FORSCHUNGSSEMINAR: DIOPHANTINE PROBLEMS AND p-ADIC PERIOD MAPPINGS (AFTER LAWRENCE-VENKATESH)

1. Introduction

The aim of this seminar is to understand a new proof of Faltings's theorem (previously Mordell conjecture) due to Lawrence-Venkatesh ([LV18]), and the application of their methods to higher dimensional situations.

As in Faltings ([Fa83]) the idea is to study variation of p-adic Galois representations in a family. However, unlike Faltings', the method of Lawrence-Venkatesh can be applied to families of not necessarily abelian varieties and they illustrate this by applying their techniques to obtain some 'finiteness' results for high degree smooth hypersurfaces in $\mathbb{P}^n$. Our goal in this seminar would be to outline the principal steps in the proof of the following results as in [LV18].

Let K be a number field and S a finite set of places of K containing all the Archimedean places. Let $\mathcal{O}_S$ be the ring of S-integers and, $\mathcal{O}_S^\times$ the group of S-units.

Theorem 1.1 (S-unit equation, the toy example)**.** *The set* $\{x \in \mathcal{O}_S^\times | 1 - x \in \mathcal{O}_S^\times\}$ *is finite.*

Theorem 1.2 (Faltings)**.** *Let* X/K *be a smooth projective curve of genus at least* 2*. Then* $X(K)$ *is finite.*

Theorem 1.3 (Lawrence-Venkatesh)**.** *Let* $F_{n,d}/\mathbb{Z}$ *be the scheme parametrizing smooth projective hypersurfaces of degree* d *in* $\mathbb{P}^n$*. Then there exists an integer* n_0 *such that for any* $n \geq n_0$*,* $d \geq d_0(n)$*, and any finite set of primes* S*, the Zariski closure of* $F_{n,d}[\mathbb{Z}[S^{-1}]]$ *is a proper subset of* $F_{n,d}$*.*

For detailed outline of the arguments in [LV18], we refer to the introduction in [LV18]. For a quick overview one can also have a look at the program at Essen ([Essen]) on the same topic.

2. A brief summary of the article

2.1. **The first part (§3–4): Controlling the p-adic period map.** After the introductory first section, the paper collects preliminary results in Section 2. In Section 3, the authors recall the notion of a Gauss-Manin connection and use it to compare p-adic and complex period maps (see Lemma 3.1). The latter can be understood in terms of monodromy. As an application, a finiteness result on Galois representations (see Lemma 2.3) is used to obtain restrictions on the image of the period map (see Proposition 3.3). In Section 4, as a toy example before the proof of Faltings theorem, the proof of Theorem 1.1 is obtained.

2.2. **The second part (§5–8): Faltings's theorem.** Section 5 outlines the proof of Faltings's theorem assuming Proposition 5.3 (essentially a variant of Proposition 3.3), and applying it to what the authors call the Kodaira-Parshin family (which replaces the modified Legendre family in the argument for S-unit equations). The proof of Proposition 5.3 is given in Section 6, and the Kodaira-Parshin family is constructed in Section 7. In Section 8, the monodromy condition for Proposition 5.3 is verified for Kodaira-Parshin families by purely topological computation.

2.3. **The third part (§9–12): Application in a higher dimensional situation.** In Section 9, Bakker and Tsimerman's Ax-Schanuel theorem is introduced and transferred to a p-adic setting. Using this, Section 10 proves the main theorem (Theorem 10.1) up to a statement from linear algebra (Proposition 10.6). Moreover an application of Theorem 10.1 to hypersurfaces is discussed in Proposition 10.2 The proof of Proposition 10.6 occupies Section 11. Section 12, the final section, gives an alternative argument to control the size of Frobenius centralizer (an alternative for Lemma 10.4), which has a potential to yield better numerical bounds.

3. Schedule

3.1. **Talk 1 (Oct. 18, 2018): Introduction.** Give an overview of the results closely following the introduction in [LV18].

