CHAPTER 10

Dynamical systems: recurrent and uniformly recurrent points

In a dynamical system X over S, we study points with special properties: recurrent and uniformly recurrent points. These are points returning to each of their neighbourhoods quite often.

Using the action of βS on X defined in 9.23, we can not only show the existence of such points but understand them really well. In particular, we will see that the uniformly recurrent points of X are exactly the minimal ones, i.e. those belonging to some minimal subsystem of X.

Finally we describe, in the example of the shift system from 9.18, which points are (uniformly) recurrent. This example can (resp. should) be read shortly after the definitions of recurrence resp. uniform recurrence.

1. Recurrent points

Let S be a monoid and X a dynamical system over S.

10.1. Definition For $x \in X$ and $U \subseteq X$, we define the *return set* of x to U by

$$R(x, U) = \{s \in S : sx \in U\}.$$

This generalizes our notation from 8.19: for the dynamical system βS over S, as defined in 9.19, the return set of $p \in \beta S$ to $U \subseteq \beta S$ coincides with that defined in 10.1.

The subsequent example shows that any subset of S is representable as a return set, for some suitably chosen dynamical system X over S and some $x \in X$.

10.2. Example We consider the abstract dynamical system given in 9.18: the shift system over S where the phase space X is the product space ${}^S\{1, \dots, r\}$, for some $r \geq 2$. For any $R \subseteq S$, there are x and U such that $R(x, U) = R$: simply pick $x \in X$ be such that $x(s) = 1$ iff $s \in R$; so x might be considered as a characteristic function of R. Put $U = \{z \in X : z(1_S) = 1\}$, a clopen subset of X. Then for $s \in S$, $s \in R(x, U)$ holds iff $(sx)(1_S) = 1$, which means that $x(s) = 1$, i.e. $s \in R$.

The following principles connecting return sets with ultrafilters are quite helpful.

10.3. Remark Assume that X is a dynamical system over S, $x \in X$, and p is an ultrafilter over S; so px is a point in X; assume that $U \subseteq X$.
(a) If U is a neighbourhood of px, then $R(x, U) \in p$: px is the p-limit of the sequence

$(sx)_{s\in S}$; so if U is a neighbourhood of px, then there is some $R \in p$ such that $sx \in U$ holds for every $s \in R$. I.e. $R \subseteq R(x, U)$ and $R(x, U) \in p$.
(b) If $R = R(x, U) \in p$ and U is closed, then $px \in U$, because $px = p - \lim_{s\in S} sx$ belongs to the closure of $\{sx : s \in R\} \subseteq U$ (cf. 2.20).
(c) Thus if U is clopen, then $R(x, U) \in p$ is equivalent to $px \in U$.

If, in 10.1, x is a point in U, then the return set $R(x, U)$ is non-empty for trivial reasons, because $1_S x = x \in U$.

10.4. Definition A point x in X is *recurrent* if $R(x, U) \neq \{1_S\}$ holds for every neighbourhood U of x.

Some effects to be studied depend on special algebraic properties of the monoid S. E.g. the following remark works for the additive monoid ω.

10.5. Remark In a discrete dynamical system (X, t), recurrence of x is equivalent to the seemingly stronger requirement that $R(x, U) = \{k \in \omega : t^k(x) \in U\}$ is infinite, for every neighbourhood U of x. Because, if x happens to be a periodic point, i.e. $t^k(x) = x$ holds for some $k \geq 1$, then $\{k, 2k, 3k, \dots\} \subseteq R(x, U)$. Otherwise we choose, for every $k \geq 1$, a neighbourhood U_k of x disjoint from $t(x), t^2(x), \dots, t^k(x)$, and an $n_k \in R(x, U_k)$ such that $n_k \geq 1$. So $n_k > k$ and $\{n_k : k \geq 1\}$ is an infinite subset of $R(x, U)$.

In a discrete dynamical system (X, t), all fixed points and, more generally, all periodic points of t are clearly recurrent; the example 9.17 however shows that a discrete system does not necessarily have periodic points. But the following theorem guarantees, under rather mild conditions on S, that dynamical systems over S have recurrent points.

