CHAPTER 4

βS as a semigroup

In this chapter, we assume that $(S, \cdot)$ is an arbitrary semigroup, equipped with the discrete topology. As explained in Chapter 3, we will consider S as a (dense) subset of its Stone-Čech compactification βS. Using the abstract universal property 3.12 of βS, we extend in Section 1 the multiplication given on S to βS in a manner which makes right and left translations on βS continuous, to a considerable extent. It turns out that βS, with the extended multiplication, is a semigroup, too, and in fact a right topological one.

For Section 2, recall that the points of βS are the ultrafilters on S. We give precise descriptions of the subsets of S contained in the product $p \cdot q$ of two ultrafilters p and q. The most concrete one, (d) of 4.7, is the one frequently used in combinatorial applications of the theory of βS; also the other ones and the notation set up here turns out to be helpful.

βS being a compact right topological semigroup, we conclude from 1.21 that it has idempotent elements. In Section 3, we draw a first and particularly simple combinatorial conclusion of this fact, Schur's theorem.

The smooth working of the first two sections may suggest that simple algebraic properties of the semigroup S carry over to βS, by continuity. But this is far from being true: in Section 4, we show that, in most cases, βS fails to be commutative, even if S is.

1. The multiplication on βS

We will define the product $p \cdot q$ of two points p, q of βS in two steps. For the sake of clarity, we slightly change the notation from Definition 1.19(a): for any $s \in S$, we have the operation $l_s : S \to S$ of left translation by s, given by $l_s(x) = sx$. Since $S \subseteq \beta S$, we also view l_s as being a map from S into the compact Hausdorff space βS.

4.1. Definition (a) For $s \in S$, $\widetilde{l_s} : \beta S \to \beta S$ is the Stone-Čech extension of $l_s : S \to \beta S$. For $q \in \beta S$, we write $s \cdot q = \widetilde{l_s}(q)$. – This defines the product $s \cdot q$, for arbitrary $s \in S$ and $q \in \beta S$.
(b) Let $q \in \beta S$ and let us write r_q for the function from S to βS defined by $r_q(s) = s \cdot q$. Let $\widetilde{r_q} : \beta S \to \beta S$ be the Stone-Čech extension of $r_q : S \to \beta S$. For $p \in \beta S$, we write $p \cdot q = \widetilde{r_q}(p)$. – This defines the product $p \cdot q$, for arbitrary $q \in \beta S$ and $p \in \beta S$. Steps (a) and (b) ensure that the multiplication thus defined on βS extends the multiplication on S.
(c) As usual, we will abbreviate $p \cdot q$ by pq.

Having defined the product of arbitrary points of βS, we pick up the notation 1.19(a). I.e. for $p \in \beta S$, we define $\lambda_p : \beta S \to \beta S$ by $\lambda_p(q) = pq$; similarly for for $q \in \beta S$, we define $\rho_q : \beta S \to \beta S$ by $\rho_q(p) = pq$. Definition 4.1 ensures that the right translation ρ_q is continuous for every $q \in \beta S$; in addition, the left translation λ_s is continuous for every $s \in S$.

4.2. Theorem *With the multiplcation defined in 4.1, βS is a compact right topological semigroup.*

PROOF. The only fact still to be checked is the associative law

$$p(qr) = (pq)r \tag{A}$$

for arbitrary elements p, q and r of βS. We prove it in three steps. To begin with, we know that (A) holds for the most special case where p, q, r are all in S, say $p = s \in S$, $q = t \in S$, and $r = u \in S$.
To prove (A) for $p = s \in S$, $q = t \in S$, and arbitrary $r \in \beta S$, we fix $s, t \in S$ and note that both sides of the equation $s(tr) = (st)r$ are continuous functions of r – more precisely, the left hand side can be written as $\lambda_s(\lambda_t(r))$ and the right hand one as $\lambda_{st}(r)$. These functions coincide on S and hence on the whole of βS, since S is dense in βS.
In a second step, we prove (A) for fixed $p = s \in S$, fixed $r \in \beta S$, and arbitrary $q \in \beta S$. Both $s(qr) = \lambda_s(\rho_r(q))$ and $(sq)r = \rho_r(\lambda_s(q))$ are continuous functions of q coinciding on S and hence on the whole of βS.
The last step proves (A) for fixed $q, r \in \beta S$ and abitrary $p \in \beta S$. Here $p(qr) = \rho_{qr}(p)$ and $(pq)r = \rho_r(\rho_q(p))$ are continuous functions of p, coinciding on S. □

In 4.1, we defined the product $p \cdot q$ of two ultrafilters p and q on S by abstract reasoning, using the theory of Stone-Čech extensions. To obtain a slightly more definite description, let us recall that, for any function $f : S \to \beta S$, its Stone-Čech extension $\tilde{f}$ was given in 3.12 by $\tilde{f}(p) = p - \lim_{s \in S} f(s)$. This gives a description of $p \cdot q$, using iterated limits in (b).

