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Improved Upper Bounds of the Oriented Diameter of Graphs

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Abstract

The diameter of a graph equals the longest distance between any pair of vertices. A graph can be oriented by assigning a direction to each edge. A strongly connected orientation includes a directed path from each vertex $u \in V(G)$ to each other vertex $v \in V(G)$. The oriented diameter of a graph G is the minimum diameter of all strongly connected orientations of G. Let $f(d)$ be the maximum oriented diameter of all graphs of diameter d. In other words, every 2-edge-connected graph of diameter d admits an orientation of diameter $f(d)$ or less. Chvátal and Thomassen [2] showed the following bounds:

$$\frac{1}{2}d^2 + d \leq f(d) \leq 2d^2 + 2d.$$

A recent paper by Babu, Benson, Rajendraprasad and Vaka improves the upper bound for all $d \geq 8$ and $d = 4$ [1]. By using a similar approach, I improve the upper bounds for all $d \geq 5$. None of these upper bounds could be shown to be tight, so there is room for further improvement. Additionally this thesis gives an overview of the current best results and techniques used to prove the current best results.

Contents

1 Introduction

For every undirected graph G, we denote $V(G)$ the set of vertices of G and $E(G)$ the set of edges. The distance between two vertices $u, v \in V(G)$ is the length of the shortest path between them and is denoted as $d_G(u, v)$. The length of a path in an unweighted graph is equal to the number of edges that are on the path. The diameter of a graph G is the longest of those distances between any two vertices u, v for all $u, v \in V(G)$. We denote the diameter of G as $d(G)$. This means that no pair of vertices u, v exists such that $d_G(u, v) > d(G)$ and a pair of vertices u, v exists such that $d_G(u, v) = d(G)$. In an undirected graph, $d_G(u, v) = d_G(v, u)$ and $d_G(u, u) = 0$ for all vertices u, v.

The eccentricity of a vertex is the largest distance between that vertex and any other vertex in the graph. The smallest of all eccentricities is the radius of a graph. Hence an undirected graph of radius r has at least one vertex u such that $d_G(u, v) \leq r$ for every vertex $v \in V(G)$. In a directed graph G we also demand that $d_G(v, u) \leq r$. The vertex u and all vertices with the same eccentricity are called center of G.

When orienting a graph we assign each edge a direction, thus imposing restrictions on the shortest path we might find between any two vertices. Every graph admits $2^{|E(G)|}$ such orientations. Many of those are not strongly connected so their diameter is not well-defined. By strongly connected we mean that for every pair of vertices $u, v \in V(G)$ a directed path from u to v exists.

Robbins [6] showed in 1939 that a connected graph admits at least one strong orientation if and only if it is a 2-edge-connected graph. A 2-edge-connected graph allows us to remove any edge without the graph becoming unconnected.

In directed graphs the equation $d_G(u, v) = d_G(v, u)$ is not generally true. The diameter of an orientation D of a graph G is the maximum of all distances, that is

$$d(D) = \max\{d_D(u, v) | u, v \in V(D)\}.$$

The oriented diameter of a graph G is the minimum of the diameters of all strong orientations of G and is denoted as $\overrightarrow{d}(G)$. Let $f(d)$ be the maximum oriented diameter of all graphs with diameter d, so every 2-edge-connected graph of diameter d admits an orientation with diameter at most $f(d)$. It is desireable to bound $f(d)$ for a generic diameter d. Chvátal and Thomassen [2] were the first to provide the following bounds in 1978:

Theorem 1. *[2]*

$$\frac{1}{2}d^2 + d \leq f(d) \leq 2d^2 + 2d.$$

This leaves a wide gap between upper and lower bounds. Kwok, Liu and West [4] studied a special case, namely $d = 3$ and showed that

$$9 \leq f(3) \leq 11,$$

which is an improvement over $7.5 \leq f(3) \leq 24$ given by Theorem 1.

Their approach was used in recent paper by Babu, Benson, Rajendraprasad and Vaka [1] who designed an algorithm that returns an oriented subgraph which improves the general upper bound to

$$f(d) \leq 1.373d^2 + 6.971d - 1.$$

This is an improvement for diameters 8 and bigger. They also showed that

$$f(4) \leq 21,$$

which is an improvement to their general upper bound and the upper bound provided by Theorem 1. The authors of that paper did not show either of those to be tight, instead they expressed their belief that these can be further improved. Using their algorithm, I will improve the upper bounds of $f(d)$ for $5 \leq d \leq 11$. I will also extend their approach for the general bounds for all diameters and improve the upper bounds to

$$f(d) \leq 1.28d^2 + 6.8d - 3.$$

2 Known Results

In this section, known results from the literature will be summarized, citing important theorems and lemmas. We will review and clarify some proofs, expand arguments that were very compact in the original paper and add illustrations. In Section 2.1, we will look into the results from the paper written by Chvátal and Thomassen [2]. Section 2.2 shortly summarizes the findings of the paper by Kwok, Liu and West dealing with the special case $d = 3$ [4]. In Section 2.3, we will take a look at the results of the paper by Babu, Benson, Rajendraprasad and Vaka that improved the upper bounds that Chvátal and Thomassen provided [1]. Finally, we will give an overview of all known results in Section 2.4.

2.1 First work by Chvátal and Thomassen

Chvátal and Thomassen were the first to provide bounds for $f(d)$ in a paper published in 1978. They showed that every 2-edge-connected graph of radius r has an orientation with radius at most $r^2 + r$ and this bound is tight. They continued to prove that $\frac{1}{2}d^2 + d \leq f(d) \leq 2d^2 + 2d$ which leaves a large gap between upper and lower bound [2].

2.1.1 Oriented Radius

Chvátal and Thomassen started by showing the following theorem which was used by other papers as well and will be used in this work too.

Theorem 2. *[2] Let $h(k) = (k-2) \cdot 2^{\lfloor (k-1)/2 \rfloor} + 2$. For every graph G there is an orientation D such that the following holds for every edge uv: If uv belongs to a cycle of length k in G, then there is a directed cycle in D including uv or vu with length at most $h(k)$.*

We will review the proof for Theorem 2 from [2] adding some intermediate steps to clarify parts that were very compact in the original. We start by constructing a series of graphs $H_3, H_4, ..., H_i$. Here H_3 is the maximum oriented subgraph of G with all edges being part of a directed triangle. To obtain H_i, take H_{i-1} and add the maximum set of vertices and edges from G such that every edge in H_i is in a cycle of length at most i. Then $D = \bigcup_{i=0}^{\infty} H_i$.
D is not necessarily an orientation of G. If G has bridges, then these bridges will never

get added to D. However, to prove the theorem, we do not need to direct the bridges. Let uv be an edge such that the smallest cycle in G that includes the edge uv is of length k. We need to show that uv or vu is in $H_{h(k)}$. This is equivalent to the following:

Lemma 3. *If neither uv nor vu belongs to H_n, then $n < h(k)$.*

Proof. Let the smallest cycle in G that contains uv be $u_1, u_2, ...u_k, u_1$ with $u_1 = u$ and $u_k = v$ and let H_n be a subgraph that does neither include uv nor vu. Let $3 \leq m \leq n$. Let H_m^* be the graph obtained from H_m be adding vu and all directed edges $u_j u_{j+1}$ unless $u_{j+1} u_j$ is already in H_m. The length of a directed cycle in H_m^* that includes vu is then $1 + \sum_{j=1}^{k-1} d_{H_m^*}(u_j, u_{j+1})$. The distance from u_j to u_{j+1} is 1 if the edge is directed from lower to higher subscript and for the edge vu. For all edges that were oriented from higher to lower subscript, we need to find an alternative path to connect the two incident vertices and close the cycle.
Let $u_{l+1} u_l$ be such an edge that is oriented against the direction of the directed cycle we are trying to find. Let i be an integer such that the edge $u_{l+1} u_l$ is in H_i but not in H_{i-1}. Then, a directed path from u_j to u_{j+1} with length exactly $i - 1$ exists, because $u_{l+1} u_l$ is in a directed cycle of length i (because it was added to H_i but is not in H_{i-1}). Let x_i (resp. y_i) be the number of such edges that were directed from higher to lower subscript (resp. from lower to higher subscript) and belong to H_i but not to H_{i-1}. In the undirected cycle $u_1, u_2, ...u_k, u_1$ there are $(k - \sum_{i=3}^{m} x_i)$ edges that are directed from u_j to u_{j+1} and $\sum_{i=3}^{m} x_i$ edges that were directed the other way, each with a distance from u_j to u_{j+1} of $i - 1$. Hence

$$1 + \sum_{j=1}^{k-1} d_{H_m^*}(u_j, u_{j+1}) = \left(k - \sum_{i=3}^{m} x_i\right) + \sum_{i=3}^{m} (i-1) x_i.$$

An integer m was chosen such that H_m does not include vu or uv. Hence the smallest directed cycle including uv or vu must be of length at least $m + 1$, otherwise it would be in H_m. Therefore

$$\begin{aligned} \left(k - \sum_{i=3}^{m} x_i\right) + \sum_{i=3}^{m} (i-1) x_i &\geq m + 1 \\ \sum_{i=3}^{m} (i-1) x_i - \sum_{i=3}^{m} x_i &\geq m + 1 - k \\ \sum_{i=3}^{m} (i-2) x_i &\geq m + 1 - k \quad \text{for all } 3 \leq m \leq n. \end{aligned} \tag{1}$$

Let $m(t) = (k-2) \cdot 2^{t-1} + 2$ for $t \in \mathbb{N}^+$ and $m(0) = 2$. We show that

$$m(t) \leq n \implies \sum_{i=3}^{m(t)} x_i \geq t. \tag{2}$$

Assume that the statement is incorrect, so $t \in \mathbb{N}_0$ exists such that $m(t) \leq n$ but $\sum_{i=3}^{m(t)} x_i < t$ so $\sum_{i=3}^{m(t)} x_i \leq t - 1$. If several such t exist, choose the smallest one, so

$$t - 1 \leq \sum_{i=3}^{m(t-1)} x_i \leq \sum_{i=3}^{m(t)} x_i \leq t - 1. \tag{3}$$

This implies equality, in particular $\sum_{i=3}^{m(t-1)} x_i = \sum_{i=3}^{m(t)} x_i$. Since x_i can not be negative, we have $x_i = 0$ for $m(t-i) < i \leq m(t)$. Therefore,

$$\begin{aligned}
\sum_{i=3}^{m(t)}(i-2)x_i &= \sum_{i=3}^{m(t-1)} (i-2)x_i \\
&= (3-2)x_3 + (4-2)x_4 + ... + (m(1)-2)x_{m(1)} + ... + (m(2)-3)x_{m(2)-1} \\
&\quad + (m(2)-2)x_{m(2)} + ... + (m(t-1)-3)x_{m(t-1)-1} + (m(t-1)-2)x_{m(t-1)} \\
&\leq (m(1)-2)x_3 + (m(1)-2)x_4 + ... + (m(1)-2)x_{m(1)} + ... + (m(2)-2)x_{m(2)-1} \\
&\quad + (m(2)-2)x_{m(2)} + ... + (m(t-1)-2)x_{m(t-1)-1} + (m(t-1)-2)x_{m(t-1)} \\
&= \sum_{s=1}^{t-1}(m(s)-2) \sum_{i=m(s-1)+1}^{m(s)} x_i \\
&= \sum_{s=1}^{t-1}(m(s)-2) \left(\sum_{i=m(s-1)+1}^{m(t-1)} x_i - \sum_{i=m(s)+1}^{m(t-1)} x_i \right) \\
&= \left(\sum_{s=1}^{t-1}(m(s)-2) \sum_{i=m(s-1)+1}^{m(t-1)} x_i \right) - \left(\sum_{s=1}^{t-1}(m(s)-2) \sum_{i=m(s)+1}^{m(t-1)} x_i \right) \\
&= \left(\sum_{s=1}^{t-1}(m(s)-2) \sum_{i=m(s-1)+1}^{m(t-1)} x_i \right) - \left(\sum_{s=2}^{t}(m(s-1)-2) \sum_{i=m(s-1)+1}^{m(t-1)} x_i \right) \\
&= \sum_{s=1}^{t-1} (m(s)-2) - (m(s-1)-2) \sum_{i=m(s-1)+1}^{m(t-1)} x_i \\
&= \sum_{s=1}^{t-1}(m(s)-m(s-1)) \sum_{i=m(s-1)+1}^{m(t-1)} x_i \\
&= \sum_{s=1}^{t-1}(m(s)-m(s-1)) \left(\sum_{i=3}^{m(t-1)} x_i - \sum_{j=3}^{m(s-1)} x_j \right) \\
&\leq \sum_{s=1}^{t-1}(m(s)-m(s-1))(t-s) \quad \text{(by (3))} \\
&= ((m(t-1)-m(t-2)) \cdot 1 + ((m(t-2)-m(t-3)) \cdot 2 + ... + ((m(1)-m(0)) \cdot (t-1) \\
&= \left(\sum_{s=1}^{t-1} m(s) \right) - (t-1)m(0) \\
&= (k-2) \left(\sum_{s=1}^{t-1} 2^{s-1} \right) + 2(t-1) - 2(t-1) \quad \left(\text{with } m(0) = 2 \text{ and } m(s) = \frac{k-2}{2} \cdot 2^s + 2 \right) \\
&= (k-2) \left(2^{t-1} - 1 \right) \\
&= (k-2) \cdot 2^{t-1} - k + 2 \\
&= m(t) - k,
\end{aligned}$$

which is a contradiction to (1).

