CHAPTER 12

Dynamical systems: Multiply recurrent points

1. e-recurrent points

In this section, we study the notion of a multiply recurrent point in a dynamical system. Unless for the notion of (uniformly) recurrent points, we can show existence of such points only under additional conditions on a dynamical system.

12.1. Definition Let X be a dynamical system over S and e a subset of S. We call a point x of X *e-recurrent* if for every neighbourhood U of x, there is some $n \geq 1$ such that $\{a^n(x) : a \in e\} \subseteq U$. (Note that the same $n \in \omega$ is required to work for all $a \in e$.)

The following remarks shed some light on this notion.

12.2. Remark Given a single continuous map $t : X \to X$ on a compact Hausdorff space X, let us consider X as a dynamical sytem over $(\omega, +)$, as explained in **??**, i.e. we put $nx = t^n(x)$, for $x \in X$ and $n \in \omega$. For $x \in X$ and $U \subseteq X$, we write $R_t(x, U) = \{n \in \omega : t^n(x) \in U\}$, the return set of x to U, with respect to t. We call x recurrent with respect to t if for every neighbourhood U of x, $R_t(x, U)$ has a non-trivial element $n \geq 1$. Note that every point recurrent with respect to t is in the closed subspace *rant* of X.
Similarly, in a dynamical system (X, S), we consider for an arbitrary $a \in S$ the left multiplication m_a by a given by $m_a(x) = ax$, a continuous map from X into itself, and we write $R_a(x, U) = R_{m_a}(x, U)$; so we call $x \in X$ recurrent for a if for every neighbourhood U of x, $R_a(x, U) = \{n \in \omega : a^n(x) \in U\}$ has a non-trivial element $n \geq 1$.
So for $e \subseteq S$, every e-recurrent point is recurrent for every $a \in e$.

12.3. Remark (a) **??** shows that, in a discrete dynamical system (X, t), there exists a uniformly recurrent, hence recurrent point, with respect to t.
In other words, if (X, S) is an arbitrary dynamical system, $a \in S$ and $e = \{a\}$, then there is some e-recurrent point of X.
(b) However, if t_1, t_2 are continuous maps from X into itself with disjoint ranges, then no point of X can be recurrent with respect to both of t_1 and t_2.
In other words, if (X, S) is an arbitrary dynamical system, $a, b \in S$ and $e = \{a, b\}$, then possibly no point of X is e-recurrent.

The notion of e-recurrence is closely connected to non-emptyness of certain subsets of X. We fix a notation for these sets.

12.4. Definition and Remark For $e \subseteq S$, $U \subseteq X$ and $n \in \omega$, we write

$$W(e, U, n) = \bigcap_{a \in e} a^{-n}[U].$$

(This set may be empty, of course.)
Clearly $y \in W(e, U, n)$ means that $\{a^n y : a \in e\} \subseteq U$. Thus x is e-recurrent iff, for every neighbourhood U of x, there is some $n \geq 1$ such that $x \in W(e, U, n)$; in particular, $W(e, U, n)$ is non-empty.
Note that, if 1_S happens to be in e, then $W(e, U, n) \subseteq 1_S^{-n}[U] = U$.

The following example shows how van der Waerden's theorem 7.5 can be used to produce points which come close to being e-recurrent, for a special case of $e \subseteq S$.

12.5. Example Let (X, S) a dynamical system and e a finite subset of S which is included in a simply generated subsemigroup of S. Say $e = \{1_S, b, b^2, \ldots, b^k\}$ where $b \in S$ and $k \in \omega$. We claim that there is some $x \in X$ such that for every neighbourhood U of x, there is $n \geq 1$ such that $W(e, U, n) \neq \emptyset$.
To this end, let x be a uniformly recurrent point for the discrete dynamical system (X, t) where $t = m_b$ is the map assigning bz to every $z \in X$. Thus the return set $R = \{n \in \omega : t^n(x) = b^n x \in U\}$ of x to U is a syndetic, hence piecewise syndetic subset of $(\omega, +)$. By van der Waerden's theorem 7.5, R includes an arithmetic progression $m, m+n, \ldots, m+kn$ of length k. We show that $y = m \cdot x = b^m x$ is in $W(e, U, n)$: for $0 \leq i \leq k$, we have $m+ni \in R$ which means that $b^{m+in}x = b^{in}y \in U$. Thus $\{a^n y : a \in e\} \subseteq U$.

