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Ideal-specific elimination orders form a star-shaped region

Hartwig Bosse [a], Christine Gärtner [b], Oleg Golubitsky [c]

[a] J.W. Goethe-Universität, Frankfurt, Germany
[b] DFG Research Center MATHEON, Freie Universität Berlin, Germany
[c] Google, Inc., 151 Charles Street West # 200, Waterloo, ON N2G 1H6, Canada

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ABSTRACT

This paper shows that for any given polynomial ideal $\mathfrak{I} \subset \mathbb{K}[x_1, \ldots, x_n]$ the collection of Gröbner cones corresponding to $\mathfrak{I}$-specific elimination orders forms a star-shaped region which contrary to first intuition in general is not convex.
Moreover we show that the corresponding region may contain Gröbner cones intersecting in the boundary of the Gröbner fan in the origin only. This implies that Gröbner walks aiming for the elimination of variables from a polynomial ideal can be terminated earlier than previously known. We provide a slightly improved stopping criterion for a known Gröbner walk algorithm for the elimination of variables.

1. Introduction

Elimination in systems of polynomial equations is a classical topic important in optimization and modeling. Given an ideal $\mathfrak{I}$ of polynomials in $\mathbb{K}[X][U] := \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_m]$ over some field $\mathbb{K}$, the task of eliminating the variables u_i can be solved by finding an ideal basis for the so-called *elimination ideal* $\mathfrak{I} \cap \mathbb{K}[X]$, where $\mathbb{K}[X] = \mathbb{K}[x_1, \ldots, x_n]$. This can be achieved using resultants (see Sylvester, 1904; Salmon, 1964, or Sederberg et al., 1984) or by calculating a Gröbner basis (GB) for $\mathfrak{I}$ with respect to some special monomial order (see Buchberger, 1988; Kalkbrenner, 1991), as for example, the pure lexicographic or block term orders. Concerning these approaches, the method using Gröbner bases has some important advantages, namely, the method is reliable and can algorithmically solve the problem in full generality.

E-mail addresses: bosse@math.uni-frankfurt.de (H. Bosse), christine.gaertner@math.fu-berlin.de (C. Gärtner), oleg.golubitsky@gmail.com (O. Golubitsky).

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In the Gröbner basis approach one calculates a Gröbner basis $G_{\prec_{\text{elim}}}$ with respect to a suitable monomial order $\prec_{\text{elim}}$, such that those polynomials in $G_{\prec_{\text{elim}}} \cap \mathbb{K}[X]$ form a Gröbner basis for $\mathfrak{I} \cap \mathbb{K}[X]$. Calculating these very specific Gröbner bases directly can in practice be rather difficult. One way to overcome this, is to calculate such a special GB by performing a Gröbner walk, a method introduced by Collart, Kalkbrener, and Mall in Collart et al. (1997).

The actual walk consists of a series of elementary Gröbner basis conversions which are easy to compute. Starting with some easily computable GB of $\mathcal{I}$ with respect to some order $\prec_{\text{start}}$, step-by-step, intermediate GBs for orders in between $\prec_{\text{start}}$ and $\prec_{\text{elim}}$ are calculated. Each basis conversion from one intermediate GB into the next is (in general) relatively cheap computationwise, keeping the overall amount of necessary calculations relatively low (see Amrhein et al., 1997).

To handle the intermediate orders in any Gröbner walk algebraically, one represents them by weight vectors and introduces the concept of a Gröbner fan:
For a fixed ideal $\mathfrak{I} \subset \mathbb{K}[X][U]$, any proper monomial order for monomials in $\mathbb{K}[X][U]$ can be represented by some weight vector in $\omega \in \mathbb{R}_{\geqslant 0}^{n+m}$. The Gröbner fan, introduced by Mora and Robbiano in Mora and Robbiano (1988), is a polyhedral complex, which subdivides the weight vectors in $\mathbb{R}_{\geqslant 0}^{n+m}$. Each cell of the Gröbner fan is an equivalence class of such weight vectors:
Two weight vectors are equivalent, if the monomial order they represent yields the same Gröbner basis for $\mathfrak{I}$. The closure of such an equivalence class is a *Gröbner cone* and the collection of these cones forms the Gröbner fan. Note that Gröbner cones are convex polyhedral cones (see Mora and Robbiano, 1988).

Concerning Gröbner walks used in elimination of variables, Tran proposes in Tran (2004) to have the target monomial order $\prec_{\text{elim}}$ dependent on $\mathfrak{I}$, combining the Gröbner walk technique with a sudden-death-algorithm.

So instead of using *the same* elimination term order for *all* ideals, Tran proposes to use *an ideal-specific* monomial order suitable (only) for elimination in the specifically given ideal. He characterizes these special ideal-specific orders via the corresponding reduced Gröbner basis.

In addition to being faster on some examined test bed cases, his approach gets rid of several algebraic technicalities usually involved in Gröbner walks, e.g. his approach simplifies the necessary perturbation of the weight vector representing the elimination order:
Gröbner walk algorithms are particularly fast, if the given path of the walk is *generic*. To achieve this, one has to perturb the target weight vector of the walk in a suitable manner (see e.g. Fukuda et al., 2007a). In Tran (2004), Tran observed that using ideal-specific elimination orders, it suffices to end a Gröbner walk in a Gröbner cone adjacent to some *elimination vector* (see below) which eases the requirements on the necessary perturbations.

We refine Tran's findings by giving a more precise classification of those Gröbner cones, which correspond to ideal-specific elimination orders.

