

CHAPTER 3

βS as a topological space

In the preceding chapter, we introduced the set βS of all ultrafilters on a (non-empty) set S. Here, we will make βS a topological space which is, in fact, compact and Hausdorff. βS can be conceived, in a natural way, as a compactification of S with the discrete topology, and as such it enjoys a universal property which explains its central importance.

1. The Stone topology on βS

We assume in this chapter that S is a non-empty set. We introduce here the Stone topology on βS and prove that βS, with this topology, is what we call a Boolean space; in particular, it is compact and Hausdorff.

3.1. Definition For a subset A of S, we denote the Stone set Ā ⊆ βS corresponding to A by Ā = {p ∈ βS : A ∈ p}.

The family B = {Ā : A ⊆ S}, consisting of all Stone sets is the Stone base of βS.

We will compute over and over with Stone sets, which is facilitated by the following lemmas. Obviously, Ā = βS and ∅ = ∅.

3.2. Lemma For all subsets A, B of S, the following equations hold.

\[ \widehat{A \cap B} = \hat{A} \cap \hat{B}, \quad \widehat{A \cup B} = \hat{A} \cup \hat{B}, \quad S \setminus A = \beta S \setminus \hat{A}. \]

Proof. The elements of the Stone sets occurring in these equations are ultrafilters p on S. Now the first equation holds since p ∈ \widehat{A \cap B} holds iff A ∩ B ∈ p, which is equivalent to A ∈ p and B ∈ p, i.e. to p ∈ \hat{A} ∩ \hat{B}. The second one follows similarly because every ultrafilter is a prime filter, and the last one by the very definition of an ultrafilter. \[ \square \]

Lemma 3.2 shows that the Stone base B is closed under finite intersections, unions, and complementation. By closedness under intersections, there is a unique topology on βS having B as an open base, the Stone topology.
3.3. Definition  The Stone topology on $\beta S$ is the topology which has $B$ as a base for open sets. We call $\beta S$, with the Stone topology, the Stone–Čech compactification of $S$. Cf. 3.11 and 3.12 for more explanation of this denotation.

3.4. Definition and Remark  (a) By the very definition of the Stone base $B$ being a base for the open sets of $\beta S$, we state that a subset $U$ of $\beta S$ is open iff there is a family $\{A_i : i \in I\}$ of subsets of $S$ such that $U = \bigcup_{i \in I} \hat{A}_i$.
(b) By passing to complements and using the fact that the Stone base is closed under complementation, it follows that a subset $Y$ of $\beta S$ is closed iff there is a family $\{A_i : i \in I\}$ of subsets of $S$ such that $Y = \bigcap_{i \in I} \hat{A}_i$.
(c) In the space $\beta S$, every point $p$ has the family $\{\hat{A} : A \in p\}$ as a canonical neighbourhood base.

Before studying the Stone topology in detail, let us note some more simple facts on Stone sets.

3.5. Lemma  For any subsets $A$ and $B$ of $S$, the following hold:
(a) $A = \emptyset$ iff $\hat{A} = \emptyset$
(b) $A \subseteq B$ iff $\hat{A} \subseteq \hat{B}$
(c) $A = B$ iff $\hat{A} = \hat{B}$
(d) $A = S$ iff $\hat{A} = \beta S$.

Proof. (a) is simply the statement of 2.9(b) that every nonempty subset of $S$ is contained in an ultrafilter. (c) follows from (b), and (d) from (c). For (b), we note that $A \subseteq B$ iff $A \setminus B = \emptyset$ iff $\hat{A} \setminus \hat{B} = \emptyset$ iff $\hat{A} \subseteq \hat{B}$. □

We begin to prove the most important properties of the space $\beta S$.