4.3. Remark Let $s \in S$ and $p, q \in \beta S$. Then
(a) $s \cdot q = q - \lim_{t \in S} st$.
(b) $p \cdot q = p - \lim_{s \in S} sq = p - \lim_{s \in S}(q - \lim_{t \in S} st)$.

The continuity properties of the multiplication on βS allow to construct homomorphisms between Stone-Čech compactifications of discrete semigroups in a straightforward manner. Assume that $f : S \to T$ is any function between non-empty discrete sets. We may view f as being a map from S into βT and thus consider its Stone-Čech extension $\tilde{f} : \beta S \to \beta T$.

4.4. Proposition *Assume $f : S \to T$ is a homomorphism between discrete semigroups. Then $\tilde{f} : \beta S \to \beta T$ is a semigroup homomorphism, too.*

PROOF. The equation

$$\tilde{f}(p \cdot q) = \tilde{f}(p) \cdot \tilde{f}(q)$$

holds if both $p = s \in S$ and $q = t \in S$. It follows for $p = s \in S$ and arbitrary $q \in \beta S$ because left translation by s respectively $\tilde{f}(s)$ is continuous. Finally it holds for arbitrary $p, q \in \beta S$ by continuity of right translation by q respectively $\tilde{f}(q)$. □

It should go without saying that we will denote the extension of a semigroup operation on S to βS by the same symbol. E.g. in a commutative semigroup S, the operation is usually written as an addition: $(S, +)$, and so is its extension to βS given by 4.1: $(\beta S, +)$.

2. Sets in $p \cdot q$

Assume p and q are two elements of βS, i.e. ultrafilters on S. In this section, we explain in a more down-to-earth way which subsets of S belong to the product ultrafilter $p \cdot q$; more specifically, we first do this for the product $s \cdot q$, where $s \in S$. The following notation turns out to be helpful here.

4.5. Definition (a) For $A \subseteq S$ and $s \in S$, we define

$$s^{-1}A = \{t \in S : st \in A\}.$$

(b) For $A \subseteq S$ and $q \in \beta S$, we put

$$A^{-q} = \{s \in S : s^{-1}A \in q\}.$$

If the semigroup S happens to be a group, then every $s \in S$ has an inverse s^{-1} in S, and $s^{-1}A$ is the result of multiplying A from the left with s^{-1}. In an arbitrary semigroup S however, elements usually don't have inverses; the definition of $s^{-1}A$ does not mean that the set A is multiplied with something called s^{-1} but just that an arbitrary $t \in S$ belongs to $s^{-1}A$ iff st belongs to A.

4.6. Proposition *Assume that $s \in S$, $q \in \beta S$, and $A \subseteq S$. Then $A \in sq$ iff $s^{-1}A \in q$.*

Proof. $A \in sq$ means that $sq = q - \lim_{t \in S} st \in \hat{A}$. Now, $\hat{A}$ being a clopen subset of βS, this means by 2.20 that $\{t \in S : st \in \hat{A}\} \in q$. And since $\hat{A} \cap S = A$, this is equivalent to $s^{-1}A = \{t \in S : st \in A\} \in q$. ☐

A similar characterization of the sets contained in pq uses the notation A^{-q} from Definition 4.5(b). The equivalence (d) seems to be the most applicable one, in many situations.

4.7. Theorem *Assume that $p, q \in \beta S$ and $A \subseteq S$. The following are equivalent:*
(a) $A \in pq$
(b) $A^{-q} \in p$
(c) $\{s \in S : \{t \in S : st \in A\} \in q\} \in p$
(d) there are $V \in p$ and a family $(W_v)_{v \in V}$ of sets in q such that $\bigcup_{v \in V} v \cdot W_v \subseteq A$.

Proof. Similarly as in the proof of 4.6, $A \in pq$ holds iff $pq = p - \lim_{s \in S} sq \in \hat{A}$, and this means by 4.6 that $\{s \in S : sq \in \hat{A}\} = \{s \in S : s^{-1}A \in q\} = A^{-q} \in p$. Thus (a) is equivalent to (b). The equivalence of (b) and (c) follows by writing out the definitions of $s^{-1}A$ and A^{-q}. Finally, (d) is a mere reformulation of (c). ☐

We state some simple consequences of Theorem 4.7. Let us recall the notation from Chapter 1: for $A, B \subseteq S$, $AB = A \cdot B$ is the set of all products ab where $a \in A$ and $b \in B$.

