CHAPTER 2

Filters and ultrafilters

The notion of a filter resp. an ultrafilter is a powerful tool both in set theory and in topology. The importance of ultrafilters in these notes lies in the fact that the set βS of all ultrafilters on a fixed semigroup S can be made into a compact right topological semigroup which will be our central object of study.
In Sections 1 and 2 of this chapter, we present the basic notions concerning filters and ultrafilters and the principal tools required throughout when dealing with them, in particular the existence theorems **??** and **??**. Section 3 explains a distinction between trivial (fixed) and non-trivial (free) ultrafilters. As a non-obvious application of free ultrafilters to set theory, we give in Section 4 a proof of Ramsey's theorem. In Section 5, we consider a notion of convergence of a sequence (indexed by S) with respect to an arbitrary ultrafilter on S.

1. Filters

A filter on a set S is a collection of subsets of S which may be viewed as being "large", with respect to some property, and compatible in the sense that any finitely many of them have non-empty intersection.

2.1. Definition Let S be a non-empty set. A *filter* on S is a subset p of the power set $\mathcal{P}(S)$ of S with the following properties:
(a) $S \in p$, and $\emptyset \notin p$
(b) $A \in p$ and $A \subseteq B \subseteq S$ implies that $B \in p$
(c) $A, B \in p$ implies that $A \cap B \in p$.

2.2. Example (a) For any non-empty subset M of S, the set $\{A \subseteq S : M \subseteq A\}$ is obviously a filter on S, the *principal filter generated by* M. It is clearly the least filter on S containing M.
(b) Let S be infinite; we call a subset A of S *cofinite* if its complement $S \setminus A$ is finite. The family of all cofinite subsets of S is then a filter, the *Fréchet filter* on S.
(c) Let X be a topological space and $x \in X$. A subset V of X is called a *neighbourhood* of x in X if there is an open set $U \subseteq X$ such that $x \in U \subseteq V$. The set $\mathcal{U}(x)$ consisting of all neighbourhoods of x is a filter on X, the *neighbourhood filter* of x.

There is a simple criterion for extendibilty of a family $\mathcal{A} \subseteq \mathcal{P}(S)$ to a filter on S.

2.3. Definition Let $\mathcal{A}$ be a family of subsets of S. We say that $\mathcal{A}$ has the *finite intersection property* if every finite subfamily of $\mathcal{A}$ has non-empty intersection.

2.4. Lemma *(a) If $\mathcal{A}$ has the finite intersection property; then*

$$p = \{A \subseteq S : \text{there are } n \in \omega \text{ and } A_1, \ldots, A_n \in \mathcal{A} \text{ such that } A_1 \cap \cdots \cap A_n \subseteq A\}$$

is the least filter on S including $\mathcal{A}$ – the filter generated by $\mathcal{A}$.
(b) There is a filter including $\mathcal{A}$ if and only if $\mathcal{A}$ has the finite intersection property.

PROOF. (a) follows by straightforward computation. For the missing implication in (b), note that every filter and hence every subset of a filter has the finite intersection property. □

Let us illustrate the filters generated by some families $\mathcal{A} \subseteq \mathcal{P}(S)$ by the examples in **??**: the principal filter in **??**(a) is the filter generated by $\mathcal{A} = \{M\}$. The Fréchet filter in (b) is generated by the family $\mathcal{A} = \{S \setminus \{a\} : a \in S\}$, the neighbourhood filter in (c) of $x \in X$ by the set of all open subsets of X containing x, and also by any neighbourhood base of x.

2. Ultrafilters

An arbitrary filter p on a set S behaves nicely with respect to intersections: for $A, B \subseteq S$, clearly $A \cap B$ is in p if and only if both A and B are in p. It follows that an intersection $A_1 \cap \ldots \cap A_r$ of finitely many subsets $A_1, \ldots, A_r$ of S is in p iff every A_i is in p.
Also, if $A \in p$ or $B \in p$, then it follows that $A \cup B \in p$, but the examples in **??** show that the converse does not hold. In this section, we define and study filters which, in addition, behave nicely with respect to unions and complements.