3.6. Definition  For a topological space $X$, we define the following.
(a) A subset $U$ of $X$ is said to be clopen if it is both closed and open – i.e. if both $U$ and $X \setminus U$ are open. E.g. every Stone set $\hat{A}$ is clopen in $\beta S$, because, by Lemma 3.2, its complement in $\beta S$ is the Stone set $S \setminus A$.
(b) $X$ is called zero-dimensional if it has an open base consisting of clopen sets. It is a Boolean space if it is Hausdorff, compact, and zero-dimensional.

3.7. Theorem  The space $\beta S$, with the Stone topology, is Boolean.
subfamily \( C' \) of \( C \) satisfying \( \bigcap_{C \in C'} C = \emptyset \). Then \( A' = \{ S \setminus C : C \in C' \} \) is a finite subfamily of \( A \) satisfying \( \bigcup_{A \in A'} A = S \), and hence \( \hat{A} : A \in A' \) is a finite subcover of \( \{ A : A \in A \} \). \( \square \)

We can use the compactness of \( \beta S \) to show that its clopen subsets are exactly the Stone sets.

3.8. Corollary A subset of \( \beta S \) is clopen iff it is the Stone set \( \hat{A} \) of some subset \( A \) of \( S \).

Proof. Assume that \( U \subseteq \beta S \) is clopen; fix a family \( A \) of subsets of \( S \) such that \( U = \bigcup_{A \in A} \hat{A} \). Now \( U \) is a closed and hence compact subspace of \( \beta S \); thus the open cover \( \{ \hat{A} : A \in A \} \) of \( U \) has a finite subcover. Pick \( A_1, \ldots, A_n \in A \) such that \( U = \hat{A}_1 \cup \cdots \cup \hat{A}_n \); so \( U \) is the Stone set of \( A = A_1 \cup \cdots \cup A_n \). \( \square \)

2. \( \beta S \) as a compactification of \( S \)

In the preceding chapter, we defined the canonical injection \( e \) from \( S \) into \( \beta S \) mapping every element \( s \) of \( S \) to the fixed ultrafilter \( \hat{s} \). We will prove that the pair \( (\beta S, e) \) is what topologists call a compactification of \( S \). Let us first note a few simple but important facts on Stone sets and specific points in \( \beta S \).

3.9. Remark (a) For \( s \in S \) and \( A \subseteq S \), clearly 
\[ s \in A \text{ iff } A \ni \hat{s} \text{ iff } \hat{s} \in \hat{A}. \]
Thus if \( A \) is infinite, then so is \( \hat{A} \).

(b) A point \( x \) of a topological space \( X \) is called isolated if the set \( \{ x \} \) happens to be open in \( X \), i.e. if \( \{ x \} \) is a neighbourhood of \( x \).

We claim that the isolated points of \( \beta S \) are exactly the fixed ultrafilters \( \hat{s} \), for \( s \in S \). First, if \( p \) is a free ultrafilter on \( S \), then every set \( A \) in \( p \) is infinite, by 2.13 and so is \( \hat{A} \); thus every neighbourhood of \( p \) is infinite and \( p \) is non-isolated. On the other hand, let \( p = \hat{s} \) where \( s \in S \); let \( A \subseteq S \) be the singleton \( \{ s \} \). The Stone set \( \hat{A} \) is the smallest neighbourhood of \( p \), and \( \hat{A} = \{ \hat{s} \} \); by finiteness of \( A \), \( \hat{A} \) contains no free ultrafilter, and the only fixed ultrafilter in \( \hat{A} \) is \( \hat{s} \), by (a).

(c) Thus the canonical map \( e \) defined by \( e(s) = \hat{s} \) is a bijection from \( S \) onto the set of isolated points of \( \beta S \).

(d) Additionally, the set \( \{ \hat{s} : s \in S \} \) of isolated points is dense in \( \beta S \) because every non-empty Stone set \( \hat{A} \subseteq \beta S \) contains some point \( \hat{s} \), by (a).

