CHAPTER 11

Dynamical systems: proximality

The notion of proximality deals with pairs (x, y) of points in a dynamical system X over S. Here X is a compact Hausdorff space, and so is its square $X^2 = X \times X$ with the product topology. We will remark below some facts on the space X^2 and neighbourhoods of its closed subset

$$\Delta = \{(x, x) : x \in X\},$$

the diagonal of X^2.

Now $(x, y) \in \Delta$ means that $x = y$; similarly for W a neighbourhood of Δ in X^2, we could say that x is W-close to y if $(x, y) \in W$. As a motivation, consider the situation where (X, d) is a metric space, $\epsilon > 0$ is a real, and $W = \{(x, y) \in X^2 : d(x, y) < \epsilon\}$; then x is W-close to y iff $d(x, y) < \epsilon$.

11.1. Remark Let X be a compact Hausdorff space.
(a) The standard base for open sets in X^2 consists of all sets of the form $U \times V$, where U and V are open subsets of X. So for $x, y \in X$, the sets $U \times V$ where U is a neighbourhood of x and and V a is a neighbourhood of y, constitute a neighbourhood base for (x, y). In particular, for $x \in X$, the sets $U \times U$, U a neighbourhood of x, are a neighbourhood base of (x, x).
(b) For every (open) covering $(U_i)_{i \in I}$ of X, the family $(U_i \times U_i)_{i \in I}$ is an (open) covering of the diagonal Δ, in the space X^2, thus $W = \bigcup_{i \in I}(U_i \times U_i)$ is a neighbouhood of Δ in X^2. Two points x and y of X are W-close iff there is some $i \in I$ such that x, y are both in U_i.
(c) Assume again that $(U_i)_{i \in I}$ is an open covering of X. Since X is compact, there is some finite $J \subseteq I$ such that $\bigcup_{i \in J} U_i = X$; and thus $\bigcup_{i \in J}(U_i \times U_i)$ is a neighbourhood of Δ. In fact, the sets of this form establish a neighbourhood base for Δ.

The main results are

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1. Proximal points and the Auslander-Ellis theorem

Assume that X is a dynamical system over S. Then X^2 becomes a dynamical system over S by coordinatewise operation:

$$s \cdot (x, y) = (s \cdot x, s \cdot y).$$

11.2. Definition and Remark (a) Let X be a dynamical system over S, $x, y \in X$ and W a subset of X^2. The set

$$JR(x, y, W) = \{s \in S : (sx, sy) \in W\}$$

is called the *(joint) return set* of x, y to W.
(In fact, it *is* the return set of (x, y) to W, in the dynamical system X^2).
(b) The points x and y are *proximal* (or, x is proximal to y) if, for every neighbourhood W of Δ in X^2, there is some $s \in S$ such that $(sx, sy) \in W$, i.e. if the orbit of (x, y), in X^2, meets every neighbourhood of Δ.
(c) Using the description of neighbourhoods of Δ from 11.1 , proximality of x, y means that for every (finite) open cover $(U_i)i \in I$ of X, there are $i \in I$ and $s \in S$ such that sx, sy are both in U_i.
(d) The relation of being proximal is clearly a reflexive and symmetric one, but not transitive, as we will see in later examples.
(e) Assume (X, t) is a discrete dynamical system; proximality of x, y means that for every neighbourhood W of Δ, there is some $n \in \omega$ such that $(t^n(x), t^n(y)) \in W$. As in **??**, it is very easy to see that in this case, the joint return set of x, y to W is actally infinite.

11.3. Example In Example **??** of a discrete dynamical system, we had $X = [0, 1]$ and $t(x) = x/2$. Here any two points x, y of X are proximal, because for every $\epsilon > 0$, there is some $n \in \omega$ such that $t^n(x)$ and $t^n(y)$ have distance less than ϵ.

11.4. Example In the discrete dynamical system of Example **??**, the phase space is the complex unit circle K ????? and t is the rotation by an angle α, where $\alpha/2\pi$ is irrational. Here two points x, y of K are proximal iff $x = y$, because for every $n \in \omega$, the distances of x, y and of $t^n(x), t^n(y)$ are the same.

There is an obvious characterization of proximality, using the action of βS on X.

11.5. Theorem *Two points x, y in a dynamical system X are proximal iff there is some $p \in \beta S$ such that $px = py$.*

PROOF. The following assertions are clearly equivalent.

- $px \neq py$ holds for every $p \in \beta S$
- the range Y of the continuous map $g : \beta S \to X^2$ given by $g(p) = (px, py)$ is disjoint from Δ
- there is an open neighbourhood W of Δ disjoint from Y
- there is an open neighbourhood W of Δ such that for all $s \in S$, $(sx, sy) \notin W$;

here the last equivalence holds since W is open. □

Similarly to **??**, we note that in a discrete dynamical system, x, y are proximal iff $px = py$ holds for some $p \in \beta S \setminus S$.