2.5. Definition Let p be a filter on S.
(a) p is an *ultrafilter* if for any $A \subseteq S$, either A or $S \setminus A$ is in p.
(b) p is a *prime filter* if for any $A, B \subseteq S$ satisfying $A \cup B \in p$, either $A \in p$ or $B \in p$.
(c) p is a *maximal filter* if every filter including p coincides with p.

Among arbitrary filters, ultrafilters are quite rare, except for trivial examples. In the examples given in **??**, we see immediately that the principal filter generated by some subset M of S is an ultrafilter iff M has exactly one element. Let us fix a notation for these trivial ultrafilters.

2.6. Notation For every element s of S, we denote by $\dot{s}$ the ultrafilter $\{A \subseteq S : s \in A\}$.

The Fréchet filter on an infinite set is not an ultrafilter, and the neighbourhood filter of a point x in a topological space X is an ultrafilter iff x is an isolated point of X, i.e. the set $\{x\}$ is open in X.

It turns out that the properties defined in **??** are equivalent.

2.7. Theorem *For any filter on S, the properties of being an ultrafilter, prime, or maximal are equivalent.*

PROOF. Every ultrafilter is prime: assume p is an ultrafilter and A, B are not in p. Then $S \setminus A, S \setminus B$ are in p, and so is their intersection $S \setminus (A \cup B)$. Hence $A \cup B \notin p$, which shows that p is prime.

Every prime filter is maximal: assume that p is prime and $A \notin p$; we claim that $p \cup \{A\}$ does not satisfy the finite intersection property, thus proving maximality of p by **??**. Now $S = A \cup (S \setminus A)$. By primeness of p, we conclude that $S \setminus A \in p$, and $A \cap (S \setminus A) = \emptyset$ shows the claim.
Every maximal filter is an ultrafilter: assume that p is maximal and $A \notin p$; we show that $S \setminus A \in p$. By maximality of p, $p \cup \{A\}$ does not have the finite intersection property. So there is $B \in p$ satisfying $A \cap B = \emptyset$, i.e. $B \subseteq S \setminus A$, and hence $S \setminus A \in p$. □

It follows that ultrafilters behave nicely with respect to unions and complements. For let p be an ultrafilter on S and $A, B \subseteq S$. Then $S \setminus A \in p$ iff $A \notin p$, by the ultrafilter property. Also $A \cup B \in p$ iff $A \in p$ or $B \in p$, since p is prime. And a union $A_1 \cup \ldots \cup A_r$ of finitely many subsets $A_1, \ldots, A_r$ of S is in p iff some A_j is in p.

The equivalence **??** allows us to prove that sufficiently many ultrafilters exist.

2.8. Theorem *Every filter is extendible to an ultrafilter.*

PROOF. A straightforward application of Zorn's lemma: assume that p is a filter on S. The set P consisting of all filters on S which extend p is a non-empty partial order under set theoretic inclusion. Clearly every non-empty chain C in P has $\bigcup_{q \in C} q$ as an upper bound in P. Thus P has a maximal element m, and by Theorem **??**, m is an ultrafilter. □

As a consequence of the last theorem and the criterion **??**, we obtain the following.

2.9. Corollary *(a) A family $\mathcal{A}$ of subsets of S is extendible to an ultrafilter if and only if it has the finite intersection property.*
(b) A subset A of S is contained in an ultrafilter iff $A \neq \emptyset$.

3. Free and fixed filters

The set βS of all ultrafilters on a semigroup S will be the central object of our studies. There is a natural map from S to βS: we map every $s \in S$ to the principal ultrafilter $\dot{s}$ defined in **??**.