3.10. Definition (a) A map \( f : X \rightarrow Y \) between two topological spaces \( X \) and \( Y \) is an embedding if it is a homeomorphism from \( X \) onto its image \( f[X] \) under \( f \); i.e. if it is one-one, continuous, and for every subset \( U \) of \( X \), \( U \) is open in \( X \) iff \( f[U] \) is open in \( f[X] \) (where \( f[X] \) is equipped with the subspace topology inherited from \( Y \)).

(b) For a topological space \( X \), a compactification of \( X \) is a pair \( (Y,f) \) where \( Y \) is a compact Hausdorff space, \( f : X \rightarrow Y \) is an embedding, and the range \( f[X] \) of \( f \) is a dense subspace of \( Y \).

3.11. Theorem Consider the set \( S \) as a topological space with the discrete topology. Then the pair \( (\beta S, e) \) is a compactification of \( S \).
We are left with proving that \( e \) is a homeomorphism from the discrete space \( S \) onto its image \( e[S] = \{ \hat{s} : s \in S \} \), i.e. that \( e[S] \) is discrete, with the subspace topology from \( \beta S \). But every point \( e(s) = \hat{s} \) of \( e[S] \) is isolated in \( \beta S \), by 3.9, hence in \( e[S] \).

Identifying every point \( s \in S \) with its image \( \hat{s} \) under the map \( e \), we shall, beginning from Chapter 4, view \( S \) as a dense subspace of the compact space \( \beta S \). Moreover we know that \( S \) is the set of isolated points of \( \beta S \).

### 3. The universal property of \( \beta S \)

We now prove a characteristic extension property of the compactification \((\beta S, e)\) of \( S \) which will frequently be used when dealing with \( \beta S \).

#### 3.12. Theorem

Assume \( X \) is a compact Hausdorff space and \( f : S \to X \) is an arbitrary map. Then there is a unique continuous map \( \hat{f} : \beta S \to X \) such that \( \hat{f} \circ e = f \), the Stone-Čech extension of \( f \).

Thus if we identify \( S \) with the dense subspace of \( \beta S \) consisting of all isolated points, i.e. we think about \( e \) as being the inclusion map from \( S \) to \( \beta S \), then \( \hat{f} \) is the unique continuous extension of \( f \) to \( \beta S \).

**Proof.** For uniqueness, just note that \( e[S] \) is a dense subset of \( \beta S \); thus every map from \( e[S] \) into a Hausdorff space has at most one continuous extension.

For existence, we rely on the notion of a \( p \)-limit introduced in 2.17. Consider the family \((f(s))_{s \in S}\) of points in \( X \), indexed by \( S \). Since \( X \) is compact Hausdorff, we can define, for \( p \in \beta S \),

\[
\hat{f}(p) = p - \lim_{s \in S} f(s).
\]

The map \( \hat{f} \) thus defined satisfies \( \hat{f} \circ e = f \) because for \( a \in S \) and \( p = \hat{a} \), we have \( p - \lim_{s \in S} f(s) = f(a) \), by Example 2.18.

We are left with showing that \( \hat{f} \) is continuous. So consider some \( p \in \beta S \) and \( x = \hat{f}(p) \); let \( U \) be a neighbourhood of \( x \) in \( X \) with the aim of finding a neighbourhood \( A \) of \( p \) in \( \beta S \) which is mapped by \( \hat{f} \) into \( U \). Pick a neighbourhood \( V \) of \( x \) with its closure included in \( U \). Since \( x = p - \lim_{s \in S} f(s) \in V \), there is some \( A \in p \) such that \( \{ f(s) : s \in A \} \subseteq V \). This set \( A \) works, since for every \( q \in A \), we conclude from \( A \in q \) and Theorem 2.19 (a) that \( \hat{f}(q) = q - \lim_{s \in S} f(s) \in \text{cl}\{ f(s) : s \in A \} \subseteq \text{cl}(V) \subseteq U \). 