1.1. Main result

The main results of this paper are the following:

For a given ideal $\mathfrak{I} \subseteq \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_m]$, the union of all Gröbner cones belonging to $\mathfrak{I}$-specific orders for the elimination of $u_1, \ldots, u_m$ from $\mathfrak{I}$ forms a star-shaped region with center $\Omega_u := \{\omega \in \mathbb{R}_{\geqslant 0}^{n+m}\colon\ \omega_1 = 0, \ldots, \omega_n = 0\}$. This means that if one wishes to eliminate the variables u_i from $\mathfrak{I}$, i.e., one wants to calculate some Gröbner basis for $\mathfrak{I} \cap \mathbb{K}[X]$, the orders $\prec$ that *do yield* such a Gröbner basis have Gröbner cones, whose union is a star-shaped region with center Ω_u.

Moreover we show that (for some ideals $\mathfrak{I}$) some of the Gröbner cones which belong to $\mathfrak{I}$-specific elimination orders intersect the boundary of the Gröbner fan in the point zero *only*, meaning that for such cones all points but the vertex lie in the relative interior of the Gröbner fan.

Both results are very useful when trying to eliminate variables using the Gröbner walk-approach: First of all, we can improve the stopping criterion for such a Gröbner walk relative to the known result of Tran (2007). Moreover, knowing the geometric shape of the target-region can help improve the step-decision process in a Gröbner walk towards an elimination basis.

Finally, in the general case, just as shown by Tran, using our algorithm, one can get rid of technicalities involved in the implementation of the Gröbner walk such as the perturbation of the target vector (see Tran, 2007).

2. Notation

In the following we introduce some general notation for polynomials and monomial orders. To avoid clashes with our distinct variables x_i and u_j, here we name all variables y_i, assuming $(y_1, \dots, y_{n+m}) = (x_1, \dots, x_n, u_1, \dots, u_m)$. So in the following we consider polynomials $f = \sum_\alpha f_\alpha y^\alpha$ where $y^\alpha := \prod_{i=1}^{n+m} y_i^{\alpha_i}$ is a monomial with exponent $\alpha \in \mathbb{N}^{n+m}$ and the coefficients f_α are from some field $\mathbb{K}$.

2.1. Monomial orders

In the following let $\prec$ be some monomial order and $f, g \in \mathbb{K}[Y]$. We denote the *leading term* of f w.r.t. $\prec$ by $\mathrm{lt}_\prec(f)$. Let $\mathfrak{I} \subset \mathbb{K}[Y]$ be some polynomial ideal, then the *initial ideal* of $\mathfrak{I}$ w.r.t. $\prec$ is the ideal $\langle \mathrm{lt}_\prec(\mathfrak{I}) \rangle$ which is generated by the set of *leading terms* of $\mathfrak{I}$, i.e., $\mathrm{lt}_\prec(\mathfrak{I}) := \{\mathrm{lt}_\prec(f)\colon\ f \in \mathfrak{I}\}$.

2.2. Reduced Gröbner bases

In this work we consider reduced Gröbner bases: Let $\mathfrak{I} \subseteq \mathbb{K}[Y]$ be some monomial ideal and let $\prec$ be some monomial order. A Gröbner basis G for $\mathfrak{I}$ w.r.t. $\prec$ is called *reduced* if for every pair $g, h \in G$, $g \neq h$, one has that $\mathrm{lt}_\prec(g)$ does not divide any monomial of h (so h cannot be reduced by g any further). Moreover G is called *normed* if for all $g \in G$ the leading coefficient is 1.

Every ideal $\mathfrak{I} \subseteq \mathbb{K}[Y]$ has a unique finite normed reduced Gröbner basis with respect to $\prec$ (see Cox et al., 1992; Buchberger, 1965), which we denote by $\underline{GB}(\mathfrak{I}, \prec)$.

2.3. Weight vectors

To algebraically work with monomial orders, it is helpful to represent them by weight vectors: The set of all *weight vectors* $\Omega := \mathbb{R}^{n+m}_{\geqslant 0}$ is the non-negative orthant. Let $f \in \mathbb{K}[Y]$ and $\omega \in \Omega$, then $\deg_\omega(f) := \max\{\omega^T\alpha\colon\ f_\alpha \neq 0\}$ is the *degree* of f w.r.t. ω. The *initial form* or *leading terms* of f w.r.t. $\omega \in \Omega$ is defined as

$$\mathrm{lt}_\omega(f) := \sum_{\alpha \in A} f_\alpha y^\alpha \quad \text{where } A := \left\{\alpha \in \mathbb{N}^n\colon\ f_\alpha \neq 0,\ \omega^T\alpha = deg_\omega(f)\right\}.$$

The *initial ideal* of $\mathfrak{I}$ w.r.t. ω is the set $\langle \mathrm{lt}_\omega(\mathfrak{I}) \rangle := \langle \{\mathrm{lt}_\omega(f)\colon\ f \in \mathfrak{I}\} \rangle$.

Definition 2.1. Let $\mathfrak{I} \subseteq \mathbb{K}[Y]$ be some fixed ideal, let $\prec$ be some monomial order and $\omega \in \Omega$. We say that

- ω *represents* $\prec$ if $\langle \mathrm{lt}_\omega(\mathfrak{I}) \rangle = \langle \mathrm{lt}_\prec(\mathfrak{I}) \rangle$ holds.
- $\prec$ *refines* ω, if for all pairs of monomials $m_1, m_2 \in [Y]$ one has that $\deg_\omega(m_1) < \deg_\omega(m_2)$ implies $m_1 \prec m_2$.