Similarly, we can show that

$$m(t) \leq n \implies \sum_{i=3}^{m(t)} y_i \geq t. \tag{4}$$

Assume that Lemma 3 is incorrect, so $n \geq h(k)$. Note that $h(k) = m(\lfloor (k+1)/2 \rfloor)$. Hence

$$\begin{aligned} \sum_{i=3}^{n} x_i + \sum_{i=3}^{n} y_i &\geq \sum_{i=3}^{m(\lfloor (k+1)/2 \rfloor)} x_i + \sum_{i=3}^{m(\lfloor (k+1)/2 \rfloor)} y_i & (5) \\ &\geq 2\left\lfloor \frac{k+1}{2} \right\rfloor \quad \text{(by (2) and (4))} & (6) \\ &\geq k. & (7) \end{aligned}$$

This is a contradiction because the smallest undirected cycle including vu is of length k but neither vu nor uv are in H_n, hence $\sum_{i=3}^{n} x_i + \sum_{i=3}^{n} y_i \leq k-1$.

□

This concludes the proof of Lemma 3. With that we can show Theorem 2. Let uv be an edge in D such that the smallest undirected cycle containing uv is of length k. By Lemma 3 we know that $uv \notin H_n \wedge vu \notin H_n \implies n < h(k)$. This is equivalent to $n \geq h(k) \implies uv \in H_n \vee vu \in H_n$, in particular $n = h(k) \implies uv \in H_n \vee vu \in H_n$ which can be simplified to $uv \in H_{h(k)} \vee vu \in H_{h(k)}$. By the definition of H_i this means that vu or uv are included in a directed cycle of length at most $h(k)$. This proves Theorem 2.

Theorem 2 admits the following corollary:

Corollary 4. *[2] Let k be an integer such that every edge in a graph G belongs to a cycle of length at most k, then G admits an orientation H such that $d_H(u,v) \leq (h(k)-1)d_G(u,v)$.*

Proof. Let $w_0, w_1, ..., w_{d_G(u,v)}$ be the shortest path between $u = w_0$ and $v = w_{d_G(u,v)}$ in G. Each edge $w_l w_{l+1}$ is part of a directed cycle of length at most $h(k)$ so either the directed edge $w_l w_{l+1}$ exists or a path with length at most $h(k)-1$ exists. □

The authors of [2] continue to show the following theorem. We will review the proof by expanding the inductive argument including a base case and adding some figures to illustrate the approach.

Theorem 5. *[2] Let G be a 2-edge-connected graph with radius r. Then, an orientation H with radius $r(H) \leq r^2 + r$ exists.*

Proof. G has radius r so a vertex u exists such that $\forall v \in G : d_G(u,v) \leq r$. By induction on r we prove that G admits an orientation H with $d_H(u,v) \leq r^2 + r$ and $d_H(v,u) \leq r^2 + r$ for each vertex v.

Base Case. Let G be a 2-edge-connected graph of radius $r = 1$. Hence a vertex u exists that is adjacent to every other vertex of G. Now we will construct an orientation of radius at most $r^2 + r = 2$. Let $N(v)$ denote the set of vertices adjacent to a vertex v.

Start with any vertex $v \neq u$ and direct the edge vu towards v. Now direct all other edges incident to v away from v. At least one such edge exists because G is 2-edge-connected. Finally, direct the edges between all $w \in N(v) \setminus \{u\}$ and u towards u. Hence v and every neighbor of v is now part of a directed triangle including u, so $d(u, w) \leq 2$ and $d(w, u) \leq 2$ for all $w \in \{v, u\} \cup N(v)$.

Repeat this for every vertex $v \neq u$ unless the edge vu is already directed. It is important to note that for such vertices v, all vertices $w \in N(v) \setminus \{u\}$ have yet undirected edges incident to u or wu is already oriented towards u (if the edge wu was oriented towards w, then all neighbors would already be part of a directed triangle including u, so the edge vu would already be oriented).

When every edge incident to u was directed, every neighbor of u is part of a triangle that includes u. Since $N(u) \cup \{u\} = V(G)$, every vertex has a distance to and from u of at most 2. Hence the oriented radius of every graph G with radius 1 is at most 2. An example is shown in Figure 1.

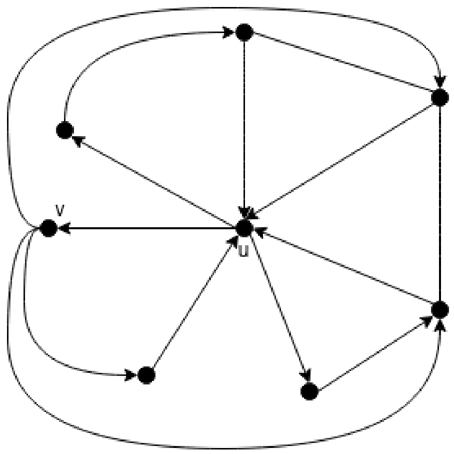

Figure 1: Orientation of a graph with radius 1. Here u is the center and vu is the first edge that gets oriented, after that we proceed clockwise. Note that some edges are not oriented yet. These can be oriented arbitrarily.

Induction hypothesis. For every 2-edge-connected graph G with radius $r-1$ and a vertex u that has distance at most $r-1$ to every other vertex in G, an orientation H with $d_H(u, v) \leq (r-1)^2 + (r-1) = r^2 - r$ and $d_H(v, u) \leq r^2 - r$ exists.

Inductive Step. For every v adjacent to u let $k(v)$ be the length of the shortest cycle including the edge uv. Such a cycle exists because G is 2-edge-connected. Consider the two vertices x, y that are part of the cycle and have the biggest distance to u. Their distance to u is at most r, so $k(v) \leq 2r + 1$.

Let A be an orientation of a subgraph of G, so $V(A) \subseteq V(G)$ and let S be a subset of the neighbors of u and C_v be a directed cycle including the edge vu or uv for every $v \in S$. A will be called admissible if

- each C_v has length $k(v)$ and
- A is the union of all C_v.

Due to the fact that $k(v) \leq 2r+1$ we know that for every vertex $w \in A$ we have $d_A(u,w) \leq 2r$ and $d_A(w,u) \leq 2r$, hence the following fact holds:

Fact 6. *[2] The radius of the directed graph A is at most $2r$.*

We also should note the following fact:

Fact 7. *[2] The largest admissible graph contains all neighbors of u.*

Proof. We assume that a vertex w exists that is adjacent to u but not in A. We know that a cycle $u, w, w_1, w_2, ..., w_{k(w)-2}, u$ exists in G. If none of the vertices $w_1, w_2, ..., w_{k(w)-2}$ are in A yet, we can easily orient all edges to form a directed cycle and add them to A. The orientation A would still be admissible but larger, a contradiction to the assumption that A is already the largest admissible graph.

When at least one vertex from $w_1, w_2, ..., w_{k(w)-2}$ is already in A, we need to consider two cases. Let w_i be the vertex with the lowest index that is already in A. There must be another vertex v that is adjacent to u and whose smallest cycle C_v contains w_i. Cycle C_v has the form $u \to v_1 \to v_2 \to \ldots \to v_{k(w)-1} \to u$ with either $v = v_1$ or $v = v_{k(w)-1}$. By symmetry, we might assume that $v = v_1$. Also $w_i = v_j$ for any $v_j \in C_v$.

Case 1. $w_{k(w)-2} = v$. In this case the edge $w_{k(w)-2}u$ is already oriented from u to $w_{k(w)-2}$. Hence we define C_w as following:

$$u \to v = w_{k(w)-2} \to v_2 \to \ldots \to v_j = w_i \to w_{i-1} \to \ldots \to w_1 \to w \to u$$

In order to show that the cycle C_w really has length $k(w)$, we only need to show that $v = w_{k(w)-2} \to v_2 \to \ldots \to v_j = w_i$ is the shortest path between $w_{k(w)-2}$ and w_i (by definition of w_l, $1 \leq l \leq i$ we know that the path $v_j = w_i \to w_{i-1} \to \ldots \to w_1 \to w \to u$ is optimal already).
Assume that a shorter path from $v = w_{k(w)-2}$ to $v_j = w_i$ exists. This shortcut would also shorten C_v, a contradiction to the fact that C_v is the smallest cycle including the edge vu.

Case 2. $w_{k(w)-2} \neq v$. Define C_w as following:

$$u \to w \to w_1 \to \ldots \to w_i = v_j \to v_{j+1} \to \ldots \to v_{k(w)-1} \to u$$

As in case 1, the cycle C_w has length $k(w)$.

We can see that in both cases, adding C_w to A creates a larger admissible graph which is a contradiction. Two examples are shown in Figure 2. □

Now, Chvátal and Thomassen create a new graph G^* that is obtained by taking G and contracting all vertices in A into a single vertex u^*. The idea of contracting vertices to create a new smaller graph is used frequently in papers dealing with this problem. By definition G had a radius r and the vertex u was chosen such that the distance between u and any other vertex was at most r. By Fact 7, the vertex u and all of its neighbors were contracted into u^*. Hence the distance from u^* to every other vertex in G^* is at most $r-1$, so G^* has radius $r-1$.

Fact 8. *By the induction hypothesis we know that an orientation H^* of G^* with $d_{H^*}(u^*, v) \leq r^2 - r$ and $d_{H^*}(v, u^*) \leq r^2 - r$ exists.*

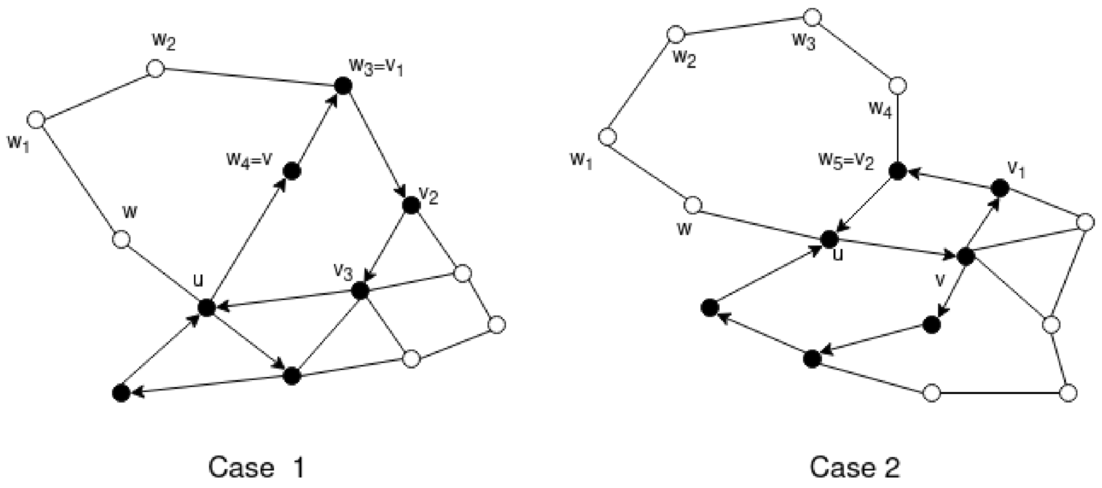

Figure 2: Vertices in A are black, vertices not in A are white.
Case 1: Adding the directed cycle $u \to w_4 \to w_3 \to w_2 \to w_1 \to w \to u$ to A creates a larger admissible graph.
Case 2: Adding the directed cycle $u \to w \to w_1 \to w_2 \to ... \to w_5 \to u$ to A creates a larger admissible graph.