Let us note that our construction of the Stone-Čech compactification \( \beta S \) of a discrete space \( S \) is a very special case of a more general fact of set theoretic topology: given a completely regular topological space \( X \), there is a (unique) compactification \((Z, \varepsilon)\) of \( X \) with the universal property described in 3.12. I.e. for every continuous map \( f : X \to Y \) from \( X \) into a compact Hausdorff space \( Y \), there is a unique continuous map \( \hat{f} : Z \to Y \) satisfying \( \hat{f} \circ e = f \). The pair \((Z, \varepsilon)\) is then called the Stone-Čech compactification of \( X \).

The universal property 3.12 of \( \beta S \) allows for a more natural explanation of Theorem 2.22. Given a sequence \((x_s)_{s \in S}\) of points in a compact Hausdorff space \( X \), we consider the function \( f \) from \( S \) into \( X \) defined by \( f(s) = x_s \) and its Stone-Čech extension \( \hat{f} : \beta S \to X \). The image \( Y \) of \( \beta S \) under \( \hat{f} \) is closed in \( X \), by compactness of...
\(\beta S\) and continuity of \(\tilde{f}\), and it includes \(\{x_s : s \in S\}\). Moreover, \(S\) is dense in \(\beta S\) and hence its image \(\{x_s : s \in S\}\) under \(f\) is dense in \(Y\), so \(Y\) is the closure of \(\{x_s : s \in S\}\) in \(X\). And by the construction of \(f\) in 3.12, \(Y = \{\lim_{s \in S} x_s : p \in \beta S\}\).

4. The relationship between \(A \subseteq S\) and \(\hat{A} \subseteq \beta S\)

In this section, we make two additional statements on how a subset \(A\) of \(S\) is related to its corresponding Stone set \(\hat{A}\), a clopen subset of \(\beta S\). Recall that \(e\) denotes the canonical embedding from \(S\) into \(\beta S\).

3.13. Proposition For any \(A \subseteq S\), \(\hat{A}\) is the closure (in \(\beta S\)) of the set \(e[A] = \{\hat{s} : s \in A\}\), and \(\hat{A} \cap \{\hat{s} : s \in S\} = \{\hat{s} : s \in A\}\). I.e. if we identify every \(s \in S\) with \(e(s) = \hat{s}\) and \(S\) with the dense set of isolated points of \(\beta S\), we obtain that \(\hat{A}\) is the closure of \(A\) and \(\hat{A} \cap S = A\).

**Proof.** The second statement has already been noted in part (a) of Remark 3.9. The first one is proved like the density of \(e[S]\) in \(\beta S\) in 3.9: for a non-empty open subset \(U\) of \(\hat{A}\), say \(U = \hat{B}\) where \(B \subseteq A\), pick some \(s \in B\); so \(s \in A\) and \(\hat{\hat{s}} \in \hat{B}\).

For the next proposition, note that we consider \(S\) and hence every subset \(A\) of \(S\) with the discrete topology, so the Stone-\v{C}ech compactification \(\beta A\) of \(A\) is well defined. On the other hand, the preceding proposition says that also \((\hat{A}, e | A)\) is a compactification of \(A\).

3.14. Proposition For any \(A \subseteq S\), \(\hat{A}\) is canonically homeomorphic to \(\beta A\).

**Proof.** The points of \(\hat{A}\) are the ultrafilters on \(S\) containing \(A\) and the points of \(\beta A\) are the ultrafilters on \(A\). There are obvious maps \(f : \hat{A} \to \beta A\) and \(g : \beta A \to \hat{A}\) defined by \(f(p) = p \cap \mathcal{T}(A)\) and \(g(q) = \{B \subseteq S : B \cap A \in q\}\); they are both continuous and inverses of each other.

**Exercises**

1. Assume that \(Y\) and \(Z\) are disjoint closed subsets of \(\beta S\). Prove that \(Y\) and \(Z\) are separated by basic sets, i.e. there are disjoint subsets \(A, B\) of \(S\) such that \(Y \subseteq A\) and \(Z \subseteq B\).