Not all weight vectors ω induce a proper monomial order. But using some monomial order as an additional tie-breaker does yield an order:

Definition 2.2. Given an ideal $\mathfrak{I} \subseteq \mathbb{K}[Y]$, a monomial order $\prec$, and some weight vector ω the monomial order $(\omega| \prec)$ is defined as follows:

Let $\prec' := (\omega| \prec)$, then

$$m_1 \prec' m_2 \quad :\Leftrightarrow \quad \begin{cases} \deg_\omega(m_1) < \deg_\omega(m_2), & \text{or} \\ \deg_\omega(m_1) = \deg_\omega(m_2) & \text{and} \quad m_1 \prec m_2. \end{cases}$$

So $(\omega|\prec)$ corresponds to first (partially) ordering the monomials by $\deg_\omega$ and using $\prec$ as a tie-breaker. Clearly, the order $(\omega|\prec)$ refines ω.

2.3.1. The Gröbner fan

Definition 2.3. Given an ideal $\mathfrak{I} \subseteq \mathbb{K}[Y]$ and a monomial order $\prec$, we define the Gröbner cone of $\mathfrak{I}$ w.r.t. $\prec$ by

$$C_\prec(\mathfrak{I}) := \mathrm{closure}\big(\big\{\omega \in \Omega\colon \big\langle \mathrm{lt}_\omega(\mathfrak{I})\big\rangle = \big\langle \mathrm{lt}_\prec(\mathfrak{I})\big\rangle\big\}\big)$$

where closure denotes the closure with respect to the standard topology in $\mathbb{R}^{n+m}$.

For complete information on Gröbner cones, we would like to refer to Mora and Robbiano (1988), here we repeat some facts of these cones, relevant to this paper:
Each Gröbner cone of $\mathfrak{I}$ is a convex polyhedral cone with non-empty interior (see Mora and Robbiano, 1988) and the set of all Gröbner cones forms a polyhedral complex, namely the Gröbner fan $\mathcal{C}(\mathfrak{I}) := \{C_\prec(\mathfrak{I})\colon \prec \text{ is some monomial order}\}$.

Moreover, each Gröbner cone corresponds to some reduced Gröbner basis, i.e., all monomial orders, which are represented by the weight vectors within the same Gröbner cone, will have the same reduced Gröbner basis. This implies that $\mathfrak{I}$ has only finitely many different Gröbner cones. Moreover, we obtain the following for a weight vector and a monomial order constructed from it:

Lemma 2.4. *For a weight vector $\omega \in \Omega$ and some order $\prec$ let $\prec_\omega := (\omega|\prec)$. With this one has $\omega \in C_{\prec_\omega}(\mathfrak{I})$.*

Reversely, if $\omega \in C_\prec(\mathfrak{I})$ holds, then $\mathrm{lt}_{\prec_\omega}(g) = \mathrm{lt}_\prec(g)$ holds for all $g \in \underline{GB}(\mathfrak{I}, \prec)$, which consequently implies $\underline{GB}(\mathfrak{I}, \prec_\omega) = \underline{GB}(\mathfrak{I}, \prec)$.

For a proof we refer to Lemma 2.15 and Corollary 2.11 in Fukuda et al. (2007b).

2.4. Geometry

In the following we prove that some special set of weight vectors is star-shaped, to this end we recall the following:

Definition 2.5. A set $S \subseteq \mathbb{R}^{n+m}$ is called *star-shaped* with center $C \subseteq S$, if for any two points $s \in S$ and $c \in C$ the segment $\overline{cs}$ is contained in S.

2.5. Universal elimination orders

In the following we assume $\mathfrak{I}$ to be some ideal in $\mathbb{K}[X][U]$. A class of monomial orders, which provides a reduced Gröbner basis for the elimination ideal $\mathfrak{I} \cap \mathbb{K}[X]$ is the set of elimination orders; these orders are traditionally used to calculate the elimination ideal via Gröbner bases.

Definition 2.6. A monomial order $\prec$ on $\mathbb{K}[X][U]$ is called ***universal*** *elimination order* for U, if

$$\mathrm{lt}_\prec(f) \in \mathbb{K}[X] \quad \Rightarrow \quad f \in \mathbb{K}[X] \quad \forall f \in \mathbb{K}[X][U].$$

So a universal elimination order for U will have to prefer *any* u-variable over some x-variable. For example, an appropriate lexicographic order is a universal elimination order. A universal elimination order can be used to calculate the GB of the elimination ideal for *any* given ideal:

Lemma 2.7. *If $\prec$ is a universal elimination order, then for* every *ideal $\mathfrak{I}$, the set $\underline{GB}(\mathfrak{I}, \prec) \cap \mathbb{K}[X]$ is the reduced Gröbner basis of the elimination ideal $\mathfrak{I} \cap \mathbb{K}[X]$ w.r.t. $\prec$.*

For a proof see Tran (2004).

2.6. Ideal-specific elimination orders

In contrast to universal elimination orders, in this paper we examine ideal-specific elimination orders, which serve to eliminate variables only for the specifically given ideal:

Definition 2.8 *(Ideal-specific elimination orders and vectors).* Let $\mathfrak{I} \subseteq \mathbb{K}[X][U]$ be an ideal and $\prec$ a monomial order with

$$\mathrm{lt}_\prec(g) \in \mathbb{K}[X] \quad \Rightarrow \quad g \in \mathbb{K}[X] \quad \forall g \in \underline{GB}(\mathfrak{I}, \prec).$$

1. Then $\prec$ is called $\mathfrak{I}$*-specific elimination order for the elimination of* U. When clear which variables are to be eliminated we abbreviate this to $\mathfrak{I}$*-specific elimination order*, or just $\mathfrak{I}$-EO.
2. Any $\omega \in C_\prec(\mathfrak{I})$ is called $\mathfrak{I}$*-specific for the elimination of* U ($\mathfrak{I}$-EV).