Now, by combining the orientations of A and H^* and orienting the remaining edges arbitrarily, we obtain an orientation H of G. By Fact 8 we know that for every $v \in V(G)$, a vertex $w \in V(A)$ exists such that $d_{H^*}(w, v) \leq r^2 - r$ and $d_{H^*}(v, w) \leq r^2 - r$. By Fact 6 we also know that for this vertex w, $d_{H^*}(u, w) \leq 2r$ and $d_{H^*}(w, u) \leq 2r$. Hence we know that for every vertex $v \in V(G)$ we have

$$d_{H^*}(v, u) \leq r^2 - r + 2r = r^2 + r,$$

and

$$d_{H^*}(u, v) \leq r^2 - r + 2r = r^2 + r,$$

hence

$$r(H) \leq r^2 - r + 2r = r^2 + r.$$

□

This bound can be shown to be tight as stated in the following theorem. Chvátal and Thomassen constructed a series of graphs with oriented radius of $r^2 + r$, but they omitted the proof that no better orientation exists. We will look into their construction and show that no orientation with a radius lower than $r^2 + r$ exists.

Theorem 9. *[2] For every $r \in \mathbb{N}^+$ there is a graph G_r of radius r such that every orientation of G_r has radius $r^2 + r$.*

Proof. Construct a series of rooted graphs $H_1, H_2, \ldots$. Let H_1 be a triangle with any vertex as root. To obtain H_r, take two copies of H_{r-1} and a cycle $u_0, u_1, \ldots, u_{2r}, u_0$. Identify the roots of the copies of H_{r-1} with u_1 (resp. u_{2r}) and set u_0 as the root. Now, to obtain G_r, identify the roots of two copies of H_r. Figure 3 shows G_3 as an example.

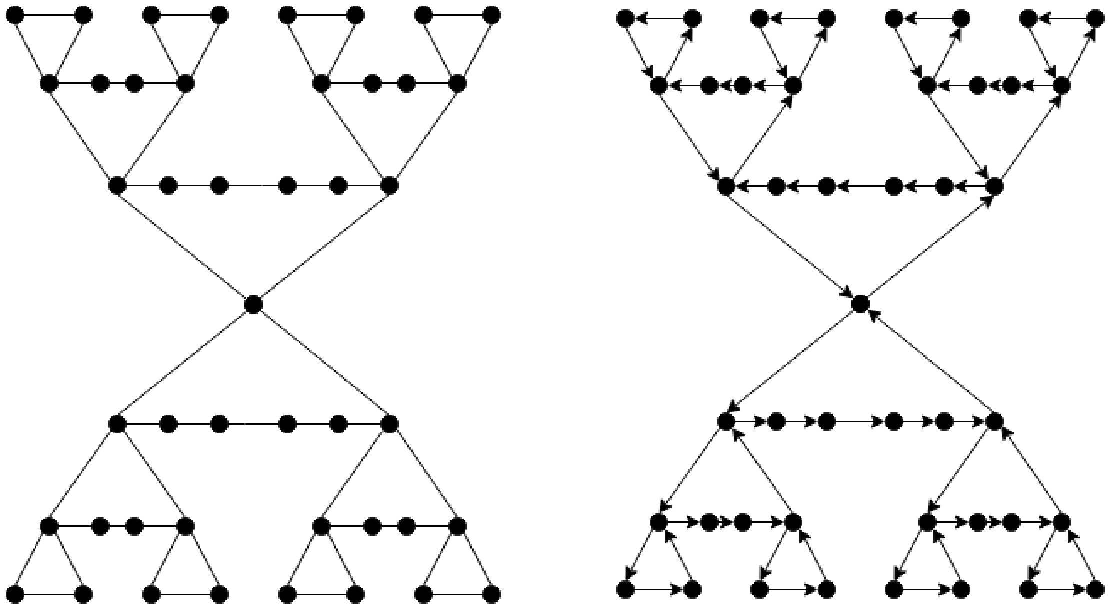

Figure 3: The graph G_3 (left) and an example orientation (right)

Let u be the root vertex of the copies of H_r. We need to show that a vertex $v \in V(G)$ exists such that for every orientation of G_r, $d(u,v) = r^2 + r$ or $d(v,u) = r^2 + r$.

We only need to consider one of the copies of H_r to find such a vertex (by symmetry the other copy has another such vertex).

Consider the vertices of all copies of H_1 that are not root and call them roots of copies of H_0 (which is a subgraph consisting of a single vertex, the root). Any path from u to a copy of H_0 must include the root of H_r (that is u), a root of one copy of H_{r-1}, a root of one copy of H_{r-2}, ... , a root of one copy of H_1 and finally the root of one copy of H_0.

The root u is adjacent to two root nodes p_{r-1}, q_{r-1} of subgraphs H_{r-1}. In order to obtain a strong orientation, one of those edges has to be directed towards u, the other one away from u. By symmetry we can assume that the edge incident to p_{r-1} and u is directed towards u. We still need to allow paths from u to p_{r-1}, hence the path between q_{r-1} and p_{r-1} needs to be directed towards p_{r-1}. This path has length $2r-1$, hence the shortest directed path from u to p_{r-1} has length $2r$. This argument can be repeated for the paths between p_{r-1} and the roots of the H_{i-2} graphs. This time the shortest path to one of the roots is of length $2(r-1)$. Hence to obtain a path from u (the root of H_r) to the copy of H_0 with the largest distance to u, we need to sum the distances between the roots of different levels. We obtain

$$d_{\overrightarrow{G_r}}(u,v) = 2r + 2(r-1) + 2(r-2) + \ldots + 2 \cdot 1 = 2\left(\frac{r^2+r}{2}\right) = r^2 + r.$$

□

2.1.2 Oriented Diameter

We know that for any graph G with diameter d and radius r, we have $d \leq 2r$. This also holds for oriented graphs. Using this fact together with Theorem 5, we get the first upper bound for $f(d)$:

Corollary 10. *[2] $f(d) \leq 2d^2 + 2d$.*

Consider again the graph G_r that was defined in the proof of Theorem 9. This graph has diameter $d = 2r$. Let u be the center vertex. As shown above, both copies of H_r have a vertex v, w with $d_{\vec{G_r}}(v,u) = d_{\vec{G_r}}(u,w) = r^2 + r$. It is obvious that any path from v to w must pass u, hence $d_{\vec{G_r}}(v,w) = 2(r^2+r) = 2r^2 + 2r$. With $d = 2r$ we get an oriented diameter of $2r^2 + 2r = 2\left(\frac{d}{2}\right)^2 + d = 2\frac{d^2}{4} + d = \frac{1}{2}d^2 + d$. Hence we get the following lemma:

Lemma 11. *[2] For every $r \in \mathbb{N}^+$, a 2-edge-connected graph of radius r exists such that every orientation of G has a diameter of at least $\frac{1}{2}d^2 + d$.*

This gives us a lower bound for the function $f(d)$:

Corollary 12. *[2] $f(d) \geq \frac{1}{2}d^2 + d$.*

Finally, Chvátal and Thomassen prove the following theorem. First, they show that every graph of diameter 2 admits an orientation with diameter at most 6 by partitioning and directing such a graph. The proof that their orientation has a diameter of at most 6 was not included. We will review their construction and show that indeed this is the case. They continue to show that the Petersen graph has an oriented diameter of exactly 6. We will review their proof and add illustrations for clarity.

Theorem 13. *[2] $f(2) = 6$.*

Proof. At first we prove that $f(2) \leq 6$, so every 2-edge-connected graph of diameter 2 admits an orientation of diameter at most 6. Let G be a 2-edge-connected graph of diameter 2. If every edge lies in a triangle, the maximum oriented diameter is 6, according to Corollary 4. Hence we only have to consider the case that G has at least one edge uv that is not contained in any triangle. Define $B = N(v) \setminus \{u\}$, $A = N(u) \setminus \{v\}$, $C = V(G) \setminus \{A \cup B \cup \{u,v\}\}$. Partition A into A_1 and A_2 where A_1 includes all vertices in A that have no neighbor in B and A_2 contains the other vertices in A. Define B_1, B_2 similarly. Consider the orientations as shown in Figure 4.

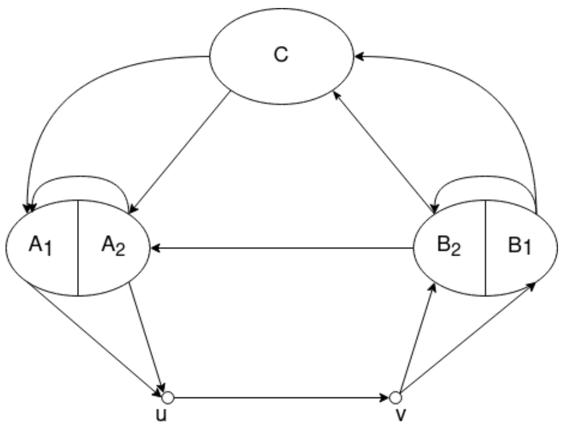

Figure 4: Orientations for a graph of diameter 2

Now we need to show that $d_{\vec{G}}(x,y) \leq 6$ for all $x, y \in V(G)$. Many distances are easy to see:

- $d_{\vec{G}}(u,v) = 1$ and $d_{\vec{G}}(v,u) \leq 4$
- $d_{\vec{G}}(A_1 \cup A_2 \cup C \cup B_2, u) \leq 2$ (the distance for $c \in C$ follows from $d = 2$, hence a path of length 2 between c and u must exist. In this orientation, that path is directed towards u)
- $d_{\vec{G}}(v, B_1 \cup B_2 \cup C \cup A_2) \leq 2$ (the distance for $c \in C$ can be explained as above)

Hence, by forming a path via u and v we get

$$d_{\vec{G}}(A_1 \cup A_2 \cup C \cup B_2 \cup \{v\}, B_1 \cup B_2 \cup C \cup A_2 \cup \{u\}) \leq 2 + 1 + 2 = 5$$

Now the only distances that we have not bounded yet are $d_{\vec{G}}(x,y)$ for $x \in B_1$ or $y \in A_1$. Let $x \in B_1$. Since $d = 2$, we know that a path from x to every $a \in A$ exists. In our orientation that path is directed towards a, hence $d_{\vec{G}}(x, A) \leq 2$. Hence $d_{\vec{G}}(x,u) \leq 3$, $d_{\vec{G}}(x,v) \leq 4$, $d_{\vec{G}}(x, B_1 \cup B_2) \leq 5$ and $d_{\vec{G}}(x, C) \leq 6$. A similar case can be made for $d_{\vec{G}}(p,q)$ where $q \in A_1$.

It was shown that for all vertices x, y we have $d_{\vec{G}}(x,y) \leq 6$ and $d_{\vec{G}}(y,x) \leq 6$, hence the oriented diameter of G is at most 6.

The fact that this upper bound is tight was shown using the Petersen graph as an example. The authors of [2] first showed the following lemma:

Lemma 14. *[2] Every strongly connected orientation admitted by the Petersen graph contains a directed cycle that has a length of 5.*

Using this lemma, we show that all orientations of the Petersen graph have diameter 6. Assume that an orientation H exists, such that $d(H) \leq 5$. Without loss of generality we may assume that the directed cycle of length 5 is as shown in Figure 5.

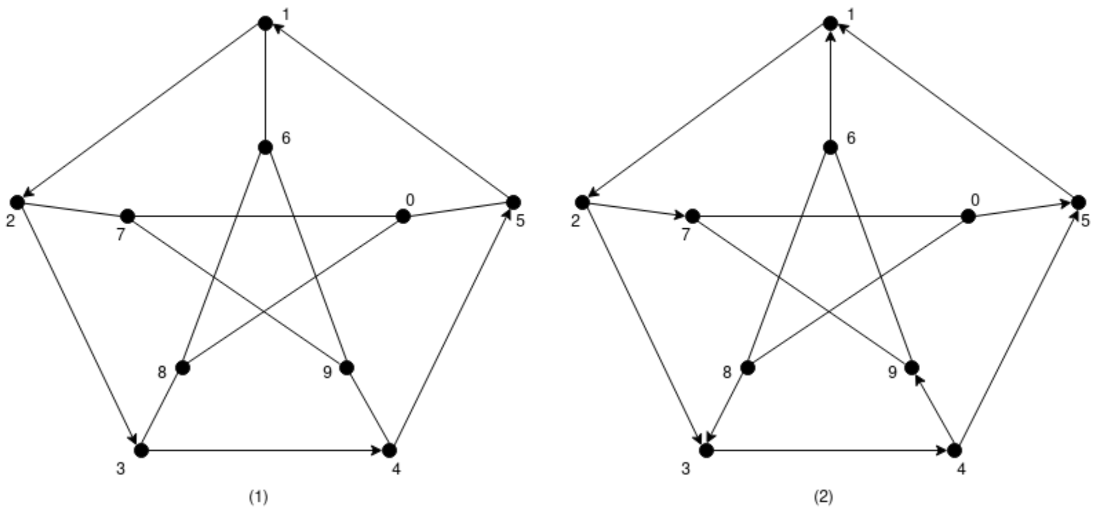

Figure 5: Every Petersen graph with a directed cycle of length 5 is equivalent to (1). With the cross edges oriented such that no 3 consecutive edges have the same direction (2), we can see that the oriented diameter is 6.