We now present an important result of topological dynamics, the Auslander-Ellis theorem. Its proof is particularly simple, using the action of βS on a dynamical system.

11.6. Remark Assume X is a dynamical system over S. Then every point x of X is proximal to some uniformly recurrent point y: simply pick any minimal idempotent ε of βS, and let $y = \varepsilon x$. Then y is uniformly recurrent by **??** and proximal to x by 11.5, since $\varepsilon y = \varepsilon\varepsilon x = \varepsilon x$.

The Auslander-Ellis theorem states that a point y as above can be chosen in a much more specific way.

11.7. Theorem *(Auslander-Ellis) Assume X is a dynamical system over S, $x \in X$, and M is an arbitrary subsystem of $\bar{x}$. Then there exists a uniformly recurrent point $y \in M$ which is proximal to x.*

Proof. To find an appropriate $y \in M$, we have to choose the idempotent ε more carefully than in 11.6. Since $\emptyset \neq M \subseteq \bar{x} = \{px : p \in \beta S\}$, the set $I = \{p \in \beta S : px \in M\}$ is non-empty; in fact, it is a closed left ideal of βS. Thus it contains a minimal idempotent ε, and $y = \varepsilon x$ is in M by the choice of $\varepsilon \in I$. □

2. Dynamically central sets

Among the sets considered as large, in Section 8, the central ones were defined by the abstract property of belonging to some minimal idempotent ultrafilter. Up to now, we have no elementary description of such sets, i.e. a description not using the arithmetic of βS. We now define the (purely mathematical but non-trivial) notion of a dynamically central subset of a monoid S and begin to prove that the dynamically central sets are exactly the central ones.

Historically, dynamically central sets were studied before central ones, which arise from the arithmetic of βS; the equivalence below was proved only much later.

11.8. Definition A subset C of a monoid S is called dynamically central if there are objects X, x, y and U which satisfy the following conditions.

- X is a dynamical system over S, and $x, y \in X$
- x is proximal to y, and y is uniformly recurrent
- U is a neighbourhood of y
- $C = R(x, U)$, i.e. C is the return set of x to U.

The second part of the next theorem wlil be proved in the next section. Let us note that it has an interesting consequence: it is by no means obvious that the family of dynamically central subsets of S is a coideal – i.e. that every superset of a dynamically central set is dynamically central, or that, if a union of two sets is dynamically central, then one of them must be dynamically central. Nevertheless this is true, because of the equivalence between being central and dynamically central, and the ultrafilter definition of central sets.

11.9. Theorem *A subset of S is dynamically central iff it is central.*

PROOF. Assume that C is dynamically central; so pick X, x, y and U such that $C = R(x, U)$ as in 11.8. As in the proof of 11.7, we see that the set

$$I = \{p \in \beta S : px = py\}$$

is non-empty (because y is proximal to x) and a closed left ideal of βS. We pick a minimal left ideal L included in I and, by 10.11, a (minimal) idempotent $\varepsilon \in L$ such that $\varepsilon y = y$. Then $\varepsilon x = \varepsilon y = y$ holds by the choice of ε in $L \subseteq I$. Thus $\varepsilon x \in U$ and, by Remark 10.3, $C = R(x, U) \in \varepsilon$, which shows that C is central. □

3. Dynamics in the shift space

We present here a few proofs and examples which refer, for an arbitrary monoid S, to one of our standard examples, the shift space $X = {}^S c$ from **??**, a dynamical system over S. For the sake of intuitiveness, we will work with the finite set $c = 2 = \{0, 1\}$, although much of the material works quite similarly for an arbitrary finite set c of colours.
Every subset C of S is then represented by its characteristic function $\chi_C \in {}^S 2$, where $\chi_C(i) = 1$ iff $i \in C$. In 10.2, we saw that $R(\chi_C, U) = C$ holds for the clopen subset $U = \{x \in X : x(1_S) = 1\}$ of X.
In particular, we will point out what these results mean for the additive monoid ω, i.e. for the discrete dynamical system ${}^\omega 2$ with the shift map t (???).

