**Master thesis**

# Complexity of recognizing generalized transmission graphs

Katharina Klost

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Betreuer und Erstgutachter:
Prof. Dr. Wolfgang Mulzer

Zweitgutachter:
PD Dr. Klaus Kriegel

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Katharina Klost

## Abstract

Given an arrangement of geometric objects with one point of each object singled out. The generalized transmission graph of this arrangement consists of one vertex per element and a directed edge $(u, v)$ if and only if the point of $v$ lies in $u$. These graphs can serve as a generalized model of antenna reachability.

The complexity class $\exists\mathbb{R}$ contains all problems that are polynomial time reducible to a sentence of the form $\exists x_1, \ldots, x_n : \phi(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are interpreted over $\mathbb{R}$. It aims to capture the complexity of the existential theory of the reals. It is well known that $\exists\mathbb{R}$ lies between **NP** and **PSPACE**.

For many geometric decision problems, such as recognition of disk graphs and of intersection graphs of lines, it is known that they are complete for $\exists\mathbb{R}$.

In this thesis we formally introduce the class of *generalized transmission graphs* and show that the recognition problem of generalized transmission graphs of $k$-spheres, regular polygons, line segments and circular sectors is complete for the complexity class $\exists\mathbb{R}$.



## Zusammenfassung

Der allgemeine Transmissionsgraph einer Menge von geometrischen Objekten, mit einem ausgezeichneten Punkt für jedes Element der Menge, hat einen Knoten pro Element und eine gerichtete Kante $(u, v)$ genau dann wenn der Punkt von $v$ in $u$ liegt. Diese Graphen sind ein mögliches Modell für Erreichbarkeit von Antennen.

Die Komplexitätsklasse $\exists\mathbb{R}$ enthält alle Sprachen, die in Polynomialzeit auf einen Satz der Form $\exists x_1, \ldots, x_n : \phi(x_1, \ldots, x_n)$ reduziert werden können. Hierbei stammen $x_1, \ldots, x_n$ aus $\mathbb{R}$. Es ist bekannt, dass $\exists\mathbb{R}$ zwischen **NP** und **PSPACE** liegt.

Für viele geometrische Entscheidungsprobleme, wie zum Beispiel das Erkennen von Kreisgraphen und Schnittgraphen von Strecken ist bekannt, dass diese $\exists\mathbb{R}$-vollständig sind.

In dieser Arbeit werden *allgemeine Transmissionsgraphen* formal definiert und es wird gezeigt, dass das Erkennen von allgemeinen Transmissionsgraphen von $k$-Kugeln, regelmäßigen Polygonen, Strecken und Kreissektoren $\exists\mathbb{R}$-vollständig ist.

# Contents

# 1 Introduction

An *intersection graph* of geometric objects has a vertex for each object and an undirected edge $\{u, v\}$ if the object associated with $u$ and the object associated with $v$ intersect. When the objects are disks in the plane, these graphs are called disk graphs. One of the practical applications of disk graphs is to model the distribution of signals of omnidirectional antennas[10]. Since this model is symmetrical it is quite limited. It is always assumed, that if antenna $a$ can reach antenna $b$ the inverse is also true. If the antennas have different transmission ranges, $a$ might reach $b$ but $b$ might not be able to reach $a$.

There are some asymmetrical models for signal distribution. In this thesis we will focus on one of these models which is called *transmission graph* [7]. Similar to a disk graph, the transmission graph represents some aspect of the structure of an arrangement of disks. Namely there is a vertex for each disk and a directed edge $(u, v)$ if the center of the disk representing $v$ is contained in the disk representing $u$.

There are two main use cases for models of signal distribution. The first is to model a given network of antennas and then examine certain properties. The second use case is to give a general description of a network and then try to realise this network with antennas. This second application is closely related to the problem of recognizing transmission or disk graphs.