In the following we will always consider the elimination of the u-variables for ideals in $\mathbb{K}[X][U]$, so all ideal-specific elimination orders and ideal-specific elimination vectors will be ideal-specific for the elimination of U.

The reduced Gröbner basis for an $\mathfrak{I}$-EO yields a Gröbner basis for the elimination ideal:

Lemma 2.9. *Let $\mathfrak{I} \subset \mathbb{K}[X][U]$ be some fixed ideal. If $\prec$ is an $\mathfrak{I}$-specific elimination order for the elimination of U, then the set $\underline{GB}(\mathfrak{I}, \prec) \cap \mathbb{K}[X]$ is the reduced Gröbner basis of the elimination ideal $\mathfrak{I} \cap \mathbb{K}[X]$ w.r.t. $\prec$.*

For a proof see Tran (2007).

So any $\mathfrak{I}$-EO yields a Gröbner basis suitable for the elimination of the variables u_i from $\mathfrak{I}$. But in contrast to *universal* elimination orders, an $\mathfrak{I}$-EO will in general not work for other polynomial ideals. However, any *universal* elimination order is – by definition – also an $\mathfrak{I}$-EO for any ideal $\mathfrak{I}$.

For our proofs we use the following characterization for an $\mathfrak{I}$-EO:

Lemma 2.10. *Let $\mathfrak{I} \subset \mathbb{K}[X][U]$ be some fixed ideal. A monomial order $\prec$ is $\mathfrak{I}$-EO for U if and only if*

$$\mathrm{lt}_\prec\big(\mathfrak{I} \cap \mathbb{K}[X]\big) = \mathrm{lt}_\prec(\mathfrak{I}) \cap \mathbb{K}[X]. \tag{1}$$

The implication "$\subseteq$" in (1) can be directly seen, for a complete proof of the converse we refer to Tran (2007).

By Definition 2.8, if $\prec$ is an $\mathfrak{I}$-EO, then any weight vector in the Gröbner cone $C_\prec(\mathfrak{I})$ is $\mathfrak{I}$-EV. Now assume (conversely) that one finds some $\mathfrak{I}$-EV ω in $C_\prec(\mathfrak{I})$ with $\omega \neq 0$. In some of these cases, one can conclude that $\prec$ is an $\mathfrak{I}$-EO, namely if ω is in the *interior* of $C_\prec(\mathfrak{I})$ (see Tran, 2007) or if ω lies on a *special* part of the boundary of Ω:

Lemma 2.11. *Let $\mathfrak{I} \subseteq \mathbb{K}[X][U]$ be some ideal and $\prec$ some monomial order. If $(\mathbf{0}, \widetilde{\omega}_u) \in C_\prec(\mathfrak{I})$ holds for some $\widetilde{\omega}_u \in \mathbb{R}^m_{>0}$, then $\prec$ is an $\mathfrak{I}$-EO.*

Lemma 2.11 and its proof can be found in Tran (2004), it is used to obtain the main result in Tran (2007). Geometrically, this lemma proves that $\prec$ is $\mathfrak{I}$-EO if its Gröbner cone intersects the *relative interior* of a special face Ω_u of the polyhedron Ω, where

$$\Omega_u := \big\{\omega \in \mathbb{R}^{n+m}_{\geqslant 0}\colon\ \omega_1 = 0, \dots, \omega_n = 0\big\}.$$

Since each $C_\prec(\mathfrak{I})$ containing some vector $(\mathbf{0}, \widetilde{\omega}_u)$ with $\widetilde{\omega}_u > 0$ comes from some $\mathfrak{I}$-EO $\prec$, and since these Gröbner cones are closed, Lemma 2.11 implies

Corollary 2.12. *All vectors $\omega \in \Omega_u$ are $\mathfrak{I}$-EVs.*

3. Main result

Our main result is the following:

Theorem 1. *Let $\mathfrak{I}$ be a polynomial ideal in $\mathbb{K}[x_1, \dots, x_n, u_1, \dots, u_m]$.*

The Gröbner cones of all $\mathfrak{I}$-specific elimination orders for the elimination of $u_1, \dots, u_m$ from $\mathfrak{I}$ form a star-shaped region, whose center is the following face of $\mathbb{R}^{n+m}_{\geqslant 0}$:

$$\Omega_u := \left\{\omega \in \mathbb{R}^{n+m}_{\geqslant 0}\colon \omega_1 = 0, \dots, \omega_n = 0\right\}.$$

Proof. Let $\tau \in \Omega_u$, and let σ be some $\mathfrak{I}$-EV, i.e., one has $\sigma \in C_{\prec'}(\mathfrak{I})$ for some $\mathfrak{I}$-EO $\prec'$. Here τ can be part of the relative boundary of Ω_u, e.g. $\tau = \mathbf{0}$ is possible. By Corollary 2.12, we know that τ is $\mathfrak{I}$-EV and thus we have to show that all other points in the segment $[\sigma, \tau]$ are $\mathfrak{I}$-EV, too. So let $\omega := \lambda\sigma + (1-\lambda)\tau$ with $\lambda \in (0, 1)$.