The existence results of this chapter will mostly be proved under the assumption that the dynamical system under consideration is minimal. They carry over to arbitrary dynamical systems as follows.

12.6. Remark (a) Assume that Y is a subsystem of X, e a subset of S, and $y \in Y$ is e-recurrent, in Y. Then y is e-recurrent, in X: for U a neighbourhood of y in X, $V = U \cap Y$ is a neighbourhood of y in Y. So there is some $n \geq 1$ satisfying $\{a^n y : a \in e\} \subseteq V$, and thus $\{a^n y : a \in e\} \subseteq U$.
(b) Similarly, if Y is a subsystem of X, $e \subseteq S$, $U \subseteq X$, $V = U \cap Y$ and $n \geq 1$ is such that $W(e, V, n)$ (computed in (Y, S)) is non-empty, then so is $W(e, U, n)$ (computed in (X, S)).

2. The Topological van der Waerden theorem

The proof of Example 12.5 generalizes to arbitrary finite subsets of commutative monoids, using the notation $W(e, U, n) = \bigcap_{a \in e} a^{-n}[U]$.

12.7. Theorem *(the Topological van der Waerden theorem) Assume that X is a minimal dynamical system over a commutative monoid $(S, \cdot)$, e is a finite subset of S, and U is a non-empty open subset of X. Then there is some $n \geq 1$ such that $W(e, U, n)$ is non-empty – i.e. there is some $y \in X$ such that $\{a^n y : a \in e\} \subseteq U$. In particular if $1_S \in e$, then, for some $n \geq 1$, we have $\emptyset \neq W(e, U, n) \subseteq U$.*

Proof. Let $U \in \mathcal{O}^+$. We fix an arbitrary point x in U and put

$$R = R(x, U) = \{s \in S : sx \in U\}.$$

By minimality of X over S, x is uniformly recurrent and hence R is syndetic; so fix a finite $f \subseteq S$ such that $S = \bigcup_{b \in f} b^{-1}R$.
By Gallai's thorem **??** for commutative semigroups, there are $b \in f$, $s \in S$ and $n \geq 1$ such that $\{sa^n : a \in e\} \subseteq b^{-1}R$ – note that we use here multplicative notation for

S. We prove that, for this choice of n, the set $W = W(e, U, n)$ is non-empty. In fact, the point $y = sbx$ is in W. This holds because, for every $a \in e$, $bsa^n \in R$ holds which means that $a^n y = bsa^n x \in U$. □

12.8. Corollary *Assume $X = \bigcup_{i \in I} U_i$ is an open cover of the phase space X, the monoid S is commutative, and e is a finite subset of S. Then there is some $i \in I$ and some $n \geq 1$ such that $W(e, U_i, n) = \bigcap_{a \in e} a^{-n}[U_i] \neq \emptyset$.*

PROOF. Fix a minimal subsystem M of X and some $i \in I$ such that $U_i \cap M \neq \emptyset$. By 12.7, applied to M and $U_i \cap M$, there is some $n \geq 1$ such that $W(e, U_i \cap M, n) = \bigcap_{a \in e} a^{-n}[U_i \cap M] \neq \emptyset$ (the latter set being computed in M). This choice of i and n works for the corollary. □

We have essentially used Gallai's theorem 7.3 for commutative semigroups in the proof of the Topological van der Waerden theorem. In fact, it can be recovered from 12.7:

12.9. Remark Assume $S = A_1 \cup \cdots \cup A_r$ is a colouring of the commutative monoid S with finitely many colours and e is a finite subset of S. In the dynamical system βS over S, we obtain the open cover $\beta S = \widehat{A_1} \cup \cdots \cup \widehat{A_r}$, and by 12.8, some $j \in \{1, \ldots, r\}$ and some $n \geq 1$ such that the clopen subset $\bigcap_{a \in e} a^{-n}[\widehat{A_j}]$ of βS is non-empty. So the subset $W = \bigcap_{a \in e} a^{-n}[A_j]$ of S is non-empty, and for $b \in W$, we have $\{ba^n : a \in e\} \subseteq A_j$ (we use multiplicative notation for S here).

3. Multiply recurrent points

Our goal, in this section, is to prove an existence theorem for points with the following strong property, under suitable conditions. Our basic tool here is 12.12.