Now consider the cross edges 16, 27, 38, 49 and 50. We can observe two things: First, at least one edge of those must be directed towards the pentagon and at least one edge must be directed away from the pentagon. Second, there must be two consecutive cross edges that are directed in the same direction. We may assume that those two edges are $0 \to 5$ and $6 \to 1$. Now consider the edge 27. If we direct it from 7 to 2, then $d(5,7) \leq 6$, a contradiction. We can find a similar example for every other orientation with 3 cross edges. Hence we can assume that no 3 consecutive cross edges oriented in the same direction exist. There is only one way to direct the other cross edges then. That is $2 \to 7$, $8 \to 3$ and $4 \to 9$. This is shown in Figure 5. We want to find an orientation with diameter 5, hence we need $d_H(3,6) \leq 5$. This forces $9 \to 6$. Similarly, to get $d_H(5,0) \leq 5$, we need $7 \to 0$. To get $d_H(9,7) \leq 5$, we need $9 \to 7$. This forces $0 \to 8$, so $d_H(7,9) \leq 5$. Now we can see that $d_H(5,6) = 6$, a contradiction. The edge 56 may by oriented arbitrarily and we obtain an orientation of diameter 6. □

2.2 First Improvement for diameter $d = 3$

In a paper published in 2010, Kwok, Liu and West improved the bounds for graphs with diameter 3 to $9 \leq f(3) \leq 11$ [4]. This is a significant improvement over $8 \leq f(3) \leq 24$ given by Chvátal and Thomassen's inequality for $d = 3$. We will shortly summarize their approach.

2.2.1 Lower Bounds

Since $f(d)$ bounds the maximum of all oriented diameters for graphs with diameter d, it is sufficient to show an example of a graph with diameter 3 and oriented diameter 9 or more to proof that $f(3) \geq 9$. Kwok, Liu and West provide such a graph and prove that all strong orientations have a diameter of 9 or more [4].

We start with $G = K_4$ with $V(G) = \{w, x, y, z\}$ and replace all edges incident to w with paths of length 3. The resulting graph has diameter 3.

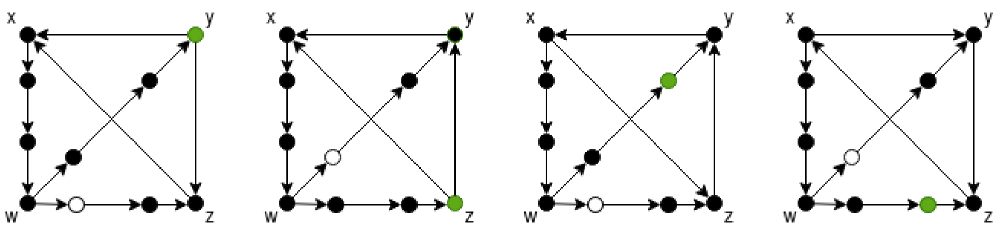

Figure 6: Possible orientation of the remaining three edges.
Not included are those that include a sink or a source

To obtain a strong orientation we have to direct all edges without creating a source or a sink, otherwise the source cannot be reached from any other point and the sink cannot reach any other point. We can reverse any strong orientation without changing its diameter, thus we can assume that w has an indegree of 2 and outdegree 1 without loss of generality. This only leaves 3 edges undirected, hence 8 possibilities. By removing those orientations that include a sink or a source, only the four possibilities shown in Figure 6 remain. In all of them we can see that a path from the white to the green vertex has length 9.

2.2.2 Upper Bounds

We will quickly summarize the steps the authors of [4] took to show the upper bound $f(3) \leq 11$ without going into detail.

At first we observe that all graphs of diameter 3 in which every edge is part of a triangle admit an orientation of diameter 9 by Corollary 4. That is lower then the upper bound we want to show, so we assume that G is a graph of diameter 3 with an edge uv that is not part of a triangle.

We partition the vertices of G into disjoint sets $S_{j,k} = \{w \in V(G) : d_G(w,u) = j \wedge d_G(w,v) = k\}$ and define $T_i(u)$ (resp. $T_i(v)$) to be the set of vertices with a distance i to u (resp. v).

Since the diameter of G is 3, we know that all sets $S_{j,k}$ with $j > 3$ or $k > 3$ are empty and since uv is not part of any triangle, $S_{1,1} = \emptyset$. So we have

$$V(G) = \{u,v\} \cup S_{1,2} \cup S_{2,1} \cup S_{2,2} \cup S_{2,3} \cup S_{3,2} \cup S_{3,3}$$

Let $N(S)$ be the set of all neighbor vertices of S, so all vertices that are incident to at least one vertex in S. We define the following disjoint sets of vertices:

$A = S_{1,2} \cap N(T_2(u))$	$A^* = S_{1,2} - A$
$B = S_{2,1} \cap N(T_2(v))$	$B^* = S_{2,1} - B$
$I = S_{2,3} \cap N(T_3(u) \cup S_{2,2})$	$A' = S_{2,3} - I$
$I = S_{3,2} \cap N(T_3(v) \cup S_{2,2})$	$B' = S_{3,2} - J$
$X = S_{3,3} \cap (N(I) - N(J))$	$Y = S_{3,3} \cap (N(J) - N(I))$
$Z = S_{3,3} \cap N(I) \cap N(J)$	$C = S_{3,3} - (X \cup Y \cup Z)$

We then oriented some edges following the scheme in Figure 7.

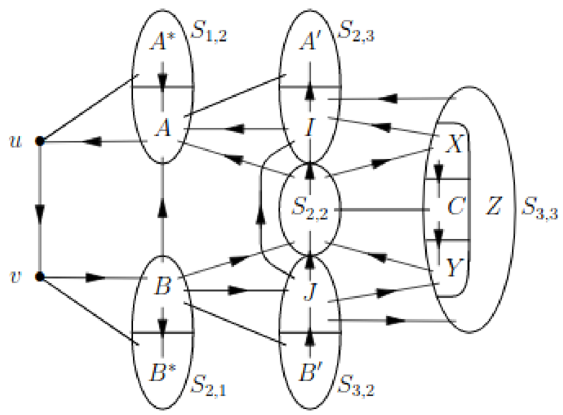

Figure 7: Partition of a graph with diameter 3 including some orientations [4].

An oriented edge from set N to set M means that all edges between $n \in N$ and $m \in M$ are oriented towards m.

From this schematic orientation the following upper bounds for distances to u and v arise:

for w in	A	B	I	J	$S_{2,2}$	X	Y	Z	C	A′	B′	A^*	B^*
$d_D(w,u) \leq$	1		2		2	3	3	3					
$d_D(v,w) \leq$		1		2	2	3	3	3					

The authors of [4] continue to prove the following lemma:

Lemma 15. *[4] In a graph H, let S and S' be disjoint vertex sets such that $S' \subseteq N_H(S)$, so all $u \in S'$ are incident to at least one $v \in S$. Let $[S,S']$ be the set of edges uv with $u \in S$ and $v \in S'$. If the induced subgraph $H[S']$ is connected and nontrivial, then there is an orientation F of $H[S'] \cup [S,S']$ such that $\forall w \in S' : d_F(S,w) \leq 2 \wedge d_F(w,S) \leq 2$.*

By considering some special cases, they obtain the following:

Lemma 16. *[4] If A' and B' are not empty, then:*

for w in	A	B	I	J	$S_{2,2}$	X	Y	Z	C	A'	B'	A^*	B^*
$d_D(w,u) \leq$	1	4	2	4	2	3	3	3	4	3	6	2	7
$d_D(v,w) \leq$	4	1	4	2	2	3	3	3	4	6	3	7	2

They now show that a path between every $x \in B^* \cup A^*$ and every vertex y of length 11 exists. Next they showed that the same is true for $x \in B'$ and $y \in A'$. For all other pairs of vertices we can use the triangle inequality to bound their distance: $d_D(x,y) \leq d_D(x,u) + 1 + d_D(v,y) \leq 11$ With this they have shown the following lemma:

Lemma 17. *If A' and B' are not empty, we get an oriented diameter of 11 at most.*

Kwok, Liu and West then continue to show the same for the case that $A' = \emptyset$ or $B' = \emptyset$. Hence they showed the following theorem:

Theorem 18. *[4]* $9 \leq f(3) \leq 11$.

2.3 Recent improvement for diameter $d \geq 8$ and $d = 4$

In their paper written in 2020 the authors Babu, Benson, Rajendraprasad and Vaka [1] describe an algorithm that uses a similar approach to Kwok, Liu and West in their study of graphs of diameter 3 [4]. Their algorithm will be used again in Section 3, so the algorithm itself will be explained. We will also summarize the results of [1] and review some proofs.

2.3.1 Algorithm OrientedCore

The algorithm designed by [1] takes as input a 2-edge-connected graph G and a specified edge $pq \in E(G)$ in the graph. It returns a 2-edge-connected oriented subgraph $\overrightarrow{H}$ of G. Similar to the approach by Kwok, Liu and West [4], we partition the vertices into sets $S_{i,j}$ which contain all vertices that have distance i to the vertex p and distance j to the vertex q in G. It is important to notice that $|i-j| \leq 1$, because if v has distance i to vertex p, a path from v to q via p with length $i+1$ exists and vice versa.

Let $L(v) = j - i$ be the *level* of vertex v. Again this can only be 1, 0 or -1. Let $W(v) = max(i, j)$ be the *width* of vertex v. An edge uv is called horizontal, if $L(v) = L(u)$ and vertical otherwise.

A few observations need to be noted before starting the algorithm:

Remark 19. [1] For every vertex $v \in S_{i,i+1}$, the shortest path from v to p consists of horizontal edges between Level 1 vertices only. Similarly, the shortest path from every $v \in S_{i+1,i}$ to q consists of horizontal edges between Level -1 vertices only and the shortest path from p to every $v \in S_{i,i}$ consists of horizontal edges in Level 1 and Level 0 and exactly one vertical edge incident to a Level 1 and a Level 0 vertex.

Proof. Consider the shortest path from a Level 1 vertex $v \in S_{i,i+1}$ to p. Its length is i, so after following the first edge, we arrive at a vertex $v_1 \in S_{i-1,k}$. Since we know that $|i - j| \leq 1$, we know that $k \in \{i - 2, i - 1, i\}$. If $k = i - 2$, so $d(v_1, q) = i - 2$ then a path from v to q via v_1 with length $d(v_1, q) + 1 = i - 2 + 1 = i - 1$ existed, a contradiction to the fact that $v \in S_{i,i+1}$ and hence by definition $d(v, q) = i + 1$. A similar case can be made for $k = i - 1$. Hence we know that $k = i$, so $v_1 \in S_{i-1,i}$ is again in Level 1, hence the edge is horizontal. We can use the same argument again for v_1 and every other vertex on the shortest vp path to prove the first case of the remark. All other cases can be explained similarly. □

Stage 1: For every edge uv with $L(u) = 1$ and $L(v) \neq 1$, do the following: Direct the shortest $p - u$ path from p to u. Direct the edge uv from u to v. Direct the shortest $v - q$ path from v to q. Hence a directed path from p to q containing the edge uv is formed. Add this path to $\overrightarrow{H}$.

Stage 2: For each edge uv with $L(u) = 0$ and $L(v) = -1$ that was not already oriented, do the following: Let P_u and P_v be the shortest $p - u$ and $q - v$ paths. Some of the vertices on P_u and P_v are already in $\overrightarrow{H}$. Let x (resp. y) be the closest vertex to u (resp. v) that is already in $\overrightarrow{H}$. Add the paths between x and u, y and v and the edge uv to $\overrightarrow{H}$ and orient them to form a directed path from x to y including the edge uv.

Stage 3: Orient the edge pq from q to p and add to $\overrightarrow{H}$.

An example of the algorithm is shown in Figure 8.