Let $\prec := (\sigma \mid \prec')$, then due to $\sigma \in C_{\prec'}(\mathfrak{I})$ one has $\underline{GB}_{\prec}(\mathfrak{I}) = \underline{GB}_{\prec'}(\mathfrak{I})$ – see Lemma 2.4. Moreover the orders $\prec$ and $\prec'$ yield the same leading terms on all $g \in \underline{GB}_{\prec}(\mathfrak{I})$ (Lemma 2.4) and so $\prec$ is $\mathfrak{I}$-EO by Definition 2.8.

We examine the monomial orders $\prec_\sigma := (\sigma \mid \prec)$, $\prec_\tau := (\tau \mid \prec)$, and $\prec_\omega := (\omega \mid \prec)$ and show that $\prec_\omega$ is $\mathfrak{I}$-EO, which together with $\omega \in C_{\prec_\omega}(\mathfrak{I})$ (see Lemma 2.4) shows that ω is $\mathfrak{I}$-EV. Note that one has $\prec_\sigma = \prec$ and thus $\prec_\sigma$ is $\mathfrak{I}$-EO.

Now we show that $\prec_\omega$ is $\mathfrak{I}$-EO. Due to $\tau \in \Omega_u$ one has $\tau = (\tau_x, \tau_u)$ with $\tau_x = \mathbf{0}$, implying that for $\omega = (\omega_x, \omega_u)$ one has $\omega_x = \lambda\sigma_x$ with $\lambda > 0$. So $\prec_\omega = (\omega \mid \prec)$ and $\prec_\sigma = (\sigma \mid \prec)$ coincide on $\mathbb{K}[X]$. This implies

$$\operatorname{lt}_{\prec}\bigl(\mathfrak{I} \cap \mathbb{K}[X]\bigr) = \operatorname{lt}_{\prec_\sigma}\bigl(\mathfrak{I} \cap \mathbb{K}[X]\bigr) = \operatorname{lt}_{\prec_\omega}\bigl(\mathfrak{I} \cap \mathbb{K}[X]\bigr),$$

we call this set $\mathrm{L}^{[X]}_{\prec}$.

Assume now, that $\prec_\omega$ is *not* $\mathfrak{I}$-EO, i.e., $\operatorname{lt}_{\prec_\omega}(\mathfrak{I}) \cap \mathbb{K}[X] \neq \mathrm{L}^{[X]}_{\prec}$. This implies $\operatorname{lt}_{\prec_\omega}(\mathfrak{I}) \cap \mathbb{K}[X] \nsubseteq \mathrm{L}^{[X]}_{\prec}$ since the reverse inclusion is always true. So there must be some $g \in \mathfrak{I}$ with $\operatorname{lt}_{\prec_\omega}(g) \in \mathbb{K}[X] \setminus \mathrm{L}^{[X]}_{\prec}$.

Let $x^\alpha := \operatorname{lt}_{\prec_\omega}(g)$ and $m := \operatorname{lt}_{\prec_\sigma}(g)$, then $m \neq x^\alpha$. This holds, since $x^\alpha = m$ leads to the contradiction $x^\alpha = m \in \operatorname{lt}_{\prec_\sigma}(\mathfrak{I}) \cap \mathbb{K}[X] = \mathrm{L}^{[X]}_{\prec}$ (the latter holds since $\prec_\sigma$ is $\mathfrak{I}$-EO).

We conclude $x^\alpha \prec_\tau m$, since $\deg_\tau(x^\alpha) = 0 \leqslant \deg_\tau(m)$ holds by choice of τ where in case of "=" the tie-breaker $\prec = \prec_\sigma$ yields $x^\alpha \prec_\sigma m$. So in total we obtain

1. $x^\alpha \prec_\sigma m$ (since $\operatorname{lt}_{\prec_\sigma}(g) = m$),
2. $m \prec_\omega x^\alpha$ (since $\operatorname{lt}_{\prec_\omega}(g) = x^\alpha$),
3. $x^\alpha \prec_\tau m$ (since $\deg_\tau(x^\alpha) = 0$ and $x^\alpha \prec m$).

By the constructions of these monomial orders we conclude

$$\left.\begin{matrix}\deg_\sigma(x^\alpha) \leqslant \deg_\sigma(m)\\ \deg_\tau(x^\alpha) \leqslant \deg_\tau(m)\end{matrix}\right\} \quad \Rightarrow \quad \deg_\omega\bigl(x^\alpha\bigr) \leqslant \deg_\omega(m),$$

where "=" in the last inequality implies "=" in all three inequalities, leading to $x^\alpha \prec m$ (due to $x^\alpha \prec_\sigma m$). This yields the contradiction $x^\alpha \prec_\omega m$, showing that g cannot exist and thus $\prec_\omega$ is $\mathfrak{I}$-EO. □

4. The geometry of ideal-specific elimination vectors

In this section we prove two further geometrical properties of the set of all ideal-specific elimination vectors for a given ideal.

Theorem 1 shows that for a given ideal $\mathfrak{I}$, the $\mathfrak{I}$-specific elimination vectors form a set that is star-shaped. Here we prove that this set in general is non-convex. Moreover, we prove by example that an ideal $\mathfrak{I}$ can have an $\mathfrak{I}$-specific elimination order $\prec$, whose Gröbner cone $C_{\prec}(\mathfrak{I})$ intersects the exterior of the Gröbner fan of $\mathfrak{I}$ in the origin only.

