CHAPTER 8

# Large subsets of semigroups

In the van der Waerden theorem 7.5, we are given a finite colouring $\omega = A_1 \cup \cdots \cup A_r$ of the commutative semigroup $(\omega, +)$; the remark 7.7(b) states that (at least) one of the sets $A_i$ is large in the sense that it includes arbitrarily long arithmetic progressions. Can we say which of the sets $A_i$ has this largeness property? Respectively, in the abstract Hales-Jewett theorem 7.2, which of the sets $B_i$ is large in the sense that it includes $\{\sigma(v) : \sigma \in \Sigma\}$, for some $v \in V \setminus W$? A simple answer is obtained by studying the proof of 7.2: we fix a minimal idempotent $q$ of $\beta W$ and see that $B_i$ is large if $B_i \in q$. This way of reasoning, however, is utterly non-constructive, using the theory of (minimal idempotent) ultrafilters on $W$, which depends heavily on Zorn's lemma. But in 8.23, we will see that piecewise syndetic subsets of $W$ are large in the sense quoted above; similarly, every piecewise syndetic subset of $\omega$ includes arbitrarily long arithmetic progressions. Moreover these sets have an elementary description.

In this chapter, we will consider subsets of an arbitrary semigroup $S$ which can be considered as large, in different respects: thick sets, central sets, IP-sets (known from Chapter 5), syndetic and piecewise syndetic sets.

As an overview, let us state here the dependencies of the notions to be studied:

- thick sets are central
- central sets are both IP-sets and piecewise syndetic.
- syndetic sets are piecewise syndetic.

The notions of thick resp. syndetic or piecewise syndetic subsets of $S$ have quite elementary definitions; consequently, in the standard example $(\omega, +)$ of a semigroup, the sets with one of these properties can be nicely described.

We denote by $\mathcal{A}_{thick}, \mathcal{A}_{cent}, \mathcal{A}_{IP}, \mathcal{A}_{synd}, \mathcal{A}_{pws}$ the families of thick, resp. central, IP-, syndetic or piecewise syndetic subsets of $S$. We will meet the following types of behaviour, for the families $\mathcal{A}_{thick}$ through $\mathcal{A}_{pws}$.

**8.1. Definition** Let $\mathcal{A}$ be a family of subsets of some set $S$.
(a) $\mathcal{A}$ is called *upward closed* if, for any $A \in \mathcal{A}$ and $A \subseteq B \subseteq S$, it follows that $B \in \mathcal{A}$.
(b) $\mathcal{A}$ is *partition regular* if, for any $A \in \mathcal{A}$ and $A = B \cup C$, $B \in \mathcal{A}$ or $C \in \mathcal{A}$ holds. Hence if $A \in \mathcal{A}$ and $A = A_1 \cup \cdots \cup A_r$ is a partition of $A$ into finitely many parts, then one of the parts $A_j$ is in $\mathcal{A}$.
(c) A family which is both upwards closed and partition regular is called a *coideal*.

In some cases, we will characterize the sets $A$ in one of the families $\mathcal{A}_{thick}$ through $\mathcal{A}_{pws}$ as those for which the corresponding Stone set $\hat{A}$ has some largeness property, in the right topological semigroup $\beta S$. E.g. by 5.5, IP-sets are just the sets belonging to some idempotent ultrafilter, i.e. those sets $A$ such that $\hat{A}$ meets $E(\beta S)$. Likewise the piecewise syndetic subsets of $S$ turn out to be exactly those belonging to some ultrafilter in the least ideal $K(\beta S)$ of $\beta S$, i.e. those sets $A$ such that $\hat{A}$ meets $K(\beta S)$ (8.17). The following easy observation sheds some light onto this connection. We say that $M \subseteq \beta S$ provides an *ultrafilter definition* of $\mathcal{A} \subseteq \mathcal{P}(S)$ if $\mathcal{A} = \{A \subseteq S : \hat{A} \cap M \neq \emptyset\}$; in other words, $A \in \mathcal{A}$ iff $A \in p$ for some $p \in M$, resp. $\mathcal{A} = \bigcup M$.