2.3.2 Distance Analysis within $\overrightarrow{H}$

Similarly to [4], the argument in [1] continues by analyzing the distance of every vertex to the specified vertex q and from the vertex p to every vertex. We will review their arguments and clarify some parts that were very compact in the original paper. After bounding the distances from every vertex to q and from p to every vertex, we can give upper bounds for the distances between any pair of vertices u, v by using the triangle inequality again:

$$d_{\overrightarrow{H}}(u, v) \leq d_{\overrightarrow{H}}(u, q) + d_{\overrightarrow{H}}(p, v) + 1,$$

and

$$d_{\overrightarrow{H}}(v,u) \leq d_{\overrightarrow{H}}(v,q) + d_{\overrightarrow{H}}(p,u) + 1.$$

Hence the maximum distance between every pair of vertices u, v, which is equal to the oriented diameter of $\overrightarrow{H}$ can be bounded from above by finding the maximum distance $d_{\overrightarrow{H}}(v,q)$ of all $v \in V(\overrightarrow{H})$ and the maximum $d_{\overrightarrow{H}}(p,u)$ of all $u \in V(\overrightarrow{H})$.

At first, consider all vertices w that were oriented in Stage 1 of the algorithm *OrientedCore*. Note that w is part of a directed path from p to q of length at most $2d$.

Case 1. $w \in S_{i,i+1}$. Then w is a Level 1 vertex. Hence the shortest path from p to w in G is directed towards w. That path has length i, hence $d_{\overrightarrow{H}}(p,w) = i$. Since the directed path from p to q via w has distance at most $2d$, we know that $d_{\overrightarrow{H}}(w,q) \leq 2d - i$.

Case 2. $w \in S_{h,h}$. In this case, w is part of a smallest cycle containing pq. Hence w has a neighbor in Level 1, so in Stage 1 the shortest path between p and w is directed towards w, thus $d_{\overrightarrow{H}}(p,w) = h$. In the same step, the shortest path from w to q is directed towards q, hence $d_{\overrightarrow{H}}(w,q) = h$.

Case 3. $w \in S_{i,i}$ for $i > h$. The directed path from p to q via w has length at most $2d$ and the shortest path between p and w and w and q are both of length at least i, hence $d_{\overrightarrow{H}}(p,w) \leq 2d - i$ and $d_{\overrightarrow{H}}(w,q) \leq 2d - i$.

Case 4. $w \in S_{i+1,i}$. Then w is a Level -1 vertex. All Level -1 vertices that were added to $\overrightarrow{H}$ in Stage 1 were added as part of a shortest path from any Level 1 vertex v to q (by Remark 19 w cannot be part of the shortest $p - v$ path). Hence $d_{\overrightarrow{H}}(w,q) = i$ and thus $d_{\overrightarrow{H}}(p,w) \leq 2d - i$.

Hence we get the following:

Lemma 20. *[1] For every vertex w that was added to $\overrightarrow{H}$ in Stage 1, we obtain the following upper bounds on the distances from p and to q:*

$$d_{\overrightarrow{H}}(p,w) \leq \begin{cases} i & w \in S_{i,i+1} \\ h & w \in S_{h,h} \\ 2d-i & w \in S_{i,i},\ i > h \\ 2d-i & w \in S_{i+1,i} \end{cases} \qquad d_{\overrightarrow{H}}(w,q) \leq \begin{cases} 2d-i & w \in S_{i,i+1} \\ h & w \in S_{h,h} \\ 2d-i & w \in S_{i,i},\ i > h \\ i & w \in S_{i+1,i} \end{cases}$$

Before analyzing the distance from p and to q for vertices added in Stage 2, we should note the following:

Fact 21. *All Level 1 vertices are either added to $\overrightarrow{H}$ in Stage 1 or they are not added to $\overrightarrow{H}$ at all.*

Proof. Assume that a Level 1 vertex x is added to $\overrightarrow{H}$ in Stage 2. This means that x is on the shortest path from p to an edge uv with $L(u) = 0$ and $L(v) = -1$. By Remark 19 we know that the shortest path from p to the Level 0 vertex u must consist of horizontal edges in Level 1 and Level 0 and exactly one vertical edge incident to a Level 1 and a Level 0 vertex. This vertical edge must have been oriented in Stage 1 already. Thus, x must have been added to $\overrightarrow{H}$ in Stage 1 already, a contradiction. □

Fact 22. *All vertices in $S_{h,h}$ are either added to $\overrightarrow{H}$ in Stage 1 or they are not added to $\overrightarrow{H}$ at all because each such vertex is adjacent to a Level 1 vertex.*

Proof. Assume that a vertex $v \in S_{h,h}$ is not adjacent to any Level 1 vertex. Let w be the vertex adjacent to v that is part of the shortest $p - v$ path (of length h). Vertex w must have a distance of $h - 1$ to p but cannot be in Level 1. Hence $w \in S_{h-1,h-2}$ or $w \in S_{h-1,h-1}$. In the first case, a shortest path from v to q via w would exist with length $h - 1$, a contradiction to the assumption that $v \in S_{h,h}$. In the second case, a cycle including the edge pq and the vertex w of length $2(h - 1) + 1 = 2h - 1$ would exist, a contradiction to the definition of $h = \lfloor k/2 \rfloor$ where k is the length of the smallest cycle including the edge pq. □

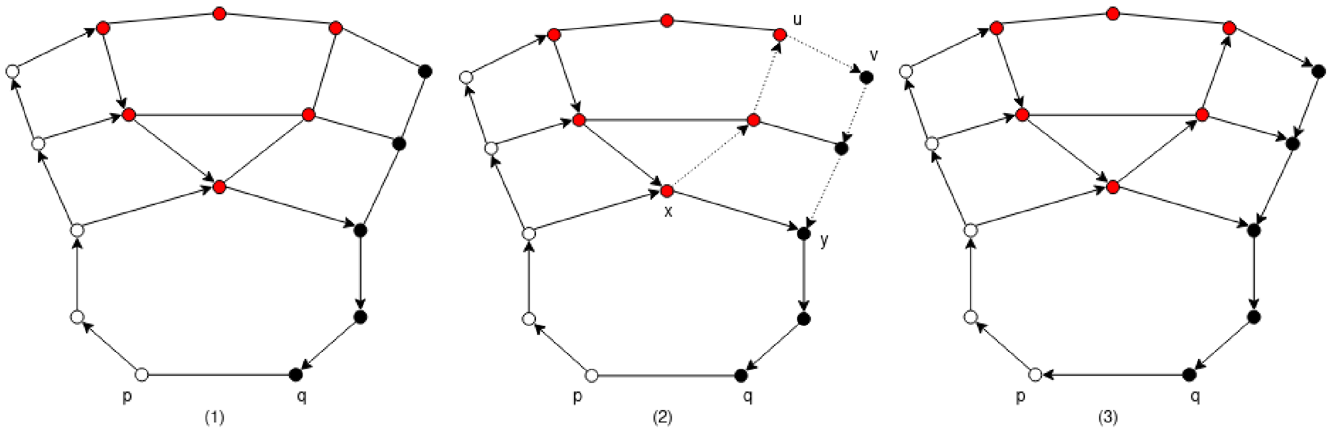

Figure 8: Example of the algorithm OrientedCore: orientations after Stage 1 (1), directing the edge uv in Stage 2 (2) and orientations after the algorithm finished (3). Notice that the algorithm returns a directed subgraph, so some edges are not directed yet.

Hence, for analyzing the distances of vertices added to $\overrightarrow{H}$ in Stage 2, we only have to consider vertices in $S_{i,i}$ with $i > h$ and vertices in $S_{i+1,i}$. The distance from w to q remains the same as in Stage 1 because for every Level -1 vertex the shortest path to q is oriented towards q. The only Level 0 vertices added have an outgoing edge incident to a Level -1 vertex by definition of the algorithm. Hence we know that

$$d_{\overrightarrow{H}}(w,q) \leq \begin{cases} 2d - i & w \in S_{i,i} \text{ for } i > h \\ i & w \in S_{i+1,i} \end{cases}$$

The distances from p to w are a little bit more complicated. Remember that in Stage 2 we formed the path $p - w$ by choosing $x \in V(\overrightarrow{H})$ such that x is the closest vertex to w that is on the shortest $p - w$ path in G as shown in Figure 8. By Remark 19 the path $p - w$ consists of horizontal edges in Level 1, one edge incident to a Level

1 and a Level 0 vertex and then only horizontal edges in Level 0. The vertical edge must already be added to $V(\overrightarrow{H})$ in Stage 1, hence the last vertex on the path from p to w that is already in $V(\overrightarrow{H})$ must be in Level 0.

Case 1. If x is in $S_{h,h}$ then $d_{\overrightarrow{H}}(p,x) \leq h$. It is important to note that in this case, $h < d$, otherwise the undirected path pw would be of length at least $d+1$, a contradiction. Hence $d_{\overrightarrow{H}}(p,x) \leq d-1$ if $x \in S_{h,h}$.

Case i. $w \in S_{i,i}$ for $i > h$. We have $d_{\overrightarrow{H}}(p,x) \leq d-1$ and we know that the shortest path between x and w is directed from x to w, so $d_{\overrightarrow{H}}(x,w) \leq i-h$, hence $d_{\overrightarrow{H}}(p,w) \leq d-1+i-h$.

Case ii. $w \in S_{i+1,i}$. Vertex w is a Level -1 vertex. Hence it is either adjacent to a Level 0 vertex $v \in S_{l,l}$ or it lies on a shortest path from a Level 0 vertex v to q. From Case i we have $d_{\overrightarrow{H}}(p,v) \leq d-1+l-h$ and with $l \leq d$ we get $d_{\overrightarrow{H}}(p,v) \leq 2d-h-1$. The shortest path from v to q is directed towards q and of length d or less. The subpath from w to q is of length i, hence $d_{\overrightarrow{H}}(v,w) \leq d-i$. Therefore, $d_{\overrightarrow{H}}(p,w) \leq 2d-h-1+d-i = 3d-h-i-1$.

Case 2. If $x \in S_{l,l}$ for $l > h$, then $d_{\overrightarrow{H}}(p,x) \leq 2d-l \leq 2d-h-1$ (by Lemma 20).

Case i. $w \in S_{i,i}$ for $i > h$. Here we have $d_{\overrightarrow{H}}(x,u) \leq i-l$ with $l \geq h+1$ so we obtain $d_{\overrightarrow{H}}(x,u) \leq i-h-1$. Therefore $d_{\overrightarrow{H}}(p,w) \leq 2d-h-1+i-h-1 = 2d-2h-2+i$.

Case ii. $w \in S_{i+1,i}$. Again w must lie on a shortest path from a Level 0 vertex $v \in S_{k,k}$ with $k \leq d$ to q. Using the distance from Case i we obtain $d_{\overrightarrow{H}}(p,v) \leq 2d-2h-2+k \leq 2d-2h-2+d = 3d-2h-2$. As above, the path from v to w is of length $d-i$ or less. Hence $d_{\overrightarrow{H}}(p,w) \leq 3d-2h-2+d-i = 4d-2h-2-i$.

We can see that $x \in S_{l,l}$ for $l > h$ always produces the worse upper bounds, so for all vertices added in Stage 2 we get:

$$d_{\overrightarrow{H}}(p,w) \leq \begin{cases} 2d-2h-2+i & w \in S_{i,i}, i > h \\ 4d-2h-2-i & w \in S_{i+1,i} \end{cases}$$

$$d_{\overrightarrow{H}}(w,q) \leq \begin{cases} 2d-i & w \in S_{i,i}, i > h \\ i & w \in S_{i+1,i} \end{cases}$$

Now we can take the worst case among vertices added in Stage 1 and Stage 2 to obtain the following:

Lemma 23. *[1] Let $\overrightarrow{H}$ be the oriented subgraph returned by the algorithm* OrientedCore, *then for every $w \in V(\overrightarrow{H})$*

$$d_{\overrightarrow{H}}(p,w) \leq \begin{cases} i & w \in S_{i,i+1} \\ h & w \in S_{h,h} \\ 2d-2h-2+i & w \in S_{i,i}, i > h \\ 4d-2h-2-i & w \in S_{i+1,i} \end{cases}$$

$$d_{\overrightarrow{H}}(w,q) \leq \begin{cases} 2d-i & w \in S_{i,i+1} \\ h & w \in S_{h,h} \\ 2d-i & w \in S_{i,i}, i > h \\ i & w \in S_{i+1,i} \end{cases}$$

Now let u and v be two vertices in $\overrightarrow{H}$. To get the distance from u to v we can use the triangle inequality:

$$d_{\overrightarrow{H}}(u,v) \leq d_{\overrightarrow{H}}(u,q) + d_{\overrightarrow{H}}(q,p) + d_{\overrightarrow{H}}(p,v).$$

The worst case for $d_{\overrightarrow{H}}(u,q)$ is when $u \in S_{1,2}$, then we have $d_{\overrightarrow{H}}(u,q) \leq 2d-1$. The worst case for $d_{\overrightarrow{H}}(p,v)$ is when $v \in S_{2,1}$, then we have $d_{\overrightarrow{H}}(u,q) \leq 4d-2h-3$.