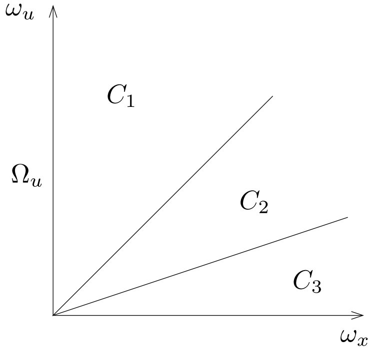


Fig. 1. Gröbner fan of $\mathfrak{I} = \langle x^2 - 1, xu^2 - x - u\rangle$.

4.1. Cones in the interior

Lemma 4.1. *There are ideals $\mathfrak{I} \subset \mathbb{K}[X][U]$ which have an $\mathfrak{I}$-EO $\prec$ whose Gröbner cone $C_\prec(\mathfrak{I})$ intersects the boundary of the Gröbner fan in the origin* 0 *only.*

Proof. Consider the following ideal $\mathfrak{I} = \langle x^2 - 1, xu^2 - x - u\rangle \subseteq \mathbb{K}[x][u]$. There are exactly three different reduced Gröbner bases of $\mathfrak{I}$, which correspond to the three Gröbner cones of the Gröbner fan:

$$G_1 = \{\mathbf{x}^2 - 1, \mathbf{u}^2 - xu - 1\},$$
$$G_2 = \{\mathbf{x}^2 - 1, \mathbf{xu} - u^2 + 1, \mathbf{u}^3 - 2u - x\},$$
$$G_3 = \{\mathbf{x} + 2u - u^3, \mathbf{u}^4 - 3u^2 + 1\}.$$

Here the leading terms are given in bold letters.

For $i = 1, 2, 3$ let C_i be the Gröbner cones corresponding to the Gröbner basis G_i and let $\prec_i$ be some corresponding monomial order.

Examining the polynomials in G_1 and G_2 in respect to Definition 2.8, one observes that $\prec_1$ and $\prec_2$ are $\mathfrak{I}$-specific elimination orders for the elimination of u. We now check that the cone C_2 must be in between the cones C_1 and C_3 (see Fig. 1).

It is easy to check that for $\overline{\omega} := (1, 0)^T$ one has $\mathrm{lt}_{\overline{\omega}}(G_3) = \mathrm{lt}_{\prec_3}(G_3)$ and $\mathrm{lt}_{\overline{\omega}}(G_i) \neq \mathrm{lt}_{\prec_i}(G_i)$ for $i = 1, 2$. This implies that $\overline{\omega} \in C_3$ holds. In the same way one proves $(0, 1)^T \in C_1$.

Since the Gröbner fan considered here is two-dimensional, C_2 must thus be in between C_1 and C_3. This proves that for $\mathfrak{I}$, there is indeed an $\mathfrak{I}$-EO ($\prec_2$) whose Gröbner cone C_2 intersects the boundary of the Gröbner fan in $(0, 0)^T$ only. □

4.2. Non-convexity

It seems intuitive at first sight that the set of all $\mathfrak{I}$-EVs should be convex, but this is in general not true.

Example 4.2. Let $\mathfrak{I} := \langle x + u + v, x^2 - 1\rangle \subseteq \mathbb{K}[x][u, v]$ and set $\sigma := (9, 12, 0)^T$, $\tau := (9, 0, 10)^T \in \Omega$, and $\omega := \frac{1}{2}\sigma + \frac{1}{2}\tau = (9, 6, 5)^T \in \overline{\sigma\tau}$. Let $\prec_\sigma$, $\prec_\tau$ and $\prec_\omega$ be monomial orders refining σ, τ, and ω respectively.

Quick calculation shows that the reduced Gröbner bases w.r.t. $\prec_\sigma$ and $\prec_\tau$ are the following

$$\underline{GB}(\mathfrak{I}, \prec_\sigma) = \{\mathbf{u} + x + v, \mathbf{x}^2 - 1\},$$
$$\underline{GB}(\mathfrak{I}, \prec_\tau) = \{\mathbf{v} + x + u, \mathbf{x}^2 - 1\}.$$

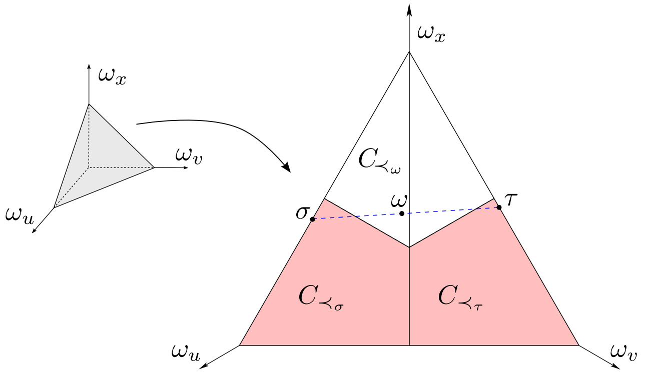


Fig. 2. Gröbner fan for $\mathfrak{I} = \langle x + u + v, x^2 - 1\rangle$.

So by Definition 2.8, both σ and τ are $\mathfrak{I}$-specific elimination vectors for elimination of the variables u and v. The reduced Gröbner basis w.r.t. $\prec_\omega$ is

$$\underline{GB}(\mathfrak{I}, \prec_\omega) = \left\{\mathbf{x} + u + v, \mathbf{u}^2 + 2uv + v^2 - 1\right\}.$$

Since one has $\mathrm{lt}_{\prec_\omega}(x+u+v) = x \in \mathbb{K}[x]$ but $x+u+v \notin \mathbb{K}[x]$, by Definition 2.8, $\prec_\omega$ cannot be $\mathfrak{I}$-specific for the elimination of u and v.