**8.2. Proposition** *A family $\mathcal{A}$ of subsets of a set $S$ is a coideal iff it has an ultrafilter definition, i.e. there is a subset $M$ of $\beta S$ such that $\mathcal{A} = \bigcup M$.*

Proof. A family with an ultrafilter definition is clearly a coideal. For the converse, assume that $\mathcal{A}$ is a coideal and put $M = \{p \in \beta S : p \subseteq \mathcal{A}\}$, i.e. an ultrafilter is in $M$ iff all of its elements are in $\mathcal{A}$. This definition of $M$ guarantees that $\bigcup M \subseteq \mathcal{A}$. Conversely, assume for contradiction that $A \in \mathcal{A}$ but there is no ultrafilter $p$ satisfying $A \in p$ and $p \subseteq \mathcal{A}$. I.e. for every $p \in \hat{A}$ there is some $B_p \in p$ such that $B_p \notin \mathcal{A}$. The compact subset $\hat{A}$ of $\beta S$ is covered by the open sets $\widehat{B_p}$, $p \in \hat{A}$; so there is a finite subset $I$ of $\hat{A}$ such that $\hat{A} \subseteq \bigcup_{p \in I} \widehat{B_p}$. It follows that $A \subseteq \bigcup_{p \in I} B_p$; therefore $\bigcup_{p \in I} B_p \in \mathcal{A}$ and there is some $p \in I$ such that $B_p \in \mathcal{A}$. A contradiction. □

We will see that besides $\mathcal{A}_{IP}$, also the families $\mathcal{A}_{cent}$ and $\mathcal{A}_{pws}$ have ultrafilter definitions, but not the families $\mathcal{A}_{thick}$ and $\mathcal{A}_{synd}$.

In many of the definitions, statements, and proofs of this chapter, we will use some of the theory of minimal left ideals and minimal idempotents in $\beta S$, developed in Section 6.

## 1. Thick sets

In the following, we work in an arbitrary semigroup $(S, \cdot)$. Recall the notation in 1.11 and 4.5: for $x \in S$ and $A \subseteq S$, we defined the (left resp. right) *forward translations* of $A$ by $x$

$$xA = \{xa : a \in A\}, \quad Ax = \{ax : a \in A\}$$

and the (left) *backward translation* of $A$ by $x$

$$x^{-1}A = \{s \in S : xs \in A\}.$$

And for any ultrafilter $p$ over $S$, we know from 4.6 that

$$\begin{aligned} x^{-1}A \in p & \text{ iff there is some } W \in p \text{ such that } xW \subseteq A \\ & \text{ iff } A \in xp \\ & \text{ iff } xp \in \hat{A}\ . \end{aligned}$$

In the case of an additive semigroup $(S, +)$, we will write correspondingly

$$-x + A = \{s \in S : x + s \in A\}.$$

**8.3. Definition** A subset $T$ of $S$ is *thick* if the familiy $\{x^{-1}\,T : x \in S\}$ of all backwards translations of $S$ has the finite intersection property.

The following characterizations of thick sets are straightforward but, nevertheless, quite helpful.

**8.4. Theorem** *For a subset $T$ of $S$, the following are equivalent.*
*(a) $T$ is thick*
*(b) for every finite subset $e$ of $S$, there is some $y \in S$ such that $ey \subseteq T$*
*(c) for some $p \in \beta S$, the (closed) left ideal $\beta S \cdot p$ of $\beta S$ is included in $\hat{T}$*
*(d) there is a minimal left ideal $L$ of $\beta S$ such that $L \subseteq \hat{T}$.*