2.10. Definition Let S be an arbitrary set.
(a) We denote by βS the set of all ultrafilters on S.
(b) The canonical map $e : S \to \beta S$ is defined by $e(s) = \dot{s}$, for $s \in S$.

The function e is one-one; thus S is usually thought of as being a subset of βS, identifying every $s \in S$ with the principal ultrafilter $\dot{s}$. For infinite S, this embedding is not a bijection, i.e. there are ultrafilters on S which don't have the form $\dot{s}$, $s \in S$. They are not easily describable but only arise from the use of Zorn's lemma in **??**, a completely non-constructive existence statement. The separating line between the trivial ultrafilters $e(s) = \dot{s}$ and the non-trivial ones is drawn in the following definition.

2.11. Definition We say that a filter p on S is *free* if the intersectionl $\bigcap p = \bigcap_{A\in p} A$ of all sets in p is empty; otherwise, p is *fixed* (by any $s \in \bigcap p$).

Thus the principal ultrafilter $\dot{s}$ is fixed (by the point $s \in S$). The Fréchet filter on an infinite set S is free, and so is every filter extending it. The neighbourhood filter p of a point x in a topological space X is fixed by x; if X happens to be a T_1-space, then $\bigcap p = \{x\}$.

It is easy to characterize free respectively fixed ultrafilters.

2.12. Proposition *For any ultrafilter p in S, the following are equivalent.*
(a) p is fixed
(b) $p = \dot{s}$, for some $s \in S$
(c) there is some $s \in S$ such that $\{s\} \in p$
(d) p contains a finite subset of S.

PROOF. (a) is equivalent to (b): for the non-trivial direction, note that if p is fixed by some $s \in S$, then $p \subseteq \dot{s}$. Now p is a maximal filter and thus $p = \dot{s}$.
(b) is equivalent to (c): if $\{s\} \in p$ for some $s \in S$, then $\dot{s} \subseteq p$ and thus $p = \dot{s}$.
(c) is equivalent to (d): if $\{s_1, \ldots, s_n\}$ is a finite set in p, then by $\{s_1, \ldots, s_n\} = \{s_1\} \cup \cdots \cup \{s_n\}$ and primeness of p, some $\{s_i\} \in p$. □

In particular, if S is finite, then every ultrafilter on S is fixed, i.e. $\beta S = \{\dot{s} : s \in S\}$, and the canonical embedding e is a bijection. On the other hand, if p is a free ultrafilter on S, then S must be infinite and every cofinite subset of S is contained in p, i.e. p extends the Fréchet filter on S.

2.13. Corollary *Let S be infinite. Then an ultrafilter p on S is free iff every set in p is infinite iff p includes the Fréchet filter on S.*

It follows that for infinite S, the map $e : S \to \beta S$ is far from being onto: by the existence theorem **??** for ultrafilters, there are ultrafilters extending the Fréchet filter on S, and none of them is in the range of e.

4. Ramsey's theorem - an application of free ultrafilters

Free ultrafilters can be very useful in combinatorial arguments; we illustrate this by a proof of the infinite Ramsey theorem. It requires some notation.

2.14. Definition (a) For any set X and a natural number n, we put

$$[X]^n = \{e \subseteq X : |e| = n\},$$

the set of all n-element subsets of X; here $|e|$ denotes the cardinality of e. Usually we will fix a linear order $<$ on X and then identify $[X]^n$ with the set of all strictly increasing sequences of length n over X, i.e. with the sequences $(x_1 \ldots, x_n)$ in X where $x_1 < \cdots < x_n$. (And we tacitly identify $[X]^1$ with X, identifying $\{x\}$ with x, for $x \in X$.)
(b) Let A be a subset of $[X]^n$ and H a subset of X. We say that H is *homogeneous for A* if $[H]^n \subseteq A$, i.e. if every increasing n-sequence with elements from H is in A.