The complexity of recognizing intersection graphs of objects in the plane and in higher dimensions is a well researched problem. It was shown by Breu and Kirckpatrick that the problem of recognizing a unit disk graph is **NP**-hard [3]. There is a second **NP**-hardness proof due to Kang and Müller [6] that implies the stronger result of $\exists\mathbb{R}$-hardness. The complexity class $\exists\mathbb{R}$ was defined by Schäfer [11]. It contains all problems that are polynomial time reducible to the existential theory of the reals. One of the first problems that was shown to be complete for $\exists\mathbb{R}$ is the problem of *line stretchability*. There is a proof due to Mnëv [9] which showed that line stretchability is equivalent to the existential theory of the reals. This proof was given before $\exists\mathbb{R}$ was formally defined and was rephrased in the nomenclature of $\exists\mathbb{R}$ by Matousek [8]. Apart from the recognition of unit disk graphs there are many other geometrical problems that were shown to be $\exists\mathbb{R}$-hard [5].

The general definition of disk graphs and transmission graphs is similar. The structural similarities lead to the assumption that the recognition problem of transmission graphs and disk graphs might have similar complexities. This assumption was the starting point of the presented thesis. By modifying the proof of Kang and Müller [6] we show in Theorem 3 that the recognition of transmission graphs is $\exists\mathbb{R}$-complete.

This is the first directed graph recognition problem we are aware of that was proved to be $\exists\mathbb{R}$-complete. Based on the problem of recognizing transmission graphs we define a new class of graphs called *generalized transmission graphs*. This class generalizes the idea of a transmission graph to arbitrary objects with one point singled out. In Theorem 4 we show that the recognition problem of generalized transmission graphs of regular polygons is ∃ℝ-complete.

As mentioned above, transmission graphs can be used as a model for omnidirectional antennas. Similarly, a circular sector can be seen as a model for a sector antenna. This similarity inspired us to consider the complexity of recognizing generalized transmission graphs of line segments and circular sectors. In Theorem 5 we prove that the recognition of generalized transmission graphs of line segments is ∃ℝ-complete. Then we consider the complexity of recognizing generalized transmission graph of circular sectors. Our ∃ℝ-completeness result in Theorem 6 holds for a class of restricted arrangements of circular sectors.

# 2 Preliminaries

In this chapter we will define some basic notions that are used throughout the thesis. We start by formally defining the two graph classes that are used.

**Definition 1** (Intersection graph)**.** *Let $k \in \mathbb{N}$ and let $x_1, \ldots, x_n \subseteq \mathbb{R}^k$. The intersection graph $G_I = (V, E)$ of $x_1, \ldots, x_n$ is an undirected graph with*

$$\begin{aligned} V &= \{x_1, \ldots, x_n\} \\ \textit{and}\ \ E &= \{\{x_i, x_j\} \mid x_i \cap x_j \neq \emptyset, 1 \leq i, j \leq n\}. \end{aligned}$$

The class of transmission graphs can be seen as a directed version of intersection graphs of disks in the plane [7]. In this thesis the former concept is generalized to arbitrary objects in $\mathbb{R}^k$. If the objects are disks and $k = 2$ this yields the definition for transmission graphs.

**Definition 2** (Generalized transmission graph)**.** *Let $k \in \mathbb{N}$ and $x_1, \ldots, x_n \subseteq \mathbb{R}^k$. Let there be a point $p(x_i) \in x_i$ for every set $x_i$. The generalized transmission graph of these sets is a directed graph $G = (V, E)$ with*

$$\begin{aligned} V &= \{x_1, ..., x_n\} \\ \textit{and}\ \ E &= \{(x_i, x_j) \mid p(x_j) \in x_i, 1 \leq i, j \leq n\}. \end{aligned}$$

At several points in this thesis we consider the intersections of unbounded simple curves or intersections of the hull of polygons. We will distinguish between *touching* and *crossing* intersection points.