Proof. (a) is equivalent to (b): $T$ is thick iff, for every finite subset $e$ of $S$, there is some $y \in S$ contained in $\bigcap_{x\in e} x^{-1}\,T$. The latter means that $xy \in T$ holds for every $x \in e$, i.e. that $ey \subseteq T$.
(a) is equivalent to (c): the family $\{x^{-1}\,T : x \in S\}$ has the finite intersection property iff there is an ultrafilter $p$ on $S$ such that $\{x^{-1}\,T : x \in S\} \subseteq p$. By the equivalences quoted above , this means that $xp \in \hat{T}$ holds for every $x \in S$, i.e. that $Sp \subseteq \hat{T}$. And by continuity of right multipliction by $p$, this is equivalent to $\beta S \cdot p \subseteq \hat{T}$.
(c) is equivalent to (d): the semigroup $\beta S$ is abundant, by 6.10, so every left ideal includes a minimal one. ☐

**8.5. Example** We work in the additive semigroup $(\omega, +)$.
(a) Here every finite subset $e$ is included in the initial segment $\{0, \ldots, k\}$ of $\omega$, for some $k \in \omega$, and for $y \in \omega$, $\{0, \ldots, k\} + y$ is the interval $\{y, \ldots, k + y\}$ of length $y$. By the theorem just proved, a subset of $\omega$ is thick iff it includes arbitrary long finite intervals.
(b) This criterion shows that there is a partition $\omega = A \cup B$ of $\omega$ in which neither $A$ nor $B$ are thick sets. So the family of thick sets is not partition regular and by 8.2, it has no ultrafilter definition. On the other hand, there is a partition $\omega = C \cup D$ of $\omega$ in which both $C$ and $D$ are thick.

## 2. Central sets and IP-sets

In contrast to many of the other largeness notions of this chapter, the notion of a central set has a short but quite abstract definition, but no simple elementary description. We will give an elementary (but not really simple) description of central sets in Chapter 11.

**8.6. Definition** A subset $A$ of an arbitrary semigroup $(S, \cdot)$ is *central* iff $A \in p$ for some minimal idempotent $p$ of $\beta S$, i.e. iff the Stone set $\hat{A}$ meets the subset $E_{min}(\beta S)$ of $\beta S$.

**8.7. Proposition** *Every thick set is central, and every central set is an IP-set.*

Proof. The first implication follows immediately from 8.4: if $A$ is a thick subset of a semigroup $S$, then its Stone set $\hat{A}$ includes a minimal left ideal $L$ of $\beta S$; $L$ is a closed subsemigroup of $\beta S$ and hence has an idempotent element $e$ by 1.21. By the abundancy property of $\beta S$ proved in 6.10, $e$ is a minimal idempotent. And the second implication holds by 5.5: IP-sets are exactly the sets contained in an idempotent ultrafilter. □

The families of central resp. of IP-sets are both coideals: this holds for $\mathcal{A}_{cent}$ by the ultrafilter definition of centrality and it follows similarly for $\mathcal{A}_{IP}$ by the ultrafilter characterization 5.5 of IP-sets.

## 3. Syndetic sets

Syndetic sets have a simple description, by their very definition, and they are closely connected to thick sets. The notion of syndetic resp. piecewise syndetic sets arises in topological dynamics, a subject we will begin to study in the next chapter.

**8.8. Definition** A subset $A$ of a semigroup $(S, \cdot)$ is syndetic if there is a finite subset $e$ of $S$ such

$$\bigcup_{t \in e} t^{-1}A = S,$$

i.e. if $S$ can be covered by finitely many backwards translations of $A$.

The equality $\bigcup_{t\in e} t^{-1}A = S$ means that, for every $s \in S$, there is $t \in e$ such that $ts \in A$. This explains, to some extent, the word *syndetic* (*connecting*, in Greek language): every $s \in S$ is connected to $A$, by left multiplication by some $t \in e$.

As an example, consider the additive semigroup $\mathbb{N}$ of natural numbers. The subset $A$ of even numbers and the set $B$ of odd numbers are both syndetic, via $e = \{1, 2\}$.

The connection between thick and syndetic sets is as follows.