Additionally we have $d_{\overrightarrow{H}}(q,p) = 1$ (oriented in Stage 3).

By adding these three distances, we obtain the following:

Lemma 24. *[1] Let $\overrightarrow{H}$ be the oriented subgraph returned by the algorithm OrientedCore, then $d(\overrightarrow{H}) \leq 6d-2h-3$, where k is the smallest cycle containing the edge pq and $h = \lfloor k/2 \rfloor$.*

Note that $\overrightarrow{H}$ is a subgraph with $V(\overrightarrow{H}) \subseteq V(G)$, hence $d(\overrightarrow{H})$ is in general not equal to the oriented diameter of G.

2.3.3 Analysis of Domination of $\overrightarrow{H}$

The authors of [1] continue to analyze the distances from every vertex not in $\overrightarrow{H}$ to the closest vertex in $\overrightarrow{H}$. We will revise their proof and add some intermediate steps to clarify their arguments.

Let L_i^c be the vertices of Level i that were captured in $\overrightarrow{H}$ and L_i^u those that are not captured. Since all vertical edges were captured, L_i^u must be separated from the rest of G by the set L_i^c. Let d_i be the maximum distance of a vertex in L_i^u to the next vertex in L_i^c. For $j \neq i$ there must be vertices $x \in L_j^u$ and $y \in L_i^u$ that each have the maximum distance (d_j, resp. d_i) to the next captured vertex, which is of Level j, resp. i. The distance between the closest captured vertices for x and for y must be at least 1, hence the distance between x and y can be bounded below by $d_i + 1 + d_j$. The upper bound is easier. The two vertices can not have a distance bigger than d. Hence we know that $d_i + 1 + d_j \leq d$ so $d_i + d_j \leq d - 1$ for $i \neq j$.

Now consider an uncaptured vertex w of Level 0 and its shortest path to q which is of length at most d. By Remark 19 this path must consist of horizontal edges in Level 0 and Level -1 and exactly one vertical edge. The vertical edge is captured into $\overrightarrow{H}$ and the Level 0 vertex incident to the vertical edge has a distance to q of at least h, hence the distance from that vertex to w is at most $d-h$, so $d_0 \leq d - h = d - \lfloor k/2 \rfloor$.

To bound d_1 and d_{-1} consider a cycle including the edge pq. This has length k by definition of k. Let this cycle be $v_1, v_2, \ldots, v_k, v_1$ with $v_1 = q$ and $v_k = p$. Now we have

$$v_i \in \begin{cases} S_{i,i-1} & 2i < k+1 \\ S_{i-1,i-1} & 2i = k+1 \\ S_{k-i,k-i+1} & 2i > k+1 \end{cases}$$

Let $t = \lceil k/4 \rceil$. The Level -1 vertex v_t has a distance of t to the next captured vertex in Level 1. Hence the distance from v_t to a vertex in Level 1 that has distance d_1 to the next captured vertex is bounded below by $d_1 + t$ and above by d. Hence $d_1 \leq d - t$ and similarly $d_{-1} \leq d - t$ with $t = \lceil k/4 \rceil$.

We can see that the largest distance from any uncaptured vertex to the next captured vertex is $d - \lceil k/4 \rceil$, hence the following lemma:

Lemma 25. *[1] Let $\overrightarrow{H}$ be the oriented subgraph returned by the algorithm* OrientedCore*, then $\overrightarrow{H}$ is a $\left(d - \left\lceil \frac{k}{4} \right\rceil\right)$-step dominating subgraph of G.*

2.3.4 Oriented Diameter of G

Now construct a graph G_0 by collapsing all vertices in $\overrightarrow{H}$ into a single vertex v_0. By Lemma 25, the graph G_0 has a radius of at most $\left(d - \left\lceil \frac{k}{4} \right\rceil\right)$. By Theorem 5 we know that $\overrightarrow{r}(G_0) \leq \left(d - \left\lceil \frac{k}{4} \right\rceil\right)^2 + \left(d - \left\lceil \frac{k}{4} \right\rceil\right)$. With $d \leq 2r$ we can get an upper bound for the oriented diameter of G_0, that is $\overrightarrow{d}(G_0) \leq 2\left(d - \left\lceil \frac{k}{4} \right\rceil\right)^2 + 2\left(d - \left\lceil \frac{k}{4} \right\rceil\right)$. By combining the orientations of $\overrightarrow{H}$ and $\overrightarrow{G_0}$, we obtain a full orientation $\overrightarrow{G}$ of G. Note the following:

Fact 26. *To get an upper bound on the diameter of the orientation, we only have to sum the upper bounds for the diameters of $\overrightarrow{H}$ and G_0.*

Proof. Consider $u \in \overrightarrow{H}$ and $v \in G_0 \setminus v_0$. We only have to show that $d_{\overrightarrow{G}}(u,v) \leq d(\overrightarrow{H}) + d(G_0)$ and $d_{\overrightarrow{G}}(v,u) \leq d(\overrightarrow{H}) + d(G_0)$. The distance from v to v_0 in G_0 is at most $d(G_0)$. By our construction of G_0 this means that a vertex $w \in \overrightarrow{H}$ exists such that $d_{\overrightarrow{G}}(v,w) \leq d(G_0)$ and $d_{\overrightarrow{G}}(w,u) \leq d(\overrightarrow{H})$. Hence $d_{\overrightarrow{G}}(v,u) \leq d(G_0) + d(\overrightarrow{H})$. Similarly we can show that $d_{\overrightarrow{G}}(u,v) \leq d(G_0) + d(\overrightarrow{H})$. □

Hence we get the following:

Theorem 27. *Let G be a 2-edge-connected graph of diameter d that has at least one edge not part of any triangle. Let k be the smallest integer such that every edge in G is contained in a cycle of length k or less. Then $\overrightarrow{d}(G) \leq 6d - 2\lfloor k/2 \rfloor - 3 + 2\left(d - \left\lceil \frac{k}{4} \right\rceil\right)^2 + 2\left(d - \left\lceil \frac{k}{4} \right\rceil\right)$.*

Now we can use another theorem that was proved by Sun, Li, Li and Huang [3]:

Theorem 28. *[3] Let G be a 2-edge-connected graph of radius r and k be the smallest integer such that every edge in G lies in a cycle of length k or less. Then $\overrightarrow{d}(G) \leq 2r(k-1)$.*

Using the fact that $r \leq d$ we obtain a second upper bound $\overrightarrow{d}(G) \leq 2d(k-1)$. Now let G be any 2-edge-connected graph of diameter d. The upper bound for the oriented diameter of G can be obtained by using one of the bounds claimed in Theorems 27 and 28. Hence we can say that

$$\overrightarrow{d}(G) \leq \min\left\{2d(k-1), 6d - 2\lfloor k/2 \rfloor - 3 + 2\left(d - \left\lceil \frac{k}{4} \right\rceil\right)^2 + 2\left(d - \left\lceil \frac{k}{4} \right\rceil\right)\right\}.$$

Note that by Lemma 24 the second term is only valid when G has an edge that is not part of any triangle. That is not a problem because if that condition is not true (so $k = 3$), then the first term is always lower than the second one.

Now let $k = 4\alpha d$, hence we get

$$\overrightarrow{d}(G) \leq \min\left\{8\alpha d^2 - 2d, 2(1-\alpha)^2 d^2 + 8d - 6\alpha d - 3\right\}.$$

The original paper [1] contains a minor mistake in this step because it has the second term as $2(1-\alpha)^2 d^2 + 8d - 6\alpha d - 1$ which is incorrect. We know that $3 \leq k \leq 2d + 1$, so

$$0 < \frac{3}{4d} \leq \alpha \leq \frac{2d+1}{4d} < 1.$$

The two dominant terms are $8\alpha d^2$ and $2(1-\alpha)^2 d^2$. We now optimize for α in the interval $[0, 1]$. By plotting the factors 8α and $2(1-\alpha)^2$ as shown in Figure 9 we can see that the worst case α is between 0.15 and 0.2 where $8\alpha = 2(1-\alpha)^2$. Solving for α we get $\alpha = 3 - 2\sqrt{2}$.

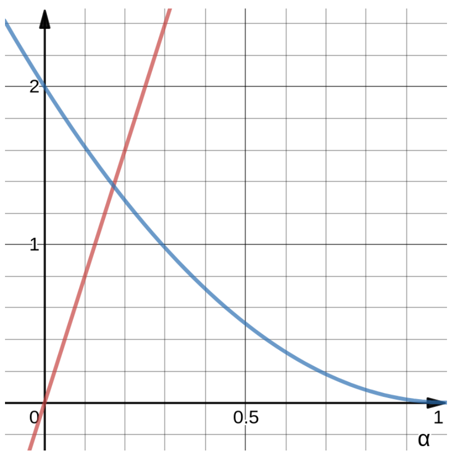

Figure 9: Plot of the dominant factors 8α (red) and $2(1-\alpha)^2$ (blue)

This is the worst case α for the dominant term. By plugging this into the term $2(1-\alpha)^2 d^2 + 8d - 6\alpha d - 3$ we get the following upper bound for the oriented diameter:

Theorem 29. *[1]* $f(d) \leq 1.373d^2 + 6.971d - 3$.

2.3.5 Special Case: Diameter 4

Babu, Benson, Rajendraprasad and Vaka [1] continued to study the special case $d = 4$. They show the following:

Theorem 30. $f(4) \leq 21$.

2.4 Overview

Figure 10 shows the current best results for bounding $f(d)$. These are

- for $d = 2$: $f(d) = 6$ [2]
- for $d = 3$: $9 \leq f(d) \leq 11$ [4]
- for $d = 4$: $f(d) \leq 21$ [1]
- for $d < 8$: $f(d) \leq 2d^2 + 2d$ [2] (dashed red line)
- for $d \geq 8$: $f(d) \leq 1.373d^2 + 6.971d - 3$ (solid red line) [1]
- for $d \geq 1$: $\frac{1}{2}d^2 + d \leq f(d)$ (solid black line) [2]

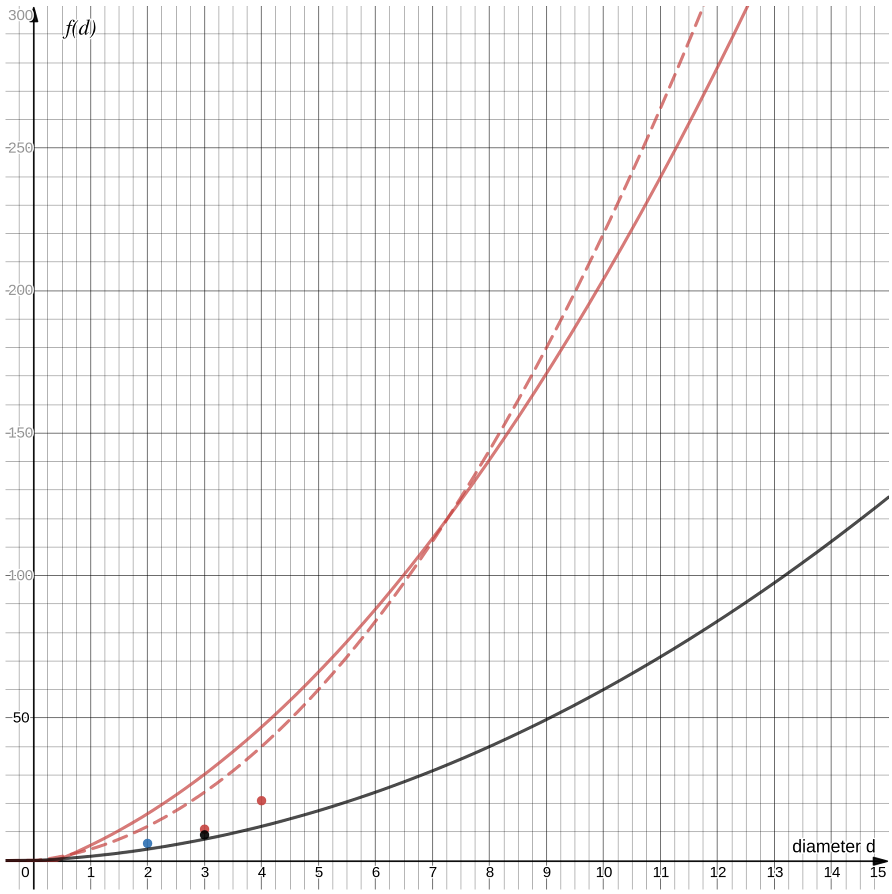

Figure 10: Current best results for $f(d)$. Upper bounds are red, lower bounds are black, exact values are blue.