2.15. Theorem *(Ramsey) Let X be an infinite set, $n, r \in \mathbb{N}$ and let $[X]^n = A_1 \cup \cdots \cup A_r$. Then there is some $j \in \{1, \ldots, r\}$ and some infinite subset H of X such that H is homogeneous for A_j.*

Before starting the proof, let us explain its intuititive content. We may assume that the sets A_i covering $[X]^n$ are pairwise disjoint, otherwise passing to disjoint subsets A'_i of A_i. We think about the numbers $1, \ldots, r$ as being finitely many colours; a sequence $\bar{x} \in [X]^n$ has colour i if $\bar{x} \in A_i$. The theorem then says that if $[X]^n$ is coloured with finitely many colours, then there is a large (i.e. infinite) subset H of X such that $[H]^n$ is monochromatic, i.e. all increasing n-sequences in H have the same colour.
We shall later meet quite a few results of this type in our text. To put it in a simplistic way, they say that if you colour a large set with finitely many colours, then there is a "large"monochromatic set – the difference between the results being how the notion of largeness is defined. In view of Ramsey's theorem, the historically first result in this line, the part of combinatorics dealing with such results is called Ramsey theory.

Proof. By passing to a subset of X, we may assume that X is countably infinite; replacing X by a set of the same cardinality, we assume that X is the set ω of non-negative integers, equipped with its natural linear ordering.
For $n = 1$, the theorem amounts to the pigeon hole principle. We prove it here for the smallest non-trivial case $n = 2$; the case of an arbitrary n is handled in an exercise below. Technically, we work with the colouring function

$$c : [\omega]^2 \to I = \{1, \ldots, r\}$$

given by

$$c(x, y) = i \text{ iff } (x, y) \in A_i,$$

and we construct an infinite set H monochromatic with respect to c.

We fix a free ultrafilter p on ω. The colouring c of $[\omega]^2$ induces, via p, a colouring c_1 of ω as follows: for $x \in \omega$ and $i \in I$, put

$$U_i(x) = \{y > x : c(x, y) = i\}.$$

The interval $(x, \infty) = \{y \in \omega : x < y\}$ of ω is in p, being cofinite, and it is the disjoint union of the finitely many sets $U_i(x)$, $i \in I$. So exactly one of the $U_i(x)$ is in p and we define

$$c_1(x) = i \text{ iff } U_i(x) \in p.$$

Reasoning in the same way, we define i^* to be the unique element of I such that

$$X = \{x \in \omega : c_1(x) = i^*\} \in p.$$

We now construct $H = \{x_k : k \in \omega\}$ with $x_0 < x_1 < \ldots$ in such a way that $x_n \in X$ holds for all $n \in \omega$ and, for $m < n \in \omega$, we have $c(x_m, x_n) = i^*$. Thus H is monochromatic with colour i^*.

To construct the x_k, pick an arbitrary $x_0 \in X$. Assume $x_0 < x_1 < \ldots x_s$ have been chosen in X such that $c(x_m, x_n) = i^*$ holds for $m < n \leq s$. The set

$$Y = X \cap \bigcap_{m \leq s} U_{i^*}(x_m)$$

is an element of p (because $x_m \in X$, which means that $c_1(x_m) = i^*$ and $U_{i^*}(x_m) \in p$). By freeness of p, we can pick $x_{s+1} \in Y$ larger than x_s. The choice of x_{s+1} guarantees that $c(x_m, x_{s+1}) = i^*$ holds for all $m \leq s$. □

By topological reasoning, we can derive a finitary version of Ramsey's theorem from the infinite one. In the following, we write $[0,\ldots,m]^n$ for $[\{0,\ldots,m\}]^n$.