**8.9. Theorem** *Let $S$ be an arbitrary semigroup, $A$ and $T$ subsets of $S$.*
*(a) $A$ is syndetic iff it meets every thick subset of $S$ iff $S \setminus A$ is not thick.*
*(b) $T$ is thick iff it meets every syndetic subset of $S$ iff $S \setminus T$ is not syndetic.*
*(c) $A$ is syndetic iff its Stone set $\hat{A}$ meeets every (minimal) left ideal of $\beta S$.*

Proof. We first prove that, if $T \subseteq S$ is thick and $A \subseteq S$ is syndetic, then $A \cap T \neq \emptyset$: fix a finite $e \subseteq S$ such that $\bigcup_{t\in e} t^{-1}A = S$. Pick some $s \in \bigcap_{t\in e} t^{-1}T$ and then some $t \in e$ such that $s \in t^{-1}A$. Thus $ts \in A \cap T$.
For the rest of the proof, we will use the equality $S \setminus x^{-1}B = x^{-1}(S \setminus B)$, for $B \subseteq S$ and $x \in S$.
(a) Assume $A$ is not syndetic; then $\bigcup_{t\in e} t^{-1}A \neq S$ holds for every finite subset $e$ of $S$. Passing to complements, we see that $\bigcap_{t\in e} t^{-1}(S \setminus A) \neq \emptyset$ holds for every finite $e$, and this means that $S \setminus A$ is thick, i.e. that $A$ is disjoint to some thick set.
(b) Assume $T$ is not thick; then $\bigcap_{t\in e} t^{-1}T = \emptyset$ holds for some finite subset $e$ of $S$. It follows that $\bigcup_{t\in e} t^{-1}(S \setminus T) = S$, i.e. that $S \setminus T$ is syndetic.
(c) We use (a) and 8.4: $A$ is syndetic iff $S \setminus A$ is not thick iff no left ideal of $\beta S$ is included in $\widehat{S \setminus A} = \beta S \setminus \hat{A}$ iff every left ideal of $\beta S$ meets $\hat{A}$. □

**8.10. Example** In the semigroup $(\omega, +)$, there is a simple elementary description of syndetic subsets.
(a) A subset $A$ of $\omega$ is syndetic iff it has the following *bounded gaps* property: there

is some $k \in \omega$ such that each interval of length $k$ in $\omega$ meets $A$. (I.e. the distance between any $x \in \omega$ and the least $y \in A$ satisfying $x < y$ is at most $k$.) This can be concluded either from the characterization of thick subsets of $\omega$. It follows also directly from the definition of syndeticity: $A$ is syndetic iff this can be verified by some finite subset of the form $e = \{0, \cdots, k\}$ of $\omega$. For such a set $e$, $\omega = \bigcup_{i \in e}(A - i)$ means that, for every $x \in \omega$, there is $0 \leq i \leq k$ such that $x + i \in A$.
(b) As noted in 8.5, there are partitions $\omega = A \cup B = C \cup D$ in which $A, B$ are both thick and $C, D$ are not thick. The preceding theorem shows that $A, B$ are non-syndetic and $C, D$ are both syndetic. In particular, the family $\mathcal{A}_{synd}$ of syndetic subsets of $\omega$ is not partition regular (but, of course, upwards closed).

## 4. Piecewise syndetic sets

Piecewise syndetic subsets of a semigroup have both a more complicated definition and, in $(\omega, +)$, a still comprehensible but less easy elementary description than syndetic ones. We will see how they are connected, in different ways, to nearly all of the other sets we have encountered so far, the deepest connection being 8.22.

**8.11. Definition** A subset $A$ of a semigroup $(S, \cdot)$ is *piecewise syndetic* if there is a finite subset $e$ of $S$ such that the set $\bigcup_{t \in e} t^{-1}A$ is thick, in $S$.

Since $S$ is a thick subset of itself, we obtain the following implication.

**8.12. Proposition** *Every syndetic set is piecewise syndetic.*

A description of piecewise syndetic sets avoiding the notion of thickness runs as follows. Put $T = \bigcup_{x \in e} x^{-1}A$, where $e \subseteq S$ is finite; then $t \in T$ iff there is some $x \in e$ such that $xt \in A$. Now $T$ is thick iff for every finite $f \subseteq S$, there is some $s \in \bigcap_{y \in f} y^{-1}T$; the latter means that for all $y \in f$, $ys \in T$ holds.
Consequently, $A$ is piecewise syndetic iff there is some finite $e$ such that for every finite $f$, there is some $s \in S$ such that for every $y \in f$ there is some $x \in e$ satisfying $xys \in A$.