Fig. 2 depicts the Gröbner fan of $\mathfrak{I}$ (intersected with some appropriate hyperplane) together with σ, ω and τ. The highlighted Gröbner cones $C_{\prec_\sigma}$, $C_{\prec_\tau}$ correspond to the $\mathfrak{I}$-EOs $\prec_\sigma$ and $\prec_\tau$.

5. Improving the elimination algorithm by Tran

The algorithm of Tran (Algorithm 1 in Tran, 2007) calculates a Gröbner basis for the elimination ideal of by means of a generic Gröbner walk.

5.1. Generic Gröbner walks

A *generic* Gröbner walk "walks" along some generic segment $\overline{\sigma\tau} \subset \Omega$. Such a segment is called generic if

- $\overline{\sigma\tau}$ only passes through the *interior* of intermediate Gröbner cones or through *interior points of their facets* and
- σ is part of the *interior* of some $C_{\prec_{\mathrm{start}}}(\mathfrak{I})$.

The walk starts with the (hopefully easy to compute) Gröbner basis G_0 of $\mathcal{I}$ w.r.t. $\prec_0 := \prec_{\mathrm{start}}$. Then sequentially, starting from $C_{\prec_0}(\mathfrak{I})$ for every cone $C_{\prec_k}(\mathfrak{I})$ through which $\overline{\sigma\tau}$ passes, the intermediate GB G_k w.r.t. $\prec_k$ is calculated. This can be done effectively by converting the previously calculated G_{k-1} into G_k. Each such basis conversion from one intermediate GB into the next is (in general) relatively cheap computationwise, keeping the overall amount of necessary calculations relatively low (see Amrhein et al., 1997).

The walk terminates returning G_ℓ when reaching a cone $C_{\prec_\ell}(\mathfrak{I})$ containing τ.

5.2. Improvement to Tran's stopping criterion

The algorithm of Tran (Algorithm 1 in Tran, 2007) which calculates a Gröbner basis for the elimination ideal by means of a Gröbner walk can be *slightly* improved, by changing the termination criterion:

Tran's algorithm performs a Gröbner walk towards some τ in the relative interior of Ω_u, i.e., a point $\tau = (\tau_x, \tau_u)$ with $\tau_x = \mathbf{0}$ and $\tau_u \in \mathbb{R}^m_{>0}$. As a stopping criterion Tran uses Lemma 2.11, which states: If some intermediate Gröbner cone $C_{\prec_k}(\mathfrak{I})$ contains τ, then the corresponding $\prec_k$ is an ideal-specific elimination order for the elimination of U. Tran then sets his Gröbner walk to terminate when reaching such a cell.

Note the following: Since $\tau \in \Omega_u$ is part of the boundary of Ω, in such a particular case $C_{\prec_k}(\mathfrak{I})$ intersects the boundary of Ω in more than just the origin. In contrast, in Lemma 4.1 we prove that there are ideals $\mathfrak{I}$, for which there are $\mathfrak{I}$-specific elimination orders, whose Gröbner cones intersect the boundary of the Gröbner fan in *just the origin*. In this regard, our Algorithm 1 is an improvement of Tran's version.

Algorithm 1 *(Improved elimination algorithm).*

Input $F = \{f_1, \ldots, f_\ell\} \subseteq \mathbb{K}[x_1, \ldots, x_n][u_1, \ldots, u_m] = \mathbb{K}[X][U]$
$\tau \in \Omega$, where $\tau^T = (0^T, \tau_u^T)$ with $\tau_u \in \mathbb{R}^m_{>0}$,
$\sigma \in \Omega$, such that $\overline{\sigma\tau}$ is generic,
$\prec_\sigma$, $\prec_\tau$ refining σ resp. τ.

Output $G \subseteq \mathbb{K}[X]$, reduced Gröbner basis of $\langle F \rangle \cap \mathbb{K}[X]$
w.r.t. some $\langle F \rangle$-specific EO for U.

Init Calculate reduced start GB $G_0 := \underline{GB}(\langle F \rangle, \prec_\sigma)$
$k := 0$, $\mathfrak{I} := \langle F \rangle$, $\omega_0 := \sigma$, $\prec_0 := \prec_\sigma$

Step 1 IF $\prec_k$ is $\mathfrak{I}$-EO RETURN $G := G_k \cap \mathbb{K}[X]$

Step 2 GB-walk: change cell
- (2.1) $k := k + 1$
- (2.2) Find next weight vector $\omega_k \in \overline{\sigma\tau}$,
  s.t. $\overline{\omega_{k-1}\omega_k} = \overline{\omega_{k-1}\tau} \cap C_{\prec_{k-1}}(\mathfrak{I})$.
- (2.3) Set $\prec_k := (\omega_k | \prec_\tau)$.
- (2.3) Convert G_{k-1} into Gröbner basis G_k w.r.t. $\prec_k$.
- (2.4) Interreduce G_k.

Step 3 GOTO Step 1

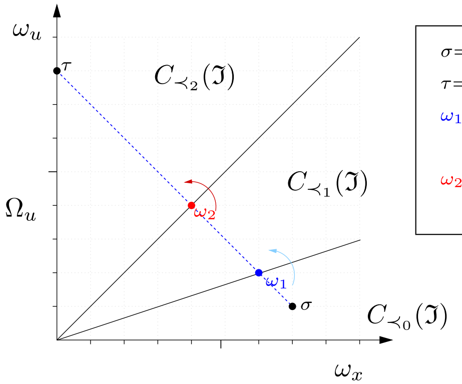
Remark 5.1. Algorithm 1 as stated above, is in fact Tran's Algorithm in Tran (2007) – the only difference being the (refined) stopping criterion in Step 1. In Tran's original version of Algorithm 1, his stopping criterion in Step 1 reads:

"IF $\tau \in C_{\prec_k}(\mathfrak{I})$ RETURN $G := G_k \cap \mathbb{K}[X]$."