3.2. **Talk 2 (Oct. 25, 2018): The Gauss-Manin connection.** §3.1 – 3.3

In the first half, introduce the Gauss-Manin connection over an arbitrary base field and explain how it can be used to compare the cohomology of nearby fibers. It would be nice if an example can be given to explicate the same. Further references [LiVHS], [ConradRH].

In the second half discuss the contents of Section 3.3 which is the first step to analyze p-adic period maps using their complex analogues.

3.3. **Talk 3 (Nov. 1, 2018): Period mappings and Galois representations-I.** §3.4

Begin with briefly recalling some basic facts about crystalline cohomology and discuss the commutative diagram (3.9). In the rest of the talk, focus on the complex and p-adic period mapping, and prove Lemmas 3.1 and 3.2.

3.4. **Talk 4 (Nov. 8, 2018): Period mappings and Galois representations-II.** Lemma 2.3 and §3.5

Recall the notion of a crystalline representation and the crystalline comparison theorem of Faltings. Recall the commutative diagram (3.9), and what it implies for the fully faithful embedding (3.12).

In the second half of the talk, discuss in detail the proof of Proposition 3.3. The style of argument is repeated several times in the article and hence is crucial. You can assume the statement of Lemma 2.3 or outline a proof depending on the constraint on time.

3.5. **Talk 5 (Nov. 15, 2018): The S-unit equation-I.** §4.1-4.2

Discuss in detail how to obtain the desired finiteness from Lemma 4.2. After this, introduce the modified Legendre family which will be the target of the aforementioned Proposition 3.3. On an unrelated note, state and prove Lemma 2.1 as a warm-up for the kind of linear algebra involved in this set-up. This will be used in the next talk.

3.6. **Talk 6 (Nov. 22, 2018): The S-unit equation-II.** §4.3 – 4.4

Assuming the statement of Lemmas 4.3 (big monodromy) and 4.4 (generic simplicity), prove Lemma 4.2.

3.7. **Talk 7 (Nov. 29, 2018): Friendly Places.** §2.4

Begin by defining the maximal CM subfield of a number field. Discuss the definition of friendly places (see Definition 2.7), its friendliness will be explained in a future lecture. Note that friendly places exist. Define what it means for a p-adic character to be locally algebraic and note that it is implied by being Hodge-Tate at p. Also discuss what it means for a character ramified at finitely many places to be of pure weight.

State and prove Lemma 2.8 to get a feel for friendly places. The aim is to show that at friendly places (above p, I think this is implicitly assumed though not mentioned) the square of a locally algebraic character, ramified at finitely many places and of pure weight is essentially the Norm character raised to the weight.

3.8. **Talk 8 (Dec. 6, 2018): Kodaira-Parshin Family.** §5 and §7

In this talk our aim will be to construct the Kodaira-Parshin family which is an analogue of the modified Legendre family from Talk 5.

State and motivate the definition of an *abelian-by-finite family* (see Definition 5.1). Discuss what it means for such a family to have *full monodromy*. Introduce the group $\mathrm{Aff}(q)$ (see Section 2.6).

In the second half of the talk begin with definition of a Prym variety and its variant the reduced Prym variety (see Section 7.2). Note the dimension of the reduced Prym variety. Show that the reduced Prym variety can also be described using (7.2), and thus has a natural relative variant. Construct the Kodaira-Parshin family assuming Proposition 7.1. Conclude by stating the key properties (i)-(iii) satisfied by the Kodaira-Parshin covers as stated in Section 5.

3.9. **Talk 9 (Dec. 13, 2018): Proof of Faltings's Theorem-I.** §5

Define the size of a Galois module at an unramified place (see Definition 5.2). State Proposition 5.3, and reduce the desired finiteness to the existence of a friendly place and a suitable Kodaira-Parshin family. The proof will be spread over two lectures. In this talk show that one can choose q and v with the desired properties (i)-(iii). We shall see that this does the job in the next lecture.