Two things are obvious when looking at Figure 10. First, we can see that the upper bound provided by Babu, Benson, Rajendraprasad and Vaka [1] (solid red line) is an improvement over the bound given by Chvátal and Thomassen [2] (dashed red line) for all diameters $d \geq 8$. Second, a big gap between upper and lower bound is obvious, leaving a lot of room for improvement.

3 New results

In this section I will present new results that improve the upper bounds for different diameters. Additionally I will describe a brute-force search for graphs with oriented diameters above the lower bounds provided by Chvátal and Thomassen [2].

3.1 Improvements of the upper bounds for $f(5)$ and $f(6)$

Chvátal and Thomassen provide a general upper bound of $f(d) = 2d^2 + 2d$ [2]. For $d = 5$ this yields $f(5) \leq 60$ and for $d = 6$ we obtain $f(d) \leq 84$. We will show the following:

Theorem 31. *$f(5) \leq 47$ and $f(6) \leq 69$.*

Proof. Let G be a 2-edge-connected graph of diameter 5. If every edge lies in a cycle of length 4 or less, G has an oriented diameter of at most 25 (Corollary 4). Since we want to show an upper bound of 47, we can assume that G has an edge pq, that is not part of any cycle of length 3 or 4. We can use the algorithm *OrientedCore* presented in Section 2.3.1 with G and the edge pq that is not part of any cycle of length 3 or 4.

Recall that the algorithm *OrientedCore* yields an oriented subgraph $\overrightarrow{H}$ with $V(\overrightarrow{H}) \subseteq V(G)$ for which $d(\overrightarrow{H}) \leq 6d - 2h - 3$ where $h = \lfloor k/2 \rfloor$ and k is the length of the smallest cycle containing the edge pq, so $k \geq 5$ and $h \geq 2$ (Lemma 24).

Thus we have $d(\overrightarrow{H}) \leq 30 - 4 - 3 = 23$.

Let $v \in V(G) \setminus V(\overrightarrow{H})$ and d_v the distance between v and the closest $u \in V(\overrightarrow{H})$. Recall that $\forall v \in V(G) \setminus V(\overrightarrow{H}) : d_v \leq d - \lceil k/4 \rceil$ so for $d = 5$ and $k \geq 5$ we get $d_v \leq 5 - 2 = 3$.

In other words, we know that $\overrightarrow{H}$ is a 3-step dominating subgraph of G.

Let G' be the graph we obtain by contracting all vertices of $\overrightarrow{H}$ into a single vertex v_0 . Since H is a 3-step-dominating subgraph of G we know that G' has a radius of 3 or less. Hence an orientation $\overrightarrow{G'}$ exists, with radius $3^2 + 3 = 12$ (Theorem 5). Since $d \leq 2r$ we know that $d(\overrightarrow{G'}) \leq 2 * 12 = 24$. By combining the orientations in $\overrightarrow{G'}$ and $\overrightarrow{H}$ we see that G admits an orientation $\overrightarrow{G}$ with $d(\overrightarrow{G}) \leq d(\overrightarrow{G'}) + d(\overrightarrow{H}) = 24 + 23 = 47$ for all graphs G of diameter 5, hence

$$f(5) \leq 47.$$

□

Following exactly the same argumentation we can show that $f(6) \leq 69$, a significant improvement to $f(6) \leq 84$ yielded by the general upper bound of Chvátal and Thomassen (Theorem 10).

3.2 Improvements for diameters 5 to 11

The approach used above can be generalized. This yields improved upper bounds for diameters 5 to 11.

Let G be a 2-edge-connected graph of diameter d and η be any integer with $3 \leq \eta \leq$

$2d+1$. Let $\eta(G)$ be the smallest integer such that every edge in G lies in a cycle with length $\eta(G)$ or less.

For all graphs G with $\eta(G) \leq \eta$ we can use Corollary 4 to obtain an upper bounds $\overrightarrow{d}(G) \leq \left[\left((\eta-2)\, 2^{\lfloor(\eta-1)/2\rfloor}\right) - 1\right] \cdot d$.

For all graphs G with $\eta(G) > \eta$ we can use the approach of Section 2.3. We know that an edge pq exists that is not part of any cycle with length η or less. With this edge we can use the algorithm *OrientedCore* to obtain an oriented subgraph $\overrightarrow{H}$ with $V(\overrightarrow{H}) \subseteq V(G)$. From Lemma 24 we know that $d(\overrightarrow{H}) \leq 6d - 2h - 3$ where $h = \lfloor k/2 \rfloor$ and k is the smallest cycle containing the edge pq, so $k \geq \eta + 1$ and hence $h \geq \lfloor (\eta+1)/2 \rfloor$. Therefore $d(\overrightarrow{H}) \leq 6d - 2h - 3 \leq 6d - 2\lfloor(\eta+1)/2\rfloor - 3$.

Let $v \in V(G) \setminus V(\overrightarrow{H})$ and d_v be the distance between v and the closest $u \in V(\overrightarrow{H})$.

By Lemma 25 we know that $\forall v \in V(G) \setminus V(\overrightarrow{H}) : d_v \leq d - \lceil k/4 \rceil$ so with $k \geq \eta + 1$ we get $d_v \leq d - \lceil (\eta+1)/4 \rceil$. Hence by contracting all vertices of $\overrightarrow{H}$ into a single vertex v_0, we obtain a new graph G' of radius $d - \lceil(\eta+1)/4\rceil$ or less. By Theorem 5 we know that an orientation of G' with radius $r^2 + r$ exists. Since $d \leq 2r$ we know that an orientation exists with diameter $2(r^2+r) = 2r^2 + 2r = 2\left(d - \lceil(\eta+1)/4\rceil\right)^2 + 2\left(d - \lceil(\eta+1)/4\rceil\right)$. By combining the orientation of G' and $\overrightarrow{H}$ we obtain an orientation of G with a diameter at most

$$d(\overrightarrow{H}) + d(G') = 6d - 2\left\lfloor\frac{\eta+1}{2}\right\rfloor - 3 + 2\left(d - \left\lceil\frac{\eta+1}{4}\right\rceil\right)^2 + 2\left(d - \left\lceil\frac{\eta+1}{4}\right\rceil\right).$$

Now we have upper bounds for all graphs G_1 with $\eta(G_1) \leq \eta$ and G_2 with $\eta(G_2) > \eta$. To get an upper bounds $f(d)$ for all graphs G with diameter d, we only need to find the maximum of all G_1 and G_2, that is

Theorem 32.

$$f_\eta(d) \leq max\left\{\left[\left((\eta-2)\,2^{\lfloor\frac{\eta-1}{2}\rfloor}\right) - 1\right] d,\ 6d - 2\left\lfloor\frac{\eta+1}{2}\right\rfloor - 3 + 2\left(d - \left\lceil\frac{\eta+1}{4}\right\rceil\right)^2 + 2\left(d - \left\lceil\frac{\eta+1}{4}\right\rceil\right)\right\}.$$

It is important to notice that we are free to choose an integer η as long as $3 \leq \eta \leq 2d+1$, because η is not a characteristic of the graphs, it is only the threshold we use to decide which of the two upper bounds to use. For smaller diameters, we can simply run a program checking all possible values for η to get the best upper bounds for $f(d)$, that is an η such that $f_\eta(d)$ is minimal, because $f(d) \leq \min\{f_\eta(d) \mid 3 \leq \eta \leq 2d+1\}$. The program code is shown in Appendix A. The results of the program are shown in Figure 11.

The solid blue line is the output of Theorem 32, optimized for η. The dashed orange line is the current best results from [1, 2, 4]. It is obvious that our approach cannot improve the current best results for large diameters. For diameters below 13 the precise values are shown in Theorem 33. The last column shows the best known results, again taken from [1, 2, 4] , $f(d)$ is the upper bounds my approach yields for the η given in the third column.

Theorem 33. *The following table shows upper bounds for $f(d)$:*

diameter d	$f(d)$	η	Best known upper bound for $f(d)$
3	15	4	11
4	29	4	21
5	**47**	4	60
6	**67**	5	84
7	**93**	5	112
8	**123**	5	142
9	**157**	5	172
10	**195**	5	206
11	**237**	5	241
12	283	5	280
13	333	5	321

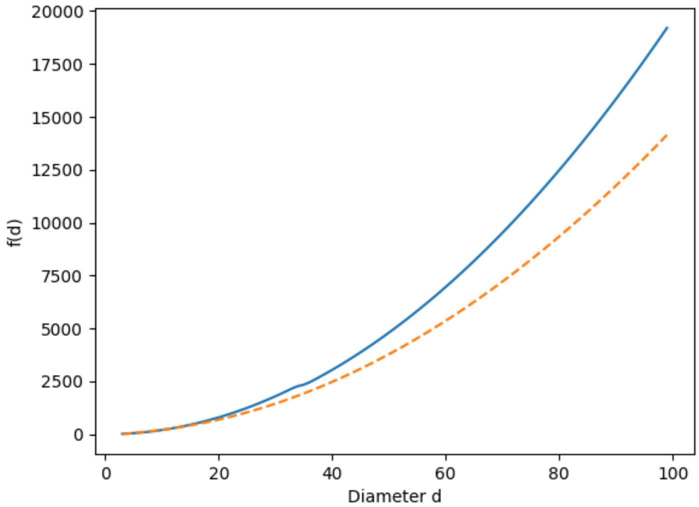

Figure 11: Comparison of our approach (solid, blue) versus the best known result (dotted, orange).

3.3 Improvement for all diameters larger than 8

As explained above, Babu, Benson, Rajendraprasad and Vaka used the two upper bounds from Theorem 27 and Theorem 28 to get the following upper bound:

$$f(d) \leq \min\left\{8\alpha d^2 - 2d, 2(1-\alpha)^2 d^2 + 8d - 6\alpha d - 3\right\}.$$

Here, $\alpha = \frac{\eta}{4d}$ and η is the smallest integer such that every edge of G lies in a cycle of length at most η. Optimizing for $0 < \alpha < 1$ they obtain

$$f(d) \leq 1.373d^2 + 6.971d - 3.$$

This result can be improved by adding a third upper bound to the minimum function. Let G be a 2-edge-connected graph and let k be the smallest integer such that every edge of G lies in a cycle of length k or less. If every edge in G is part of a triangle, we can again use Theorem 28 to obtain the upper bound $\vec{d}(G) \leq 2r(k-1) = 4d$ (with $k = 3$ and $r \leq d$). This is lower than the upper bounds that I will prove, hence we can assume that an edge pq exists that is not part of a triangle. With that edge we use the algorithm *OrientedCore* again to obtain an oriented subgraph $\vec{H}$ that has an oriented diameter of at most $6d - 2\lfloor k/2 \rfloor - 3$. By contracting all the vertices in this subgraph into a single vertex v_0 (that might have multi edges with some vertices), we obtain a new graph G_0.

Fact 34. *G_0 is 2-edge-connected.*

Proof. By definition, G is 2-edge-connected. We also know that a graph is 2-edge-connected if, and only if every edge is part of a cycle. So we only need to show that every edge of G_0 is part of a cycle. Let $pq \in E(G_0)$. Since no edges were added when we created G_0, we know that pq is also an edge in G. G is 2-edge-connected so pq must be part of a cycle in G. Let that cycle be $p, q, ..., u_k, p$. Let u_i (resp. u_j) be the first (resp. last) vertex in that cycle that is in $\vec{H}$. In G_0, both vertices u_i and u_j were contracted into v_0, hence the edges $u_{i-1}v_0$ and v_0u_{j+1} exist. Then $p, q, ..., u_{i-1}, v_0, u_{j+1}, ..., u_k, p$ is a cycle containing the edge pq. □

Let k_0 be the smallest integer such that every edge in G_0 is contained in a cycle of length at most k_0. We know that $k_0 \leq k$. To obtain an upper bounds on the oriented diameter of G_0 we can use Theorem 28 which gives us $\vec{d}(G_0) \leq 2r(k_0 - 1) \leq 2r(k-1)$. According to Lemma 25, $\vec{H}$ is a $\left(d - \left\lceil \frac{k}{4} \right\rceil\right)$-step dominating subgraph of G, hence the radius of G_0 is $d - \left\lceil \frac{k}{4} \right\rceil$. By putting these together we obtain $\vec{d}(G_0) \leq 2\left(d - \left\lceil \frac{k}{4} \right\rceil\right)(k-1)$. Now to get the oriented diameter of G we have to add the oriented diameters of G_0 and $\vec{H}$.