PROOF. **(Proof of 11.9: central sets are dynamically central.)** Let $C \subseteq S$ be central; so fix a minimal idempotent ε of βS such that $C \in \varepsilon$. We work in the shift system ${}^S 2$ and consider the point of $x = \chi_C \in X$. The point $y = \varepsilon x$ of X is proximal to x and uniformly recurrent, as shown in 11.6. For the clopen subset $U = \{z \in X : z(1_S) = 1\}$ of X, we have $R(x, U) = C$, and thus $R(x, U) \in \varepsilon$. We are left with showing that U is a neighbourhood of y, i.e. that $y \in U$.
Otherwise, $y(1_S) = 0$, and $V = \{z \in X : z(1_S) = 0\}$ is a neighbourhood of $y = \varepsilon x$; thus $R(x, V) \in \varepsilon$ holds by 10.3. But U and V are disjoint and so are their return sets $R(x, U)$ and $R(x, V)$, a contradiction to $R(x, U), R(x, V) \in \varepsilon$. □

The proof of 11.9 gives a more concrete information: the objects X, x, y, and U from 11.8 can be chosen very specifically.

11.10. Theorem *Let S be a monoid and $C \subseteq S$; in the dynamical system $X = {}^S\{0, 1\}$, we consider the point $\chi_C \in X$. Then C is central iff there exists some $y \in X$ proximal to χ_C such that y is uniformly recurrent and $y(1_S) = 1$.*

PROOF. If C is central, then the above proof of dynamical centrality of C given in 11.9 gives an y as desired and shows that $y(1_S) = 1$. Conversely, assume $y \in X$ is as reqired in our theorem. Then $U = \{z \in X : z(1_S) = 1\}$ is a neighbourhood of y and $R(\chi_C, U) = C$ was shown in 10.2. Thus C is dynamically central, hence central. □

For the monoid $(\omega, +, 0)$, it is quite easy to describe which points of $X = {}^{\omega}\{0,1\}$ are proximal, which will somewhat facilitate the characterization of central subsets of ω. We first establish a particularly simple neighbourhood base for the diagonal $\Delta \subseteq X^2$.

11.11. Remark For $n \in \omega$, we abbreviate by S_n the set of all sequences $v = (v_0, \ldots, v_n)$ of length $n+1$ in $2 = \{0,1\}$. Recall from **??** that, for $x \in X = {}^{\omega}2$, $\mathrm{seg}_n(x)$ is the initial segment of x of length $n+1$. For every $v \in S_n$, we can define a clopen subset U_v of X by $U_v = \{x \in X : \mathrm{seg}_n(x) = v\}$; the sets U_v (where $n \in \omega$ and $v \in S_n$) constitute a base for the space X.
Now for $n \in \omega$, the family $(U_v)_{v \in S_n}$ is a finite cover of X by clopen sets. Thus

$$W_n = \bigcup_{v \in S_n} (U_v \times U_v)$$

is a (clopen) neighbourhood of Δ, in X^2, and $\{W_n : n \in \omega\}$ is a neighbourhood base of Δ.

11.12. Example Two points x and y of $X = {}^{\omega}\{0,1\}$ are proximal iff for every $n \geq 1$ there is some $i \in \omega$ such that the segments $\mathrm{seg}_{i,i+n}(x)$ and $\mathrm{seg}_{i,i+n}(y)$ (of length $n+1$) coincide. (Note that these segments appear in the same position, in x and y.) I.e. x and y are proximal iff they agree on arbitrarily long segments (which appear in the same position).
This follows immediately from considering the neighbourhood base $\{W_n : n \in \omega\}$ of Δ constructed in 11.11: proximality of x, y means that for each $n \in \omega$ there is some $i \in \omega$ and some $v \in S_n$ such that $t^i(x)$ and $t^i(y)$ are both in U_v. I.e. $\mathrm{seg}_n(t^i(x)) = \mathrm{seg}_n(t^i(y))$ and thus $\mathrm{seg}_{i,i+n}(x) = \mathrm{seg}_{i,i+n}(y)$.

The most precise information on central subsets of $(\omega, +)$ we can obtain here is the following.

11.13. Example A subset C of ω is central iff there is some uniformly recurrent $y \in {}^{\omega}\{0,1\}$ which satisfies $y(0) = 1$ and coincides with χ_C on arbitrarily long intervals.

Exercises

(1) In the discrete dynamical system from 9.14, i.e. $X = [0,1]$ and $t(x) = x/2$, show that any two points x, y are proximal.
(2) In the discrete dynamical system from 9.17, i.e. X is the unit circle in the complex plane and $t(x) = x \cdot x_\alpha$ where $\alpha/2\pi$ is irrational, show that two points x, y are proximal iff $x = y$.
(3) Assume that the topology on a compact Hausdorff space X is induced by a metric d.
(a) Prove that the sets $U_\epsilon = \{(x,y) : d(x,y) < \epsilon\}$, where $\epsilon > 0$, constitute a neighbourhood base of Δ, in X^2.
(b) Let (X, t) be a discrete dynamical system. Prove that $x, y \in X$ are proximal iff there is a increasing sequence $(n_k)_{k \in \omega}$ of natural numbers such that $\lim_{k \to \infty} t^{n_k}(x) = \lim_{k \to \infty} t^{n_k}(y)$.