There is a very simple way of constructing recurrent points: if βS has an idempotent p distinct from 1_S, then X has recurrent points: for any point y of X, $x = py$ is recurrent, by the following theorem.
Such an idempotent exists iff there is a closed subsemigroup T of βS such that $1_S \notin T$, because T has idempotents, by Theorem 1.21. More specifically, if for all s, t in S, $st = 1_S$ implies that $s = 1_S$ or $t = 1_S$, then $A = S \setminus \{1_S\}$ is a subsemigroup of S and $\hat{A}$ is a closed subsemigroup of βS not containing 1_S. Cf. Exercise 1 below for more precise information.

10.6. Theorem *Let x be a point of a dynamical system X over S.*
(a) x is recurrent, i.e. $R(x, U) \neq \{1_S\}$ for every neighbourhood U of x, iff there is some $p \in \beta S$ such that $p \neq 1_S$ and $px = x$.
If, in addition, $st \neq 1_S$ for $s, t \in S \setminus \{1_S\}$, then p satisfying $px = x$ can be taken as an idempotent in βS.
(b) Similarly, $R(x, U)$ is infinite for every neighbourhood U of x iff there is some $p \in \beta S \setminus S$ such that $px = x$.

PROOF. (a) Assume that x is recurrent; so all sets in the family

$$\mathcal{R} = \{R(x, U) \setminus \{1_S\} : U \text{ a closed neighbourhood of } x\}$$

are non-empty. Moreover, $\mathcal{R}$ has the finite intersection property, because for any neighbourhoods U and V of x, $R(x, U \cap V) \subseteq R(x, U) \cap R(x, V)$. So pick $p \in \beta S$

including $\mathcal{R}$; clearly $p \neq 1_S$. Now 10.3 shows that $px \in U$ holds for every closed neighbourhood U of x, and thus $px = x$.
Conversely, assume that $p \neq 1_S$ and $px = x$. Given any neighbourhood U of x, we show that $R(x,U) \neq \{1_S\}$. Now U is a neighbourhood of px; by 10.3, $R(x,U) \in p$. The set $R(x,U) \setminus \{1_S\}$ is in p, hence non-empty.
In addition, assume that $st \neq 1_S$ holds for $s,t \in S \setminus \{1_S\}$ and that $p \neq 1_S$ satisfies $px = x$. Then $T = \{p \in \beta S \setminus \{1_S\} : px = x\}$ is a closed subsemigroup of $\beta S \setminus \{1_S\}$, and every idempotent of T works.
(b) This works exactly like the proof of (a); p can be taken to be free if the sets in $\mathcal{R}$ are infinite. □

2. Uniformly recurrent points

In this section, we define uniformly recurrent points in a dynamical system; under very weak conditions on S, they turn out to be recurrent. The main result is a characterization of uniformly recurrent points (10.9 and 10.11) which in particular guarantees their existence.

10.7. Definition Let X be a dynamical system over S. A point x of X is *uniformly recurrent* if the return set $R(x,U)$ is syndetic, for every neighbourhood U of x.

Let us recall from 8.19 the notation $t^{-1}U = \{x \in X : tx \in U\}$, for $U \subseteq X$ and $t \in S$. The following remarks closely parallel those in **??**; they are quite applicable for the study of uniformly recurrent points. The proof is by straightforward computation.

10.8. Remark Assume X is a dynamical system over S, $U \subseteq X$, and $x \in X$.
(a) For $t \in S$, we have $t^{-1}R(x,U) = R(x, t^{-1}U)$.
(b) Let e be an arbitrary subset of S; then $Sx \subseteq \bigcup_{t\in e} t^{-1}U$ is equivalent to $S = \bigcup_{t\in e} t^{-1}R(x,U)$.
(c) Let $t, y \in S$ and put $s = ty$. Then $t^{-1}R(x, y^{-1}U) = s^{-1}R(x,U)$.
(d) Let e be a subset of S, $y \in S$, and define the subset f of S by $f = e \cdot y = \{ty : t \in e\}$. Then
$$\bigcup_{t\in e} t^{-1}R(x, y^{-1}U) = \bigcup_{s\in f} s^{-1}R(x,U).$$

We begin to characterize the uniformly recurrent points of X. Let us point out that condition (c) of the next theorem ensures the existence of such points, because of the existence theorem 9.11 for minimal subsystems.

10.9. Theorem *For any point x in a dynamical system X over S, the following are equivalent.*
(a) x is uniformly recurrent
(b) the orbit closure $\bar{x}$ of x is a minimal subsystem of X
(c) there is a minimal subsystem of X containing x.