12.10. Definition A point x of X is *multiply recurrent* in (X, S) if it is e-recurrent, for every finite subset e of S.

12.11. Definition and Remark Let X be a topological space.
(a) We write $\mathcal{O}$ respectively $\mathcal{O}^+$ for the family of all open respectively of all non-empty open subsets of X.
(b) A π-base of X is a subfamily $\mathcal{D}$ of $\mathcal{O}^+$ such that every $U \in \mathcal{O}^+$ includes some $D \in \mathcal{D}$. (E.g., every base of X is a π-base.)
(c) For $\mathcal{D}$ a π-base of X, the union $\bigcup \mathcal{D} = \bigcup_{D \in \mathcal{D}} D$ of $\mathcal{D}$ is a dense open subset of X.

12.12. Lemma *Assume that X is a minimal dynamical system over a commutative monoid $(S, \cdot)$, e is a finite subset of S containing 1_S, and $B \in \mathcal{O}^+$. Then the family*

$$\mathcal{D}_e(B) = \{W = W(e, V, n) : n \geq 1, V \in \mathcal{O}^+, V \subseteq B \text{ or } V \subseteq X \setminus B, W \neq \emptyset\}$$

is a π-base for X.

PROOF. Given any $U \in \mathcal{O}^+$, we put $V = U \cap B$ if non-empty and $V = U$ otherwise; so V is as required in the definition of $\mathcal{D}_e(B)$. By 12.7, we pick $n \geq 1$

such that $W = W(e, V, n)$ is non-empty, hence an element of $\mathcal{D}_e(B)$. Since $1_S \in e$, we obtain that $W \subseteq V \subseteq U$. □

12.13. Theorem *Let X be a minimal dynamical system over the commutative monoid S. Moreover assume that S is count ableand that X has a countable base. Then there are multiply recurrent points in X – in fact, the set MR of all multiply recurrent points of X is dense in X.*

Proof. We fix a countable base $\mathcal{B}$ for X. For every finite subset e of S containing 1_S and every $B \in \mathcal{B}$, put

$$U_e(B) = \bigcup\{W : W \in \mathcal{D}_e(B)\},$$

a dense open subset of X, since $\mathcal{D}_e(B)$ is a π-base. By countabilty of S and $\mathcal{B}$, we conclude from Baire's theorem (in a compact Hausdorff space, the intersection of countably many dense open subsets is dense) that

$$Y = \bigcap_{B \in \mathcal{B}, e \text{ finite}, 1_S \in e} U_e(B)$$

is a dense subset of X.
We claim that $Y \subseteq MR$, which finishes the proof of our theorem. So assume that $y \in Y$, B is a neighbourhood of y, and $e \subseteq S$ is finite; we may assume that $1_S \in e$ and $B \in \mathcal{B}$. Now $y \in U_e(B)$; say $y \in W = W(e, V, n)$ where $n \geq 1$ and $V \subseteq B$ or $V \subseteq X \setminus B$.
Note that $y \in W = W(e, V, n) \subseteq V$ (since $1_S \in e$); moreover $y \in B$, so V meets B and thus $V \subseteq B$. Finally $y \in W = W(e, V, n)$ means that $\{a^n y : a \in e\} \subseteq V \subseteq B$, as desired. □

12.6 shows that a multiply recurrent point y in an S-subsystem Y of X is multiply recurrent in X. This gives the following consequence.

12.14. Corollary *Assume that the monoid S is countable and that X has a (minimal) subsystem with a countable base. Then X has a multiply recurrent point.*

For readers somewhat experienced in set theory, let us finally state a straightforward generalization of 12.14.

12.15. Definition (a) A topological space X is said to satisfy the countable chain condition if every family of non-empty pairwise disjoint open subsets of X is countable. (E.g. this holds if X has a countable π-base.)
(b) For a cardinal κ, *Martin's axiom for κ* (MA(κ)) is the following statement:
If X is a compact Hausdorff space satisfying the countable chain condition, $(U_i)_{i \in I}$ a family of dense open subsets of X and the cardinality $|I|$ of I is at most κ, then the intersection $\bigcap_{i \in I} U_i$ is dense in X.

For $\kappa \geq 2^\omega$, MA(κ) contradicts the axioms of Zermelo-Fraenkel set theory (ZFC), but it is known that if ZFC is consistent, then so is the theory ZFC plus "MA(κ) holds for all $\kappa < 2^\omega$".

With exactly the same proof as 12.14, we conclude the following.

12.16. Theorem *Assume that MA(κ) holds. Let X be a dynamical system over S satisfying the countable chain condition. Moreove assume that S is commutative, $|S| \leq \kappa$ and that X has a (minimal) subsystem with a base of size at most κ. Then X has a multiply recurrent point.*