4.8. Remark Assume that A, B, C are subsets of S and $p, q \in \beta S$.
(a) If $A \in p$ and $B \in q$, then $AB \in pq$: in 4.7(d), we put $V = A$ and, for all $v \in V$, $W_v = B$; then $AB = \bigcup_{v \in V} v \cdot W_v \in pq$.
(b) Another formulation of (a) is that $\hat{A} \cdot \hat{B} \subseteq \widehat{AB}$. It follows that if $A \cdot B \subseteq C$, then $\hat{A} \cdot \hat{B} \subseteq \hat{C}$. In particular if T is a subsemigroup of S, then $\hat{T}$ is a subsemigroup of $\hat{S} = \beta S$.
(c) Let T be a subsemigroup of S; so $\hat{T}$ is a subsemigroup of βS. Proposition 3.14 describes the canonical homeomorphism f from $\hat{T}$ onto βT. It is easily checked that f is, in addition, a semigroup isomorphism from $\hat{T}$, with the multiplication inherited from βS, and βT, with the multiplication constructed in 4.1 from the semigroup structure of T.

3. Schur's theorem

We know from 1.21 that βS, being a compact right topological semigroup, has idempotent elements. We use this to prove a combinatorial fact which, for the semigroup $(\mathbb{N}, +)$, is a classical theorem due to I. Schur. It is the first and probably the most simple nontrivial application of idempotent elements in βS. In the next chapter, we will prove a powerful generalization of Schur's theorem along the same lines – Hindman's finite sums theorem.
Let us recall a notion from Chapter 2: we consider a partition $S = A_1 \cup \cdots \cup A_r$ of S into r parts A_i, $1 \leq i \leq r$, as being a colouring of S with r colours named $1, \ldots, r$ – i.e. we say that $s \in S$ has colour i if $s \in A_i$.

4.9. Theorem *Assume $(S, \cdot)$ is a semigroup and $S = A_1 \cup \cdots \cup A_r$ is a colouring of S with r colours. Then there are $j \in \{1, \ldots, r\}$ and elements v, w of S such that $v, w, vw \in A_j$ – i.e. v, w and vw have the same colour j.*

Proof. Pick any idempotent ultrafilter p in βS. One of the sets $A_1, \ldots, A_r$ has to be in p, say $A = A_j \in p$. Now $A \in p \cdot p = p$, and by Theorem 4.7, there are sets V in p and, for $v \in V$, W_v in p such that $\bigcup_{v \in V} vW_v \subseteq A$. We may assume that V and the sets W_v are subsets of A, otherwise passing to $V \cap A \in p$ and $W_v \cap A \in p$. Thus for any $v \in V$ and any $w \in W_v$, we see that v, w, and vw are in $A = A_j$. □

The proof of 4.9 works for any idempotent ultrafilter p on S, be it fixed or free. For p fixed, i.e. $p = \dot{s} \in S$, however, where $s \in S$, the theorem has a completely trivial proof – in this case, s is idempotent, and we can put $v = w = s$. In fact, the following very special case shows that putting $v = w = s$ may be the only possibility to obtain $v, w, vw \in A_j$. Let S be the semigroup considered in 1.3; i.e. we have an element $z \in S$ such that $xy = z$ holds for all $x, y \in S$. Then z is the only idempotent element of S and also of βS, as is easily checked. Now assume that $A_j = \{z\}$ happens to be one of the blocks of the given partition; clearly $v = w = z$ is the only solution for $v, w, vw \in A_j$.
On the other hand, if we start with a free idempotent ultrafilter p, in the proof of 4.9, then there are j and $v, w, vw \in A_j$ such that $v \neq w$. To achieve this, pick, V and W_v as in the above proof, and an arbitrary $v \in V$. The set W_v is infinite, since

p is free, and we can pick an arbitrary $w \in W_v \setminus \{v\}$.
In the case $(S, \cdot) = (\mathbb{N}, +)$, there are no idempotents in $\mathbb{N}$, and we obtain the following classical version of 4.9.

4.10. Corollary *(Schur's theorem) Assume $\mathbb{N} = A_1 \cup \cdots \cup A_r$ is a colouring of $\mathbb{N}$ with r colours, Then there are $j \in \{1, \ldots, r\}$ and $v < w$ in $\mathbb{N}$ such that $v, w, v + w \in A_j$ – i.e. v, w and $v + w$ have the same colour j.*