Proof. Equivalence of (b) and (c) is obvious.
(c) implies (a): assume that M is a minimal subsystem of X and $x \in M$; so $M = \beta S \cdot x$. Fix a neighbourhood U of x. By minimality of M, there is some finite $e \subseteq S$ such that $M \subseteq \bigcup_{t\in e} t^{-1}U$. Thus $Sx \subseteq \beta S \cdot x = M \subseteq \bigcup_{t\in e} t^{-1}U$, which by

12.5 means that $S = \bigcup_{t \in e} t^{-1}R(x,U)$. The latter shows that $R(x,U)$ is syndetic. *(a) implies (b):* we assume that x is uniformly recurrent and prove that the orbit closure M of x is minimal. So let V be an open subset of X intersecting M, with the intent of proving that M is covered by finitely many backwards translates of V. Pick an open subset W of X whose closure $\overline{W}$ is included in V and such that W intersects M. Since M is the orbit closure of x, we can pick some $r \in S$ such that $rx \in W$. Then $U = r^{-1}W$ is an open neighbourhood of x in X; by uniform recurrence of x, the set $R = R(x,U)$ is syndetic, say $S = \bigcup_{t \in e} t^{-1}R$ for some finite $e \subseteq S$. Put $f = e \cdot r$, a finite subset of S. Using the computing rules in 12.5, we obtain $S = \bigcup_{t \in e} t^{-1}R(x, r^{-1}W) = \bigcup_{s \in f} s^{-1}R(x,W) \subseteq \bigcup_{s \in f} s^{-1}\overline{W}$. This means that $Sx \subseteq \bigcup_{s \in f} s^{-1}\overline{W}$; since $\overline{W}$ is closed, it follows that $M = \beta S \cdot x \subseteq \bigcup_{s \in f} s^{-1}\overline{W} \subseteq \bigcup_{s \in f} s^{-1}V$, as desired. □

In particular, in the universal dynamical system βS, the minimal subsystems are the minimal left ideals of the semigroup βS, so the uniformly recurrent points are those in $K(\beta S)$. This gives a more intuitive explanation for 8.21 (if $A \in p$ for some $p \in K(\beta S)$, then the return set of p to $\hat{A}$ is syndetic): in the dynamical system βS, $\hat{A}$ is a neighbourhood of the uniformly recurrent point p.

The next theorem continues the characterization of uniformly recurrent points of X, the subtle detail being that the minimal left ideal L of βS referred to can be chosen independently of the point x. For its proof, we note how (minimal) subsystems behave under S-homomorphisms.

10.10. Lemma *Assume that X and Y are dynamical systems over S and that $f : X \to Y$ is an S-homomorphism. Let Z be a subsystem of X and T a subsystem of Y*
(a) The image $f[Z]$ of Z under f is a subsystem of Y; likewise, the preimage $f^{-1}[T]$ of T is a subsystem of X.
(b) If Y is a minimal system, then f is onto.
(c) If Z is minimal (as a subsystem of X), then so is $f[Z]$ (as a subsystem of Y).

Proof. (a) and (b) are easily checked. For (c), consider the restriction $f \restriction Z$ of f to Z, an S-homomorphism from Z onto $f[Z]$. For any proper subsytem T of $f[Z]$, the preimage of T under $f \restriction Z$ is a proper subsystem of Z; contradiction. □

10.11. Theorem *Assume that X is a dynamical system over S and that $x \in X$. Let L be any minimal left ideal of βS. The following assertions are equivalent to (a) through (c) from 10.9, i.e. to uniform recurrence of x:*
(d) there is an idempotent ε in L such that $x = \varepsilon x$
(e) there is some $p \in L$ such that $x = px$
(f) there are an idempotent ε in L and $y \in X$ such that $x = \varepsilon y$
(g) there are $r \in K(\beta S)$ and $y \in X$ such that $x = ry$.

Proof. By (b) resp. (c), we mean assertions (b), (c) from 10.9, i.e. the statements that $\bar{x}$ is a minimal subsystem of X resp. that x is contained in some minimal subsystem of X; we will show that (b) implies (e), (e) implies (d), (d) implies (f), (f) implies (g), and (g) implies (c). The implications from (d) to (f) and from (f) to (g) are trivial.