**8.13. Example** We work in the semigroup $(\omega, +)$.
(a) For an (infinite) subset $A$ of $\omega$ and $x \in \omega$, write $d(x, A)$ for the least $i \in \omega$ such that $x + i \in A$, the distance from $x$ to the next larger element of $A$.
(b) Thus for $e = \{0, \ldots, k\}$, a finite subset of $\omega$, and $T = \bigcup_{x \in e}(-x + A)$, $t \in T$ means that $d(t, A) \leq k$.
(c) So for $n, s \in \omega$, $s + \{0, \ldots, n\} \subseteq T$ means that each of the numbers $s, \ldots, s + n$ has distance at most $k$ to $A$. Such an $s$ exists (for given $n$) iff there is an interval $I \subseteq \omega$ of length $n$ such that every element of $I$ has distance at most $k$ to $A$.
(d) We say that $A$ has gaps of length at most $k$ in an interval $I \subseteq \omega$ if every element $x$ of $I$ satisfying $x + k \in I$ has distance at most $k$ to $A$, i.e. if at least every $k$'th element of $I$ is in $A$.
(e) It follows that $A$ is piecewise syndetic in $(\omega, +)$ iff there is some $k$ such that for every $n$ there is an interval $I \subseteq \omega$ of length $n$ such that $A$ has gaps of length at most $k$ in $I$. Or, in a simpler way: $A$ is piecewise syndetic iff for some $k$, $\omega$ has arbitrarily long intervals in which $A$ has gaps of length at most $k$.

For further reference, we point out a canonic construction which associates, with every piecewise syndetic set $A$, objects $e, T, L, \varepsilon$ and $x$. Their properties seem to be what makes piecewise syndetic sets work.

**8.14. Remark** *(the first piecewise syndetic construction)* Assume $A$ is a piecewise syndetic subset of $S$. Then fix a finite subset $e$ of $S$ such that the set $T = \bigcup_{t\in e} t^{-1}A$ is thick. By 8.4, fix a minimal left ideal $L$ of $\beta S$ such that $L \subseteq \hat{T}$. By 1.21, fix an idempotent element $\varepsilon$ in $L$; by 6.13, it is a minimal idempotent of $\beta S$. Then

$$\varepsilon \in L \subseteq \hat{T} = \bigcup_{t\in e} t^{-1}A$$

and thus we can fix some $x \in e$ such that $\varepsilon \in \widehat{x^{-1}A}$. The latter says that $x^{-1}A \in \varepsilon$, resp. that $A \in x\varepsilon$.

This yields part of the main theorems on piecewise syndetic sets. Recall the theory in Chapter 6, in particular that $K(\beta S)$ is the least two sided ideal of $\beta S$, the union of all minimal left ideals.

**8.15. Corollary** *If $A$ is piecewise syndetic, then there are $x \in S$ and a minimal idempotent $\varepsilon$ such that $x^{-1}A \in \varepsilon$, so $x^{-1}A$ is central. Both $\varepsilon$ and $x\varepsilon$ are in the least ideal $K(\beta S)$ and $x\varepsilon \in \hat{A}$, thus the Stone set $\hat{A}$ of $A$ meets $K(\beta S)$.*

