta function associated to a hypersurface [DL00, §3.4].

2.13. **Monodromy conjectures and proof of Veys' conjecture.** (16.2. Marco)

Introduce local monodromy and Bernstein polynomials [Den91, §2.1–2.2].

State Igusa's monodromy conjectures [Den91, §2.3], [Nic10, §3.2].

State the motivic monodromy conjecture [DL00, Conjecture 3.4.1], [Nic10, §5.2].

Explain the relation between the motivic and the p-adic zeta functions [Nic10, §5.3].

Explain the relation between the Igusa zeta function and the topological zeta function [Den91, §4.3].

Introduce the log canonical threshold and Veys' conjecture References: [Den91, §6], [LV99, Introduction], [NX15, (1.2)].

Present [NX15, §3].

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