(e) implies (d). Assume $p \in L$ satisfies $px = x$, so the set

$$G = \{r \in L : rx = x\}$$

is non-empty. G is a closed subsemigroup of L, and every idempotent ε of G works for (d).
(b) implies (e). We consider the minimal subsystem $M = \bar{x}$ of X and the S-homomorphism f from βS to X mapping 1_S to x (cf. **??**), i.e. $f(q) = qx$ holds for every $q \in \beta S$. Now L is a subsystem of βX and M is minimal; by (b) in 12.5, $f \restriction L$ maps L onto M. Pick $p \in L$ such that $f(p) = x$, so $px = x$.
(g) implies (c). Let $x = ry$ where $y \in X$ and $r \in K(\beta S)$. We consider the minimal left ideal L' containing r and the S-homomorphism $f : \beta S \to X$ given by $f(q) = qy$, for every $q \in \beta S$. Now L' is a minimal subsystem of βS, thus the image M of L' under f is a minimal subsystem of X. And $r \in L'$, so $x = ry = f(r) \in M$. □

3. Recurrent and uniformly recurrent points in the shift system $^{\omega}c$

We describe in a very elementary way which points of the (discrete) shift system (X, t) from 9.18 are recurrent, resp. uniformly recurrent. Here we had $X = {}^{\omega}c$ for some finite set c of colours; we write the elements of X as sequences $x = (x_0, x_1, x_2, \ldots)$ where $x_i \in c$, indexed with natural numbers. The shift map $t : X \to X$ sends $(x_0, x_1, x_2, \ldots)$ to $(x_1, x_2, x_3, \ldots)$. Here the fixed points of t are those points x satisfying $x_i = x_j$ for all $i, j \in \omega$; similarly the periodic points are those x for which there is some $k \geq 1$ such that $x_{i+k} = x_i$ holds for all $i \in \omega$.

We fix some notation.

10.12. Notation (a) For $y = (y_0, y_1, y_2, \ldots) \in X$ and $k \leq n$ in ω, we put

$$\mathrm{seg}_{kn}(y) = (y_k, \ldots, y_n),$$

the segment of y between k and n,

$$\mathrm{seg}_n(y) = (y_0, \ldots, y_n),$$

the initial segment of y of length $n + 1$, and

$$\mathrm{Seg}(y) = \{\mathrm{seg}_{kn} : k, n \in \omega, k \leq n\},$$

the family of all segments of y. Similarly, for any segment $v = (y_j, \ldots, y_{j+k})$ of y,

$$\mathrm{Seg}(v) = \{\mathrm{seg}_{im}(y) : j \leq i \leq m \leq j + k\}$$

denotes the family of all subsegments of v.
(b) The sets $U_n(y) = \{z \in X : \mathrm{seg}_n(z) = \mathrm{seg}_n(y)$, $n \in \omega$, constitute a neighbourhood base of y in X. A natural number i belongs to the return set of y to $U_n(y)$ iff $\mathrm{seg}_{i,i+n}(y) = \mathrm{seg}_n(y)$, i.e.

$$R(y, U_n(y)) = \{i \in \omega : \mathrm{seg}_{i,i+n}(y) = \mathrm{seg}_n(y)\}.$$

10.13. Example Using this notation, we see that y is a recurrent point of X iff for every $n \in \omega$ there is some $i \geq 1$ such that $\mathrm{seg}_{i,i+n}(y) = \mathrm{seg}_n(y)$, i.e. iff every finite (initial) segment u of y appears once more in y – even infinitely often (cf. 10.5).

10.14. Example Similarly, y is uniformly recurrent iff for every $n \in \omega$, the return set $R(y, U_n(y))$ of y to its neighbourhood $U_n(y)$ is syndetic, i.e. iff every finite (initial) segment $u = \mathrm{seg}_n(y)$ of y appears "syndetically often" in y. By the characterization 8.10 of syndeticity, this means that there is some $k \geq n$ such that for every $i \in \omega$, the segment $seg_{i,i+k}(y)$ has u as a subsegment, i.e. that every segment v of y of length k has u as a subsegment.

Exercises

(1) This continues Exercise 4 of Chapter 8. For a monoid S, prove that the following are equivalent:
(a) there is a dynamical system over S without recurrent points
(b) there is a dynamical system over S such that every point x has a neighbourhood U satisfying $R(x, U) = \{1_S\}$
(c) 1_S is the only idempotent of βS
(d) S is a finite group.