**8.16. Remark** *(the second piecewise syndetic construction)* Assume that $A$ is a piecewise syndetic set. Using 8.14 and its notation, there is a minimal left ideal $L$ of $\beta S$ such that $L \subseteq \bigcup_{t\in S} \widehat{t^{-1}A}$. For every $p \in L$, it follows that $p \in K(\beta S)$ and $L = \beta S \cdot p$.
But we can prove more: given *an arbitrary* $p \in K(\beta S)$ such that $A \in p$, the unique minimal left ideal $M = \beta S \cdot p$ of $\beta S$ containing $p$ satisfies $M \subseteq \bigcup_{t\in S} \widehat{t^{-1}A}$.
To prove this, let $q$ be any point of $M$. Since $M$ is a minimal left ideal, we conclude that $M = \beta S \cdot q$. Now $p \in M$, $M$ is the topological closure of $Sq$, and $\hat{A}$ is a neighbourhood of $p$. Thus there is some $t \in S$ such that $tq \in \hat{A}$, i.e. that $q \in \widehat{t^{-1}A}$. Finally, since $M$ is a compact subset of $\beta S$ and the sets $\widehat{t^{-1}A}$ are open, there is a finite subset $f$ of $S$ such that $p \in M \subseteq \bigcup_{t\in f} \widehat{t^{-1}A}$ holds.

Theorem 8.17 gives an ultrafilter definition of piecewise syndetic sets, which proves that the family of piecewise syndetic sets is a coideal.

**8.17. Theorem** *A subset $A$ of $S$ is piecewise syndetic iff $A \in p$ for some ultrafilter $p \in K(\beta S)$, i.e. iff $\hat{A} \cap K(\beta S) \neq \emptyset$.*

Proof. One direction was already stated in the corollary 8.15, and the reverse one follows from 8.16: if $p$ is a point in $K(\beta S) \cap \hat{A}$, then the minimal left ideal $\beta S \cdot p$ is included in $\bigcup_{t\in f} \widehat{t^{-1}A}$ for some finite $f \subseteq S$. By 8.4, the set $\bigcup_{t\in f} t^{-1}A$ is thick, and thus $A$ is piecewise syndetic. □

The minimal idempotents of $\beta S$ are in $K(\beta S)$, so 8.17 yields another implication between our notions of largeness.

**8.18. Corollary** *Every central set is piecewise syndetic.*

The final results of this subsection show a close connection between piecewise syndetic and central sets. We start with two additional pieces of notation and a few simple computational rules resp. remarks which will be helpful in the rest of the section.

**8.19. Definition** (a) For an arbitrary subset $U$ of $\beta S$ and arbitrary $x \in S$, we write $x^{-1}U$ for the set $\{p \in \beta S : xp \in U\}$, the *backward (left) translation* of $U$ by $x$. This generalizes the notion $x^{-1}A = \{s \in S : xs \in A\}$ from 4.5, for $A \subseteq S$.
(b) For $U \subseteq \beta S$ and $p \in \beta S$, the *return set* $R(p, U)$ of $p$ to $U$ is defined as $\{s \in S : sp \in U\}$.

**8.20. Remark** (a) For $A \subseteq S$, $x \in S$, and $p \in \beta S$, the following are equivalent: $p \in x^{-1}\hat{A}$; $xp \in \hat{A}$, i.e. $x \in R(p, \hat{A})$; $A \in xp$; $x^{-1}A \in p$; $p \in \widehat{x^{-1}A}$.
(b) This equivalence shows that $\widehat{x^{-1}A} = x^{-1}\hat{A}$ – note that here the set $x^{-1}A$ is computed in $S$, and $x^{-1}\hat{A}$ in $\beta S$.
(c) Assume that $U \subseteq \beta S$, $p \in \beta S$, and $t \in S$. Then for any $s \in S$, the following are equivalent: $sp \in t^{-1}U$; $tsp \in U$; $ts \in R(p, U)$; $s \in t^{-1}R(p, U)$. This shows that

$$t^{-1}R(p, U) = R(p, t^{-1}U).$$

(d) It follows that for $U \subseteq \beta S$, $p \in \beta S$, $R = R(p, U)$, and $e$ an arbitrary subset of $S$, $Sp \subseteq \bigcup_{t \in e} t^{-1}U$ is equivalent to $S = \bigcup_{t \in e} t^{-1}R$.
(e) As in (c), assume that $U \subseteq \beta S$ and $p \in \beta S$. Then for arbitrary $y, t \in S$ and $s = ty$, we have

$$t^{-1}R(p, y^{-1}U) = s^{-1}R(p, U),$$

because $t^{-1}R(p, y^{-1}U) = y^{-1}t^{-1}R(p, U) = (ty)^{-1}R(p, U) = s^{-1}R(p, U)$.
(f) It follows from (e) that for $e \subseteq S$ and $f = e \cdot y = \{ty : t \in e\}$,

$$\bigcup_{t \in e} t^{-1}R(p, y^{-1}U) = \bigcup_{s \in f} s^{-1}R(p, U).$$

We now see that being an element of $K(\beta S)$ is closely connected to the behaviour of return sets. We will meet a more general version of the next theorem in Chapter 10.