**Definition 3** (Touching and crossing intersection)**.** *Let $c_1$ and $c_2$ be two simple curves in the plane. Moreover for $x \in c_1 \cap c_2$, let $c_1'$ and $c_1''$ be the components of $c_1 \setminus x$ and define $c_2'$ and $c_2''$ analogously. Then if $x$ is the endpoint of any of the curves, it is called a* touching point.

*For a point $x$ that is not an endpoint of any of the curves, let $l$ be the line through $x$ with the slope defined by two points $p_1 \in c_2'$ and $p_2 \in c_2''$ with $p_1$ and $p_2$ having an infinitely small distance to $x$. Then $x$ is called a* touching point *of $c_1$ and $c_2$, if $q_1 \in c_1'$ and $q_2 \in c_1''$ with infinitely small distance to $x$ lie in the same halfplane induced by $l$. Otherwise $x$ is called a* crossing point. *Both cases of intersection points are depicted in Figure 2.1.*

One of the basic problems that is used for reductions throughout this thesis is based on arrangements of pseudolines.

**Definition 4** (Pseudoline arrangement)**.** *A* pseudoline *in the euclidean space is a simple unbounded curve in the plane. A set of $n$ such pseudolines is called an* arrangement of

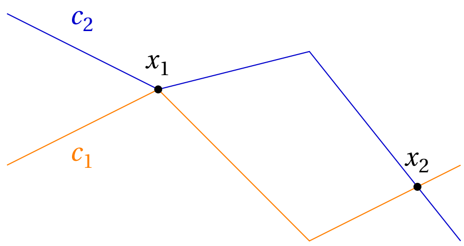



Figure 2.1: A touching point $x_1$ and a crossing point $x_2$ of two curves.

pseudolines *if each pair of pseudolines intersects at most once and all intersection points are crossing points. Such an arrangement is called* simple *if there is no point in which more than two pseudolines intersect and each pair of pseudolines intersects [1].*

Pseudolines can be seen as a non-straight generalization of lines. In this thesis we will need the notion of *oriented lines* and pseudolines. An oriented line is a line with a direction, thus the halfplanes induced by this line can be uniquely classified as the "left" and "right" halfplane. For an oriented pseudoline this concept is adapted to the connected components of the plane induced by the pseudoline.

When considering arrangements of lines or pseudolines, they divide the plane into facets. There are some divisions of the plane into facets that can be realized with pseudolines but not with lines. The commonly given example of this situation is derived from the *Pappus hexagon theorem*. This theorem states that given two sets $\{A, B, C\}$ and $\{a, b, c\}$ of collinear points, the intersections of the line pairs $(\overline{Ab}, \overline{aB})$, $(\overline{Ac}, \overline{aC})$ and $(\overline{Bc}, \overline{bC})$ are collinear. This fact can be used to construct the arrangement that can be seen in Figure 2.2. In this arrangement the intersection point $e$ does not lie on the pseudoline through $d$ and $f$. This is not possible with lines due to the Pappus hexagon theorem [8].

In the constructions in chapter 3 and chapter 4, we have to construct hyperplanes separating two convex sets. The following theorem shows that such hyperplanes do always exist.

**Theorem 1** (Hyperplane separation theorem [2])**.** *Let $C$ and $D$ be two disjoint nonempty convex subsets of $\mathbb{R}^k$. Then there exists $a \in \mathbb{R}^k$, $a \neq \mathbf{0}$ and $b$ such that for all $c \in C$ and $d \in D$*

$$a^T c \leq b$$
$$a^T d \geq b.$$

*This means that the hyperplane $h = \{x \mid a^T x = b\}$ separates $C$ and $D$.*

Now we introduce the complexity class that is considered throughout this thesis. Let $\phi$ be quantifier free boolean formula without negations over the signature $(0, 1, +, \cdot, <, \leq)$, interpreted over the universe of the real numbers. Then the *existential theory of the reals* is the set of all true sentences of the form $\exists x_1, \ldots, x_n : \phi(x_1, \ldots, x_n)$[11].