2.16. Corollary *(the finite Ramsey theorem) Assume $n,k\in\mathbb{N}$ and $I=\{1,\ldots,r\}$ is a finite set of colours. Then there exists a natural number $N=N(n,k,r)$ such that for every colouring $c:[0,\ldots,N]^n\to I$ of $[0,\ldots,N]^n$ with r colours, there is a subset H of $\{0,\ldots,N\}$ such that $|H|=k$ and H is monochromatic with respect to c.*

Proof. Assume for contradiction that there is no such number N. We consider the finite set I with the discrete topology and the product space $X={}^{[\omega]^n}I$, a compact space by Tychonoff's theorem. I.e. the elements of X are the colourings $c:[\omega]^n\to I$. For $m\in\omega$, we define the closed subset

$$B_m=\{c\in X:\text{no } H\in[0,\ldots,m]^k \text{ is monochromatic, under } c\}$$

of X. Clearly $B_m\subseteq B_i$ holds for $i<m$, and by our assumption, every B_m is non-empty. By compactness of X, we can pick some $c\in\bigcap_{m\in\omega}B_m$. Then under c, no k-element subset of ω is monochromatic, a contradiction to Ramsey's theorem. □

5. The p-limit of a sequence

We define a notion of convergence for sequences $(x_s)_{s\in S}$ in a topological space, indexed by S, with respect to an arbitrary filter p on S. It will be applied quite frequently in the following chapters.

2.17. Definition Assume that X is a topological space, $(x_s)_{s\in S}$ is a sequence of points in X, p is a filter on S and x is a point in X. We say that x is a *p-limit* of $(x_s)_{s\in S}$ if, for every neighbourhood U of x in X, there is some $A\in p$ such that $\{x_s:s\in A\}\subseteq U$.

To put it in a slightly different way, x is a p-limit of $(x_s)_{s\in S}$ iff, for every neighbourhood U of x, the set $\{s\in S:x_s\in U\}$ is in p. Clearly, a p-limit of $(x_s)_{s\in S}$ is also a q-limit of $(x_s)_{s\in S}$, if q is a filter extending p.

2.18. Example (a) Let S be the set ω of all natural numbers, i.e. we consider a sequence $(x_n)_{n\in\omega}$ in X. Let p be the Frèchet filter on ω consisting of all cofinite subsets of ω (cf. **??**). Then a p-limit x of $(x_n)_{n\in\omega}$ in the sense of **??** is simply a limit of the sequence $(x_n)_{n\in\omega}$ as defined in the context of metric spaces.
(b) Let a be an element of S and $p=\dot{a}$ the principal ultrafilter on S fixed by a (cf. **??**, **??**). In this case, $\{a\}$ is the smallest set in p, and the point x_a is a p-limit of $(x_s)_{s\in S}$.

A p-limit of a sequence $(x_s)_{s\in S}$ does not necessarily exist, nor will it be unique. But under appropriate assumptions, there is a unique p-limit. In the following results, we denote the closure of a subset M of X by $\operatorname{cl}M$.

2.19. Theorem *Assume $(x_s)_{s\in S}$ is a sequence in X and p is a filter on S.*
(a) If $A\in p$, then every p-limit of $(x_s)_{s\in S}$ is contained in $\operatorname{cl}\{x_s:s\in A\}$.
(b) If p is an ultrafilter, then the converse holds: every point in $\bigcap_{A\in p}\operatorname{cl}\{x_s:s\in A\}$ is a p-limit of $(x_s)_{s\in S}$.
(c) If X is a Hausdorff space, then $(x_s)_{s\in S}$ has at most one p-limit.

(d) If X is compact and p is an ultrafilter, then $(x_s)_{s\in S}$ has at least one p-limit.
(e) Hence for X compact Hausdorff and p an ultrafilter, there is a unique p-limit of $(x_s)_{s\in S}$ – the unique point of $\bigcap_{A\in p} \mathrm{cl}\{x_s : s \in A\}$.