2. Prove that, in the space \(\beta S\), the closure of every open subset is clopen. (This very special property of \(\beta S\) is called extreme disconnectedness.)

3. Let \(A_i\) and, for \(i \in I\), \(A_i\) be subsets of \(S\). Prove that \(\hat{A} = \bigcup_{i \in I} \hat{A}_i\) if \(A_i \subseteq A\) holds for all \(i \in I\) and there is a finite subset \(J\) of \(I\) such that \(A = \bigcup_{i \in J} A_i\).

Similarly, \(\hat{A} = \bigcap_{i \in I} \hat{A}_i\) if \(A \subseteq A_i\) holds for all \(i \in I\) and there is some finite \(J \subseteq I\) such that \(A = \bigcap_{i \in J} A_i\).

This can be proved as the compactness statement of Theorem ?? or directly concluded from it.

4. Prove that in the space \(\beta S\), the \(p\)-limit of the sequence \((\hat{s})_{s \in S}\) is the point \(p\).
(5) Prove that for a free ultrafilter $p$ on $S$, the point $p$ of $\beta S$ does not have a countable neighbourhood base.

(6) Assume that $f : S \to T$ is a map between the discrete spaces $S$ and $T$. We consider $f$ as a mapping from $S$ into the compact Hausdorff space $\beta T$ and thus obtain its Stone-Čech extension $\tilde{f} : \beta S \to \beta T$. So for $p \in \beta S$, $\tilde{f}(p)$ is an ultrafilter on $T$. Prove that a subset $B$ of $T$ is in $\tilde{f}(p)$ iff its preimage $f^{-1}[B]$ under $f$ is in $p$.

(7) Give a more detailed proof of Proposition 3.14: for $A \subseteq S$, the Stone-Čech compactification $\beta A$ of the discrete space $A$ is homeomorphic to the Stone set $\hat{A}$.

(8) We call a compactification $(C, g)$ of a discrete space $S$ universal if it has the property proved for $(\beta S, e)$ in Theorem 3.12, i.e. for any compact Hausdorff space $X$ and any map $f : S \to X$, there is a unique continuous map $k : C \to X$ such that $k \circ g = f$.

Prove that any two universal compactifications $(C, g)$ and $(D, h)$ of $S$ are homeomorphic, more precisely there is a unique homeomorphism $k : C \to D$ such that $k \circ g = h$.

(9) (For model theorists.) Let $\mathcal{L}$ be a first order language. We denote the set of all sentences (i.e. closed formulas) of $\mathcal{L}$ by $F$ and the set of all maximally consistent $\mathcal{L}$-theories by $X$. For $\mathcal{A}$ an $\mathcal{L}$-structure, $Th(\mathcal{A}) = \{ \alpha \in F : \mathcal{A} \models \alpha \}$, the first order theory of $\mathcal{A}$, is maximally consistent, and

$$X = \{ Th(\mathcal{A}) : \mathcal{A} \text{ an } \mathcal{L}\text{-structure} \}.$$  

(a) For $\alpha \in F$, define $\hat{\alpha}$ to be the subset $\{ T \in X : \alpha \in T \}$ of $X$. Prove that $X$ is a Boolean space under the topology with base $\mathcal{B} = \{ \hat{\alpha} : \alpha \in F \}$.

(b) Assume that for every $i$ in a set $I$, $\mathcal{A}_i$ is an $\mathcal{L}$-structure and $T_i = Th(\mathcal{A}_i)$. Let $p$ be an ultrafilter on $I$, $\mathcal{A}$ the ultraproduct of the family $(\mathcal{A}_i)_{i \in I}$ with respect to $p$, and $T = Th(\mathcal{A})$. Prove that $T$ is the $p$-limit of the sequence $(T_i)_{i \in I}$, in the space $X$. 