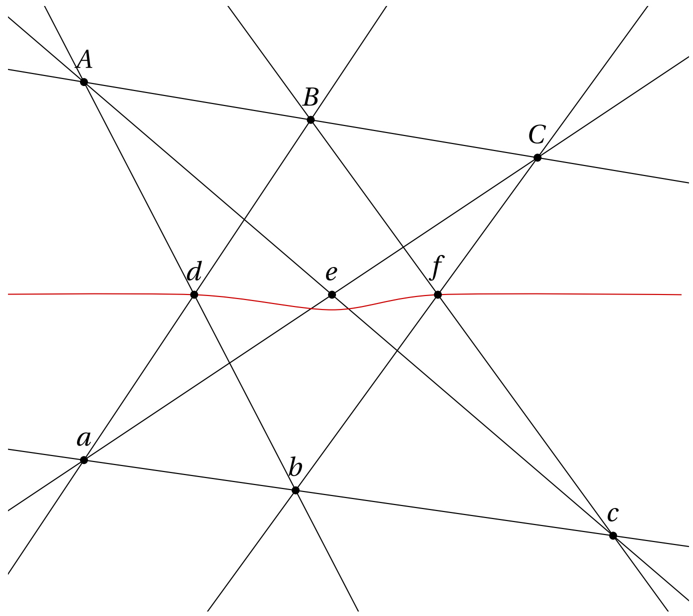



Figure 2.2: An arrangement of pseudolines that is not realizable by lines.

It can be shown that the existential theory of the reals is equivalent to the set that is defined analogously but has a reduced signature of the form $(0, 1, +, \cdot, <)$[11].

The complexity class $\exists\mathbb{R}$ was formally defined by Schäfer [11]. It contains all problems that are *polynomial-time reducible* to the existential theory of the reals. This means that for each instance of a problem in $\exists\mathbb{R}$ there is a polynomial time construction giving a sentence in the existential theory of the reals that is true if and only if the instance is a yes-instance.

It is currently known that **NP** $\subseteq \exists\mathbb{R} \subseteq$ **PSPACE** [12, 4]. A problem is called $\exists\mathbb{R}$*-complete* if it lies in $\exists\mathbb{R}$ and if all problems in $\exists\mathbb{R}$ are polynomial time many-to-one reducible to to it. There are some problems that are known to be complete for $\exists\mathbb{R}$, most of them are geometric problems. Examples of $\exists\mathbb{R}$-complete problems are:

- Stretchability of pseudolines [9, 8] (see section 3.1)
- Realizability of order types [8]
- Recognizing intersection graphs of line segments [11]
- Recognizing unit disk graphs and $k$-sphere graphs [6] (see section 3.2)

Some of these problems will be introduced in detail in chapter 3.

At many points throughout this thesis we will give sentences in the existential theory of the reals in order to show that a problem lies in $\exists\mathbb{R}$. For most problems we are considering the distances between points. For better readability we will write expressions of the form $(x_1 - x_2)^2$. This notation is explicitly not covered by the standard signature, as expressions of the form $(a+b)^k$ for arbitrary $k$ might have exponential size when written as monomials. Note that since in our case the exponent is constant, the size of this phrase remains constant when using the standard signature.

In the sentences constructed in the proofs of Theorem 4 and Lemma 7 we will need the value of $\cos\left(\frac{2\pi\cdot k}{m}\right)$ and $\cos\left(\frac{2\pi\cdot k}{m}\right)$ for constant $m$. We will now show that we can use these values in the existential theory of the reals.