PROOF. For any $A \subseteq S$, let us abbreviate the set $\{x_s : s \in A\}$ by $M(A)$. So x is a p-limit of $(x_s)_{s\in S}$ iff every neighbourhood of x includes $M(A)$, for some $A \in p$.
(a) Otherwise, x is an element of the open set $U = X \setminus \mathrm{cl}\, M(A)$, so there is some $B \in p$ satsfying $M(B) \subseteq U$. A contradiction, since A and B and hence $M(A)$ and $M(B)$ cannot be disjoint.
(b) Assume that $x \in \bigcap_{A\in p} \mathrm{cl}M(A)$ and let U be a neighbourhood of x. We may assume U to be open and want to prove that $\{s \in S : x_s \in U\} \in p$. Otherwise, the set $A = \{s \in S : x_s \in X \setminus U\}$ is in p, since p is an ultrafilter. Then $x \in \mathrm{cl}M(A) \subseteq X \setminus U$, since $X \setminus U$ is closed; a contradiction to $x \in U$.
(c) Assume X is Hausdorff. For x and y two distinct points of X, we pick disjoint neighbourhoods U of x and V of y. Then the sets $\{s \in S : x_s \in U\}$ and $\{s \in S : x_s \in V\}$ are disjoint and hence cannot both be elements of the filter p.
(d) Assume X is compact; by (a) and (b), we have to prove that $\bigcap_{A\in p} \mathrm{cl}M(A)$ is non-empty. But $\mathrm{cl}M(A)$ is a non-empty closed subset of X, for every $A \in p$, and $B \subseteq A$ in p implies that $\mathrm{cl}M(B) \subseteq \mathrm{cl}M(A)$. Hence the family $\{\mathrm{cl}M(A) : A \in p\}$ has the finite intersection property; by compactness of X, its intersection is non-empty. □

In all further applications of p-limits, we will work in compact Hausdorff spaces and, by Part (e) of Theorem **??**, speak about *the* p-limit of $(x_s)_{s\in S}$. We will write $p - \lim_{s\in S} x_s$ for this limit.
For subsets of X which are both open and closed, the definition of $p - \lim_{s\in S} x_s$ and **??**(a) give the following equivalence.

2.20. Remark Assume that X is compact and Hausdorff and $U \subseteq X$ is both open and closed. Then $p - \lim_{s\in S} x_s \in U$ iff $\{s \in S : x_s \in U\} \in p$ iff there is some $A \in p$ such that $\{x_s : s \in A\} \subseteq U$.

Taking p-limits commutes with continuous maps, a fact which will be helpful in later chapters.

2.21. Proposition *Assume that $f : X \to Y$ is a continuous map between compact Hausdorff spaces, $(x_s)_{s\in S}$ is a sequence in X, and p is an ultrafilter on S. Then*

$$f(p - \lim_{s\in S} x_s) = p - \lim_{s\in S} f(x_s).$$

PROOF. Writing $x = p - \lim_{s\in S} x_s$, we show that $f(x)$ is the p-limit of the sequence $(f(x_s))_{s\in S}$ in Y. Assume V is a neighbourhood of $f(x)$ in Y and let U be the preimage of V under f, a neighbourhood of x in X. So pick $A \in p$ such that $\{x_s : s \in A\} \subseteq U$; it follows that $\{f(x_s) : s \in A\} \subseteq V$. □

Our last theorem demonstrates the usefulness of ultrafilters in topology – more exactly, of the set βS of *all* ultrafilters on S: we can represent the closure of a set indexed by S by using βS.

2.22. Theorem *For a compact Hausdorff space X and $(x_s)_{s\in S}$ a family of points in X, we have*

$$cl\{x_s : s \in S\} = \{p - \lim_{s\in S} x_s : p \in \beta S\}.$$

Proof. The special case $A = S \in p$ in (a) of Theorem **??** shows that, for every ultrafilter p on S, the p-limit of $(x_s)_{s\in S}$ is in the closure of $\{x_s : s \in S\}$.
Conversely, assume that x is a point in the closure of $\{x_s : s \in S\}$. Then every neighbourhood U of x meets $\{x_s : s \in S\}$, i.e. the set $A(U) = \{s \in S : x_s \in U\}$ is non-empty. The family $\mathcal{A} = \{A(U) : U \text{ a neighbourhood of } x\}$ has the finite intersection property, because $V \subseteq U$ implies that $A(V) \subseteq A(U)$. By Corollary **??**, pick an ultrafilter p including $\mathcal{A}$. It follows that $x = p - \lim_{s\in S} x_s$ because for every neighbourhood U of x, the set $A(U)$ is in p. □