Theorem 2. *Algorithm 1 terminates and is correct.*

Proof. In each ω_k the section $\overline{\sigma\tau}$ crosses from some Gröbner cone into another. Since there are only finitely many such transition-points ω_k, the algorithm can only perform a finite number of steps.

The algorithm terminates as soon as it passes some ω_ℓ where $\prec_\ell$ is $\mathfrak{I}$-EO. Such a point ω_ℓ must exist, due to the following:
One has $\tau \in C_{\prec_\tau}(\mathfrak{I})$ and thus $\overline{\sigma\tau} \cap C_{\prec_\tau}(\mathfrak{I}) \neq \emptyset$. Let ω be the first point on $\overline{\sigma\tau}$ that is in $C_{\prec_\tau}$, and assume the algorithm does not terminate on any point in $\overline{\sigma\omega} \setminus \{\omega\}$. Then $\omega = \omega_k$ holds for some k, since either $\omega = \sigma = \omega_0$ holds or ω is on the boundary of some $C_{\prec_{k-1}}(\mathfrak{I})$ of the examined ω_{k-1}. In



$\sigma=$	(7,1)	Start of the walk.
$\tau=$	(0,8)	Target of the walk, $\prec_\tau:=\prec_2$.
$\omega_1=$	(6,2)	Endpoint of our algorithm, order is $\prec_1=(\omega_1\mid\prec_\tau)$
$\omega_2=$	(4,4)	Endpoint of Tran's algorithm, order is $\prec_2=\prec_\tau=(\omega_2\mid\prec_\tau)$

Fig. 3. Gröbner walk for $\mathfrak{I}=\langle x^2-1, xu^2-x-u\rangle$.

either case, the algorithm will calculate $\underline{GB}_{\prec_\omega}(\mathfrak{I})$ for $\prec_\tau=(\omega\mid\prec_\tau)$ and then terminate, since $\omega\in C_{\prec_\tau}$ and thus $\prec_\tau$ is $\mathfrak{I}$-EO by Lemma 2.11.

With $\prec_\ell$ being an $\mathfrak{I}$-EO, due to Lemma 2.9 $G_\ell\cap\mathbb{K}[X]$ is a Gröbner basis for $\langle F\rangle\cap\mathbb{K}[X]$ (see Lemma 2.9). □

Example 5.2. In Lemma 4.1 we present the ideal $\mathfrak{I}=\langle x^2-1, xu^2-x-u\rangle$ where the difference in stopping criteria actually matters – see Fig. 3. For $\mathfrak{I}$ there are three reduced Gröbner bases, namely (leading terms in bold letters)

$$\underline{GB}_{\prec_0}(\mathfrak{I})=\{\mathbf{x}-u^3+2u,\ \mathbf{u}^4-3u^2+1\},$$
$$\underline{GB}_{\prec_1}(\mathfrak{I})=\{\mathbf{x}^2-1,\ \mathbf{xu}-u^2+1,\ \mathbf{u}^3-2u-x\},$$
$$\underline{GB}_{\prec_2}(\mathfrak{I})=\{\mathbf{x}^2-1,\ \mathbf{u}^2-xu-1\}.$$

We now start our walk with the Gröbner basis $\underline{GB}_{\prec_0}(\mathfrak{I})$ at $\sigma=(7,1)$, in the interior of $C_{\prec_0}$. We then walk towards $\tau=(0,8)\in C_{\prec_2}$. With this setting, our algorithm stops after reaching $\omega_1=(6,2)$ with the Gröbner basis $\underline{GB}_{\prec_1}(\mathfrak{I})$ while Tran's algorithm stops after reaching $\omega_2=(4,4)$ with the Gröbner basis $\underline{GB}_{\prec_2}(\mathfrak{I})$. So Tran's algorithm calculates one more basis conversion than our Algorithm 1.

6. Conclusion and outlook

The work in both Tran (2007) and Tran (2004) provides proper algorithms to make use of Gröbner walks in the elimination of variables from polynomial ideals. Tran's approach even simplifies perturbing the corresponding walk in order to obtain a generic walk.

Our results refine this work. We provide a geometric interpretation of the set of ideal-specific elimination vectors. More precisely we prove that these weight vectors form a star-shaped region. More surprisingly, we show that the corresponding region in general is not convex.

Finally, we redefine Tran's stopping criterion and show that this yields some improvement over Tran's original stopping criterion. Tran's criterion stops the Gröbner walk when reaching a Gröbner cone containing the target weight vector τ, which in turn is part of the boundary of the Gröbner fan. In contrast to this, we show that for some polynomial ideals one can terminate the Gröbner walk in some "interior" Gröbner cone, namely a cone whose intersection with the boundary of the Gröbner fan is just the origin. Whether this improvement yields a significant improvement for the average running time of Tran's algorithm is not clear, and should be subject to further research.

A possible improvement to our work would be to check whether the star-shapedness of the region of interest gives rise to cleverly changing the direction of the walk, leading to a more efficient zig-zag-walk. More precisely one would like to answer the following:

If, in some step of the Gröbner walk, the current Gröbner cone borders (via a facet) to some cone of an ideal-specific elimination order, one could of course terminate the walk with a single step. Is it possible to cheaply determine such situations from the current Gröbner basis?

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