It may come as a surprise that the seemingly most general result 4.9 can be easily derived from its special case 4.10. For assume that $S = A_1 \cup \cdots \cup A_r$ is a colouring of an arbitrary semigroup $(S, \cdot)$ with finitely many colours. We fix an arbitrary element $a \in S$ and consider the semigroup homomorphism $f : (\mathbb{N}, +) \to (S, \cdot)$ given by $f(n) = a^n$ (the n'th power of a in S). The map f induces a colouring $\mathbb{N} = B_1 \cup \cdots \cup B_r$ of $\mathbb{N}$ where B_i is the preimage of A_i under f – i.e. $n \in \mathbb{N}$ obtains the same colour as $f(n) \in S$. Now pick m, n in $\mathbb{N}$ such that m, n and $m+n$ have the same colour in $\mathbb{N}$; it follows that $v = f(m), w = f(n)$, and $vw = f(m) \cdot f(n) = f(m + n)$ have the same colour in S.
A little abstract reflection, however, shows why 4.9 is so easily derivable from 4.10. The semigroup $(\mathbb{N}, +)$ is free over its generating set $\{1\}$, and therefore easily allows homomorphisms into arbitrary semigroups S – the map f above is simply the unique homomorphic extension of the map from $\{1\}$ to S mapping 1 to an arbitrary $a \in S$.

4. Non-commutativity in βS

Unless the associative law $x(yz) = (xy)z$, other algebraic laws hardly ever carry over from S to βS. The most prominent example of this effect is the failure of the commutative law $xy = yx$ in βS – see the following proposition.
Let us, however, first note one exception: if S is commutative, then for $s \in S$ and $q \in \beta S$, we have $sq = qs$. This holds for all $q \in S$ and extends to arbitrary $q \in \beta S$ because, for fixed s, both sq and qs are continuous functions of q.

4.11. Proposition *Assume that, in the semigroup S, there are two sequences $(x_k)_{k\in\omega}$ and $(y_n)_{n\in\omega}$ such that the sets $A = \{x_k y_n : k < n\}$ and $B = \{x_k y_n : n < k\}$ are disjoint. Then βS is not commutative.*

Proof. Fix an ultrafilter p on S containing the sets $X_k = \{x_n : n \geq k\}$, for all $k \in \omega$; this is possible since the family $\{X_k : k \in \omega\}$ has the finite intersection property. Similarly, fix some $q \in \beta S$ containing the sets $\{Y_k = \{y_n : n \geq k\}$.
Then by 4.7, the set A defined above is in pq, because it can be written as $\bigcup_{v\in V} vW_v$ where $V = X_0 = \{x_k : k \in \omega\}$ is in p, and for $v = x_k \in V$, $W_v = Y_{k+1} = \{y_n : n > k\}$ is in q. It follows similarly that $B \in qp$, and thus $pq \neq qp$. □

It is a simple exercise to verify the assumption of the proposition for several popular commutative semigroups, e.g. for $(\omega, +)$, $(\omega, \cdot)$, and even for seemingly more trivial ones, like the semigroup of all finite subsets of an infinite set X, under the operation of taking the union of two sets.
Let us note that if S is commutative but βS is not, then for some $p \in \beta S$, the left translation λ_p by p fails to be continuous (cf. Exercise 2.)

Exercises

(1) This exercise shows how to define the multiplication on βS without using topology. For $p, q \in \beta S$, we define $p \cdot q$ to be the set $r \subseteq \mathcal{P}(S)$ such that

$$A \in r \text{ iff there are } V \in p \text{ and, for } v \in V, W_v \in q \text{ such that } \bigcup_{v \in V} v \cdot W_v \subseteq A.$$

Prove that
(a) the set r thus defined is an ultrafilter on S
(b) the product thus defined on βS is associative.

(2) Assume that S is a commutative semigroup but βS is non-commutative. Explain that, in this case, not all left translations by elements of βS are continuous.

(3) Prove the converse of 4.11: if βS is non-commutative, then there are sequences $(x_k)_{k\in\omega}$ and $(y_n)_{n\in\omega}$ in S such that the sets $A = \{x_k y_n : k < n\}$ and $B = \{x_k y_n : n < k\}$ are disjoint.

(4) Working in the semigroup $(\mathbb{N}, +)$, assume that $p \in \beta\mathbb{N}$ is idempotent, i.e. $p+p = p$. Prove that the set E of even natural numbers is an element of p. – Similarly, for $k \in \mathbb{N}$, we have $\mathbb{N} = A_1 \cup \cdots \cup A_k$ where A_j is the set of natural numbers congruent to j modulo k; prove that $A_k \in p$.

(5) Assume that $(S, \cdot)$ is a semigroup and $S = A_1 \cup \cdots \cup A_r$ is a colouring of S with finitely many colours. Following the proof of 4.9, show that there are $j \in \{1, \ldots, r\}$ and elements $u, v, w \in A_j$ such that uv, uw, vw, uvw are in A, too.

(6) Give a more detailed proof of 4.8(c), i.e. check that $f : \hat{T} \to \beta T$ is a semigroup isomorphism.