**8.21. Theorem** *For $A \in p \in K(\beta S)$, the return set $R = R(p, \hat{A}) = \{x \in S : xp \in \hat{A}\}$ of $p$ to $\hat{A}$ is syndetic.*

Proof. By 8.17, we conclude that $A$ is piecewise syndetic. Applying the second piecewise syndetic construction, we obtain $S \cdot p \subseteq \beta S \cdot p \subseteq \bigcup_{t \in f} \widehat{t^{-1}A}$, for some finite $f \subseteq S$. By remark 8.20 (d), this means that $S = \bigcup_{t \in f} t^{-1}R$, and thus $R$ is syndetic. □

We finally establish the intimate connection between central sets, syndetic sets, and piecewise syndetic sets.

**8.22. Theorem** *For any subset $A$ of a semigroup $S$, the following are equivalent.*
*(a) $A$ is piecewise syndetic*
*(b) for some $x \in S$, $x^{-1}A$ is central*
*(c) the set $\{x \in S : x^{-1}A$ is central$\}$ is syndetic.*

PROOF. Trivially, (c) impies (b).
(b) implies (a): assume $x^{-1}A$ is a central set, e.g. $x^{-1}A \in \varepsilon$ holds for some minimal idempotent $\varepsilon$ in $K(\beta S)$. Then $A \in x\varepsilon$, an ultrafilter in $K(\beta S)$; hence $A$ is piecewise syndetic by 8.17.
(a) implies (c): assume that $A$ is piecewise syndetic. By the first piecewise syndetic construction, fix $y \in S$ and a minimal idempotent $\varepsilon$ such that $y^{-1}A \in \varepsilon$. The return set $R(\varepsilon, \widehat{y^{-1}A})$ is syndetic, by 8.21, so fix a finite subset $e$ of $S$ such that $S = \bigcup_{t\in e} t^{-1}R(\varepsilon, \widehat{y^{-1}A})$. By 8.20(f), this means that $S = \bigcup_{s\in f} s^{-1}R(\varepsilon, \widehat{A})$, where $f = e \cdot y$ is a finite subset of $S$; thus $Z = R(\varepsilon, \widehat{A})$ is syndetic, too. But $x \in Z$ means that $x\varepsilon \in \widehat{A}$; i.e. $\varepsilon \in x^{-1}\widehat{A} = \widehat{x^{-1}A}$, so $x^{-1}A$ is central for every $x \in Z$. □

## 5. Combinatorial largeness of piecewise syndetic sets

We prove that piecewise syndetic sets are combinatorially large, in the sense sketched in the beginning of the chapter.

**8.23. Theorem** *Assume that $V, W$ and $\Sigma$ are given as in the abstract Hales-Jewett theorem 7.2, i.e. $W$ is a nice subsemigroup of $V$, and $\Sigma$ is a finite set of retractions from $V$ to $W$. Then for every piecewise syndetic subset $B$ of $W$, there is some $v \in R = V \setminus W$ such that $\{\sigma(v) : \sigma \in \Sigma\} \subseteq B$.*