Theorem 35. $\vec{d}(G) \leq 6d - 2\lfloor k/2 \rfloor - 3 + 2\left(d - \left\lceil \frac{k}{4} \right\rceil\right)(k-1)$.

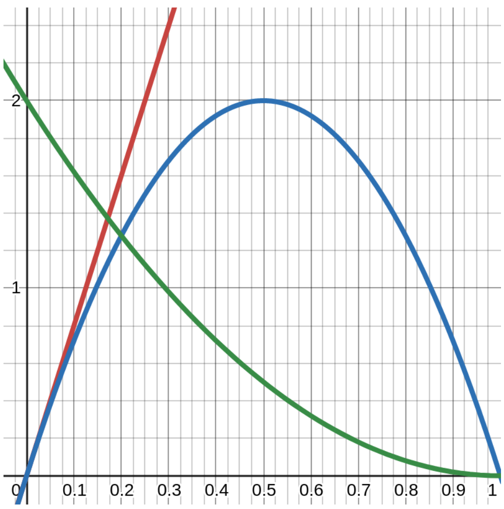

Figure 12: Plot of the three dominant terms 8α (red), $2(1-\alpha)^2$ (green) and $8\alpha - 8\alpha^2$(blue)

Using Theorem 35 with $k = 4\alpha d$, we obtain $\overrightarrow{d}(G_0) \leq \left(8\alpha - 8\alpha^2\right) d^2 + 4d - 3$ which is the third function we add to the minimum function. Hence we get

Theorem 36. $\overrightarrow{d}(G) \leq \min\left\{8\alpha d^2 - 2d, 2(1-\alpha)^2 d^2 + 8d - 6\alpha d - 3, \left(8\alpha - 8\alpha^2\right) d^2 + 4d - 3\right\}.$

We can see that the dominant terms are $8\alpha d^2$, $2(1-\alpha)^2 d^2$ and $\left(8\alpha - 8\alpha^2\right) d^2$. By plotting the factors as shown in Figure 12, we can see that the worst case α changed a little bit and can be obtained by solving $\left(8\alpha - 8\alpha^2\right) d^2 = 2(1-\alpha)^2 d^2$ for α. This yields $\alpha = 0.2$.

By plugging this into the second term, we get the following theorem which is an improvement to the upper bound of Theorem 29:

Theorem 37. $f(d) \leq 1.28d^2 + 6.8d - 3.$

This is an improvement over the upper bounds $f(d) \leq 2d^2 + 2d$ provided by Chvátal and Thomassen [2] for all diameter d larger than or equal to 6. For all graphs of diameter 10 or more, this also improves the upper bounds shown in Section 2.3 and 3.2. Figure 13 gives an overview over all bounds for $f(d)$.

3.4 Brute Force Approach for Lower Bounds

The current best lower bound was provided by Chvátal and Thomassen [2], that is $f(d) \geq \frac{1}{2}d^2 + d$. Kwok, Liu and West [4] studied the special case $d = 3$ and showed that $f(3) \geq 9$. To improve these lower bounds for diameter d, we need to find a graph of diameter d such that every orientation has a diameter above the lower bounds. The nature of this problem allows a brute-force approach, since we only need to find a single example to improve the lower bound for a specific diameter. I will describe a brute-force approach surveying all graphs with at most 14 edges. We assume that all edges (resp. vertices) are numbered $e_1, e_2, ..., e_m$ (resp. $v_1, v_2, ...v_n$). We can iterate through all orientations by using an integer i that is incremented for each orientation. In every iteration, we translate i into a binary string B and add leading zeros such that the length of the string equals the number of edges. Now we can iterate through each edge and direct e_j from the vertex with the lower index to the vertex with the higher index, if the j-th bit in B is a zero, and from higher to lower index otherwise.

The first problem is that the number of possible orientations of a graph is growing exponentially with the number of edges of the graph. In many cases, we don't have to consider all of those orientations. If we are only interested in finding graphs with oriented diameters above the known lower bounds, then we can discard any graph once we find an orientation H with $d(H) \leq \frac{1}{2}d^2 + d$. Additionally we can improve performance by only considering those orientations where the last bit in B is zero. This is due to the fact that the diameter does not change when we reverse all orientations.

The second problem is the larger number of possible graphs. For 9 vertices, there are $2^{36} = 68,719,476,736$ different graphs, not allowing loops and multi-edges. This large number of graphs makes it impossible to check all of them, at least with the resources at my disposal. A solution is to only consider non-isomorphic graphs. Two graphs G_1, G_2 are isomorphic if we can obtain G_2 from G_1 by renaming the vertices. Isomorphic graphs have the same oriented diameter, so we only have to consider one graph of every isomorphism class. For graphs with 9 vertices, this reduces the

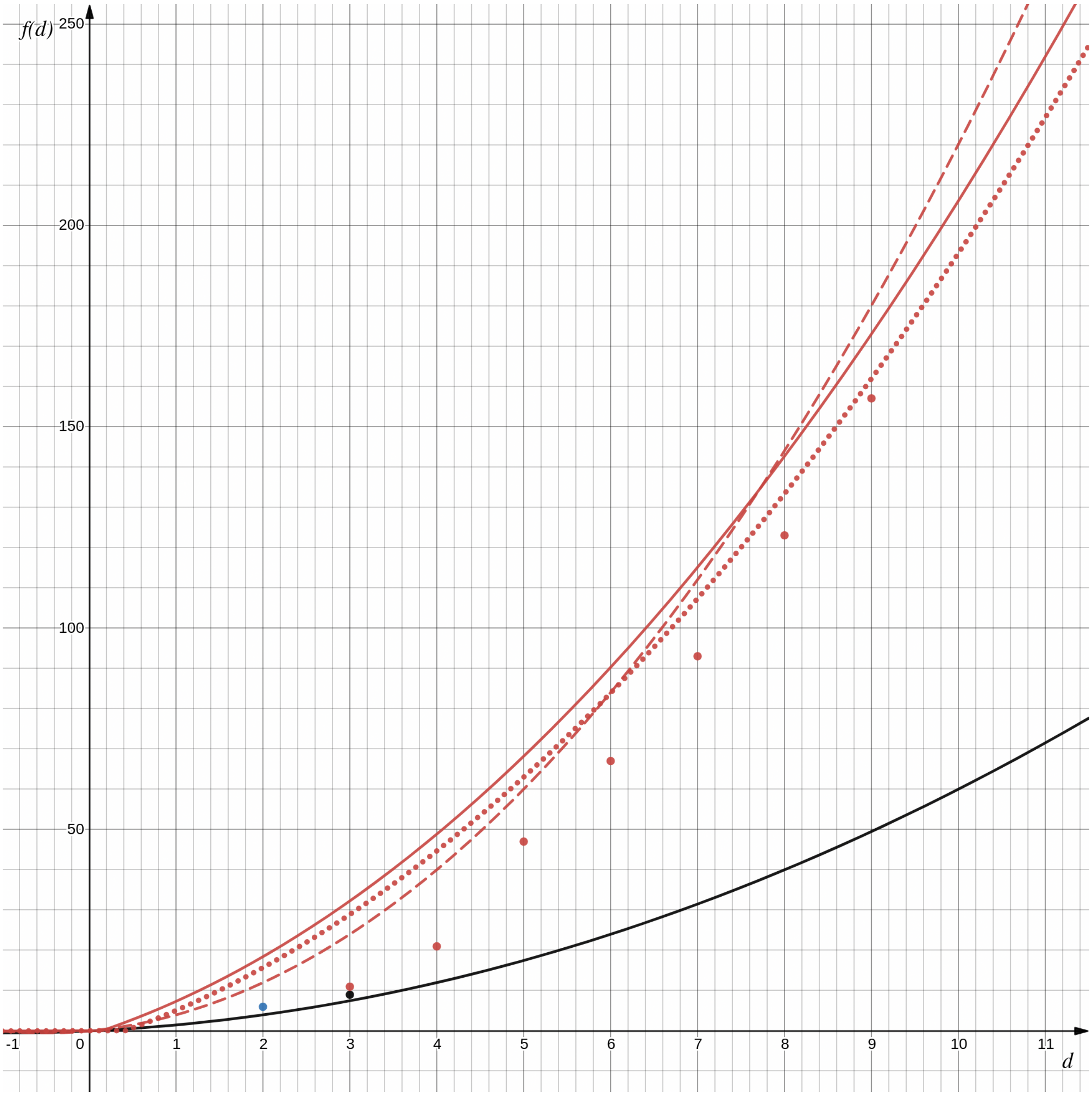

Figure 13: Overview over best bounds for $f(d)$. Upper bounds are red, lower bounds are black, exact values are blue. The dotted red line is from Theorem 37. Red dots for $d > 4$ are from Theorem 33. The other bounds are as in Figure 10.

number of graphs to 274,668 [5]. We can further reduce this number to 261,080 by only considering connected graphs. Data sets with all such graphs, separated by number of vertices and number of edges are provided by [5]. With this we can check the oriented diameter of all graphs with at most 14 edges. We can filter all graphs that have a diameter of 2 because Chvátal and Thomassen have already shown that $f(2) = 6$ and also provided an example [2]. This further reduces the number of graphs that we need to find the oriented diameter of.

After running all data sets we can see that no graph with an oriented diameter above $\frac{1}{2}d^2 + d$ (resp. above 8 for diameter 3) was found, hence we get the following:

Theorem 38. *Let G be a graph with diameter d such that $\overrightarrow{d}(G) > \frac{1}{2}d^2 + d$ for $d > 3$ or $\overrightarrow{d}(G) > 9$ for $d = 3$, then G must have at least 15 edges.*

Bibliography

[1] Jasine Babu, Deepu Benson, Deepak Rajendraprasad, and Sai Nishant Vaka. An improvement to chvátal and thomassen's upper bound for oriented diameter. *Computer Science - Theory and Applications Lecture Notes in Computer Science*, pages 117–129, 2020.

[2] Vašek Chvátal and Carsten Thomassen. Distances in orientations of graphs. *Journal of Combinatorial Theory, Series B*, 24(1):61–75, 1978.

[3] Xiaolong Huang, Hengzhe Li, Xueliang Li, and Yuefang Sun. Oriented diameter and rainbow connection number of a graph. *Discrete Mathematics and Theoretical Computer Science*, 16, 11 2011.

[4] Peter K. Kwok, Qi Liu, and Douglas B. West. Oriented diameter of graphs with diameter 3. *Journal of Combinatorial Theory, Series B*, 100(3):265–274, 2010.

[5] Brendan McKay. Combinatorial data. *URL: http://cs. anu. edu. au/bdm/data/graphs. html*, 2011.

[6] Herbert Ellis Robbins. A theorem on graphs, with an application to a problem of traffic control. *The American Mathematical Monthly*, 46(5):281–283, 1939.

A Code for Section 3.2

```
from math import floor, ceil

# if every edge is in a cycle with length at most eta this is the upper bounds
def lower_than_eta(d,eta):
  return floor(d * ((eta-2) * (2 ** ((eta-1)/2)) - 1))

# if one edge is in no cycle of length <=eta this yields the upper bounds
def higher_than_eta(d,eta):
  eta+=1
  return 6*d-2*floor(eta/2)-3 + 2*((d-ceil(eta/4))**2) + 2*(d-ceil(eta/4))

# checks all possible values for eta  and finds the minimum eta
def f(d):
  min, min_eta = None, None
  for eta in range(3, 2*d + 2):
    k = max(lower_than_eta(d, eta), higher_than_eta(d,eta))
    if min is None or min>k:
      min, min_eta = k, eta
  return min, min_eta

# bounds according to the recently published paper by Babu, Benson, Rajendraprasad
    and Vaka
def recent_improv(d):
  return floor(1.373 * d**2 + 6.971 * d - 1)

# original bounds from Chvatal and Thomassen
def chvatal(d):
  return 2 * d**2 + 2 * d

for d in range(3,100):
  min, min_eta = f(d)
 # use Chvatal and Thomassen only on diameters below 8, otherwise the recent
     improvement is better, additionally use f(3)<=11 and f(4)<=21
  if d==3:
     r = 11
  elif d==4:
      r = 21
  elif d<8:
    r = chvatal(d)
  else:
    r = recent_improv(d)

print("d={}, f(d)={}, eta={}, Current Best Result={}".format(d, min, min_eta, r))
```