Exercises

(1) Prove that every filter f on a set S is the intersection of the family $\{p \in \beta S : f \subseteq p\}$ of ultrafilters.

(2) Let $\mathcal{A}$ be a family of subsets of S. Prove that $\mathcal{A}$ can be extended to a free ultrafilter iff it has the *ω-intersection property*, i.e. if the intersection of every finite subfamily of $\mathcal{A}$ is infinite.

(3) (A generalization of Exercise 2.) Let κ be an infinite cardinal; an ultrafilter p on S is said to be *κ-uniform* if every set in p has size at least κ. – Assume $\mathcal{A}$ is a family of subsets of S. Prove that $\mathcal{A}$ can be extended to a κ-uniform ultrafilter iff it has the *κ-intersection property*, i.e. if the intersection of every finite subfamily of $\mathcal{A}$ has size at least κ.

(4) Let X be an infinite linear order. Conclude from Ramsey's theorem that there is an infinite sequence $(x_i)_{i\in\omega}$ in X which is either strictly increasing (i.e. $x_0 < x_1 < \dots$) or strictly decreasing (i.e. $x_0 > x_1 > \dots$).

(5) Prove Ramsey's theorem for arbitrary n as follows. Fix a free ultrafilter p on ω and assume that $c : [\omega]^n \to I$ is a colouring of $[\omega]^n$ with colours in a finite set I. By induction on m, we define for every $1 \le m \le n$ a colouring $c_m : [\omega]^m \to I$ of $[\omega]^m$ as follows.
For $m = n$, put $c_n = c$. Now assume that the colouring $c_{m+1} : [\omega]^{m+1} \to I$ has been constructed. Then for $\bar{x} = (x_1, \dots, x_m) \in [\omega]^m$ and $i \in I$, put $U_{im}(\bar{x}) = \{y > x_m : c_{m+1}(\bar{x}, y) = i\}$ (where $(\bar{x}, y)$ is the $m+1$-sequence $(x_1, \dots, x_m, y)$). Put $c_m(\bar{x}) = i$ iff $U_{im}(\bar{x}) \in p$; this defines $c_m : [\omega]^m \to I$.
After $c_1 : \omega \to I$ has been defined, let i^* be the element of I such that $X = \{x \in \omega : c_1(x) = i^*\}$ is in p.
The required monochromatic set is defined by $H = \{x_k : k \in \omega\}$ where $x_0 < x_1 < \dots$ in X. Here we choose the elements x_k in such a way that for every $1 \le m \le n$ and every $\bar{x} \in [H]^m$, we have $c_{i^*m}(\bar{x}) = i^*$.

(6) For $N, n, r, k \in \mathbb{N}$, we write $N \to (k)^n_r$ for the assertion that for every colouring $c : [1, \dots, N]^n \to \{1, \dots, r\}$ of an N-element set $\{1, \dots, N\}$ with r colours, there is a monochromatic subset of $\{1, \dots, N\}$ of size k; Corollary **??** says that such an N exists for given n, r, k.
Prove that $6 \to (3)^2_2$ is true but $5 \to (3)^2_2$ fails. Moreover, if $N \le M$ and $N \to (k)^n_r$ holds, then so does $M \to (k)^n_r$.

(7) Let X be the compact real interval $[-1, 1]$ and, for $n \in \omega$, let $x_n = (-1)^n$. Characterize the filters p on ω for which $p - \lim_{n\in\omega} x_n$ exists.