3.10. **Talk 10 (Dec. 20, 2018): Proof of Faltings's Theorem-II.** §5

Begin by recalling Propostion 5.3. Prove Lemma 2.11, construct the morphism in (5.5), and use Lemma 2.11 to explicitly describe (5.5). Now complete the proof of Theorem 5.4.

3.11. **Talk 11 (Jan. 10, 2019): Rational points on the base of an abelian-by-finite family-I.** §6

The aim of this and the next two talks will be to prove Proposition 5.3, which implies Faltings's Theorem (Theorem 5.4). These will be the technically more intensive than the rest.

Begin by discussing the restrictions on the p-adic period mapping at a base point of an abelian-by-finite family. In the next part of the talk discuss the Equations (6.1)-(6.7) explicitly, and show that Proposition 5.3 can be deduced from Lemmas 2.3, 6.1 and 6.2.

3.12. **Talk 12 (Jan. 17, 2019): Rational points on the base of an abelian-by-finite family-II.** §6

Begin by recalling the statements of Lemmas 2.1 and 3.2. Also recall the fully faithful embedding (3.12) and the Crystalline comparision theorem, and apply it to the situation in hand. Complete the proof of Lemma 6.2.

3.13. **Talk 13 (Jan. 24, 2019): Rational points on the base of an abelian-by-finite family-III.** §6

The aim of this talk is to prove Lemma 6.1 (generic simplicity).

Begin by stating Lemma 2.10 and if possible indicate a proof. Use this to prove the *Sublemma*. Recall the equation (6.7) to conclude the proof of Lemma 6.1 modulo the general position Lemmas 6.3 and 6.4, which can be stated without proof.

4. Application of variation of p-adic Galois representations to higher dimensional varieties

Let $\pi : X \to Y$ be a smooth proper morphism of finite type schemes over $\mathbb{Z}[S^{-1}]$, where S is a finite set of primes. The goal of the next three lectures is to find sufficient conditions (on π) so that $Y(\mathbb{Z}[S^{-1}])$ is not Zariski dense in Y and to obtain some interesting Corollaries. The basic idea is similar to the proof of Faltings's theorem but applying them requires more work than before.

4.1. **Talk 14 (Jan. 31, 2019): The case of high degree hypersurfaces.** §10

Set up the general problem at hand as in the introduction to §10. Define the monodromy representation of interest (10.1) (it appears to me the authors are assuming π to be projective, but they only say proper) and the T function (10.3). State the principal Theorem 10.1 and in particular explain what it means to have large monodromy. Finally show that the family of high degree hypersurfaces satisfy the assumptions of Theorem 10.1 (Proposition 10.2).

4.2. **Talk 15 (Feb. 7, 2019): Proof of Theorem 10.1-I.** §10.3 and §10.5

Discuss the preliminary set-up as in the beginning of §10.3. In particular note that we need finiteness results analogous to ones we have used before (Lemma 2.3), but for more general reductive groups (Lemma 2.6).

Describe the period mapping in this setting, and also the action of crystalline Frobenius on the primitive cohomology of reduction mod ℓ. Now state Lemmas 10.4 and 10.5, and explain how to deduce Theorem 10.1 from these Lemmas.

In the final part of the talk explain the proof of Lemma 10.5 assuming Proposition 10.6 and Lemma 9.3 (the p-adic transcendence of period mapping).

4.3. **Talk 16 (Feb. 14, 2019): Proof of Theorem 10.1-II.** §10.4

The aim of the last talk is to outline a proof of Lemma 10.4. Begin by recalling the finiteness result on Galois representation (Lemma 2.6). Define the notion of a very regular element using the condition (*). Show that very regular elements are regular, and that being very regular is an open condition. Use Cebotarev to conclude (10.18).

Recall Lemmas 2.4, 2.5 and a result of Sen to get the dimension inequality in (10.20). Show that centralizer dimension of the crystalline Frobenius at ℓ and the Frobenius at ℓ can be calculated the same way (10.21) to obtain the desired inequality in Lemma 10.4 using (10.20).

References

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