PROOF. By 8.15, we fix some $x \in W$ such that $x^{-1}B$ is central, say $x^{-1}B \in q$ where $q$ is a minimal idempotent of $\beta W$. We proceed as in the proof of 7.2: fix a minimal idempotent $p$ of $\beta V$ such that $p \leq q$ and observe that $\sigma(p) = q$ holds for all $\sigma \in \Sigma$. Take some

$$a \in R \cap \bigcap_{\sigma\in\Sigma} \sigma^{-1}[x^{-1}B];$$

(a set in $p$, thus non-empty); so $a \in R$ and $x\sigma(a) \in B$ holds for all $\sigma \in \Sigma$. Then $v = xa$ works for the theorem: $v \in R$ because $R$ is a (two sided) ideal of $V$. And for all $\sigma \in \Sigma$, we have $\sigma(v) = x\sigma(a) \in B$ because $x \in W$ and thus $\sigma(x) = x$. □

Let us point out that the theorem immediately implies the abstract Hales-Jewett theorem 7.2, because for $W = B_1 \cup \cdots \cup B_r$ a finite partition of $W$, at least one $B_j$ is piecewise syndetic. It also implies refined versions of the consequences of 7.2 in Chapter 7.

**8.24. Corollary** *Assume $(S, +)$ is a commutative semigroup, $A \subseteq S$ is piecewise syndetic, and $E$ is a finite subset of $S$. Then there are $a \in S$ and $d \in \mathbb{N}$ in such that $\{a + de : e \in E\} \subseteq A$.*

**8.25. Corollary** *Every piecewise syndetic subset of $(\omega, +)$ includes arithmetic progressions of arbitrary length.*

**8.26. Corollary** *Assume that $k \in \mathbb{N}$, $A$ is a piecewise syndetic subset of $(\omega^k, +)$, and that $n \in \mathbb{N}$. Then $A$ includes a homothetic copy of the $k$-dimensional cube $\{0, 1, \ldots, n\}^k$.*

The following example demonstrates that in 8.25, the consequence (including arbitrarily long arithmetic progressions) is weaker than the assumption of being piecewise syndetic.

**8.27. Example** We define a subset $A$ of $\omega$ as follows. Fix disjoint subintervals $I_n \subseteq \omega$ of length $n^2$, for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $m_n$ be the least element of $I_n$ and put

$$A_n = \{m_n + in : 0 \leq i < n\},$$

i.e. $A_n$ contains exactly every $n$'th element of $I_n$. Define

$$A = \bigcup_{n \in \mathbb{N}} A_n.$$

So $A$ includes arithmetic progressions of arbitrary length. But, by the characterization in 8.27, it is not piecewise syndetic. In fact, for every $k \in \mathbb{N}$, $A$ has gaps at most $k$ in the interval $I_k$ of size $k^2$, but there is no fixed $k$ such that $A$ has gaps at most $k$ in arbitrarily long intervals.

## Exercises

(1) Prove Proposition 8.2 as follows. For a coideal $\mathcal{A}$, put $f = \{A \subseteq S : S \setminus A \notin \mathcal{A}\}$. Show that $f$ is a filter on $S$; so by Exercise 1 in Chapter 2, there is a subset $M$ of $\beta S$ satisfying $f = \bigcap M$. Conclude that $\mathcal{A} = \bigcup M$.

(2) Prove that every thick subset of the semigroup $(\omega, +)$ can be partitioned into countably many thick subsets. Similarly, every syndetic subset of $(\omega, +)$ can be partitioned into countably many syndetic subsets.

(3) Let $A \subseteq S$ be syndetic, resp. piecewise syndetic, and $x \in S$. Prove or disprove by a counterexample that $xA$ resp. $x^{-1}A$ is syndetic resp. piecewise syndetic.

(4) Let $(S, \cdot, 1)$ be a finite group; let $A$ and $T$ be subsets of $S$. Prove that
(a) $T$ is thick iff $T = S$
(b) $A$ is central iff it is an IP-set iff $1 \in A$
(c) $A$ is syndetic iff it is piecewise syndetic iff it is non-empty.
(d) Conversely, show that if a semigroup $S$ has an identity 1 and $\{1\}$ is syndetic, then $S$ is a finite group.

(5) Prove the converse of 8.21: if $p \in \beta S$ has the property that for every $A \in p$, the return set $R = R(p, \hat{A})$ of $p$ to $\hat{A}$ is syndetic, then $p \in K(\beta S)$.