**Lemma 1.** *Let $m$ be a constant, then there exists a true sentence of constant size in the existential theory of the reals with the variables $c_0, \ldots, c_{m-1}, s_1, \ldots, s_{m-1}$, having $c_k = \cos\left(\frac{2\pi k}{m}\right)$ and $s_k = \sin\left(\frac{2\pi k}{m}\right)$, $0 \leq k < m$.*

*Proof.* In order to prove this lemma we take a detour to the complex numbers. We know by the fundamental theorem of algebra, that the equation $(a+bi)^m = 1$ has exactly $m$ complex roots. These $m$ solutions are the roots of unity in the complex planes. For $0 \leq k \leq m$ they have the form

$$a = \cos\left(\frac{2\pi k}{m}\right)$$
$$b = \sin\left(\frac{2\pi k}{m}\right).$$

Using the binomial theorem we can rewrite $(a+bi)^m$ as follows:

$$\begin{aligned}(a+bi)^m &= \sum_{k=0}^{m}\binom{m}{k}(bi)^k + a^{m-k} \\ &= \begin{array}{l}\sum_{k=0}^{\lceil\frac{m-1}{2}\rceil}(-1)^k\binom{m}{2k}b^{2k} + a^{m-2k} \\ + i\cdot\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\binom{m}{2k+1}b^{2k+1} + a^{m-(2k+1)}\end{array}\end{aligned}$$

Now we reconsider the equation $(a+bi)^m = (1+0i)$. By splitting the equation into its real and complex part we get

$$1 = \sum_{k=0}^{\lceil\frac{m-1}{2}\rceil}(-1)^k\binom{m}{2k}b^{2k} + a^{m-2k} \tag{2.1}$$

$$0i = i\cdot\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^k\binom{m}{2k+1}b^{2k+1} + a^{m-(2k+1)}. \tag{2.2}$$

We want to use (2.1) and (2.2) to calculate the values $s_1$ and $c_1$, where $s_1 = \sin\left(\frac{2\pi}{m}\right)$ and $c_1 = \sin\left(\frac{2\pi}{m}\right)$. Since there are $m$ solutions of this system we have to find additional constraints to enforce the uniqueness of the solution.

When considering the complex plane, all solutions of the equation lie on the unit circle. For $m \geq 2$ the following inequalities hold:

$$\begin{aligned}
&\sin\left(\frac{2\pi}{m}\right) \geq 0 \\
&\cos\left(\frac{2\pi}{m}\right) < 1 \\
&\cos\left(\frac{2\pi}{m}\right) \geq \cos\left(\frac{2\pi k}{m}\right) \qquad\qquad 1 \leq k < m
\end{aligned}$$

With these considerations the values of $s_1$ and $c_1$ can be uniquely determined with the following sentence in the existential theory of the reals.

$$\begin{aligned}
&\exists s_0, s_1, \dots, s_{m-1}, c_0, c_1, \dots, c_{m-1} : \\
&\qquad \sum_{i=0}^{\lceil \frac{m-1}{2} \rceil} (-1)^i \binom{m}{2i} s_1^{2i} \cdot c_1^{m-2i} = 1 \\
&\wedge \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^i \binom{m}{2i+1} s_1^{2i+1} \cdot c_1^{m-(2i+1)} = 0 \\
&\wedge \qquad s_1 \geq 0 \\
&\wedge \bigwedge_{2 \leq k < m} c_1 \geq c_k.
\end{aligned}$$

Now that these values are uniquely determined we can use the relation

$$(\cos(\alpha) + i\sin(\alpha))^k = ((\cos(k\alpha) + i\sin(k\alpha))$$

to find the values of $s_2, \dots, s_{m-1}$ and $c_2 \dots, c_{m-1}$. The values of $s_0$ and $c_0$ are set directly. This can be achieved by adding the following inequalities to the system.

$$\begin{aligned}
&\wedge \bigwedge_{2 \leq k < m} \left( \sum_{i=0}^{\lceil \frac{k-1}{2} \rceil} (-1)^i \binom{k}{2i} s_1^{2i} \cdot c_1^{k-2i} = c_k \right) \\
&\wedge \bigwedge_{0 \leq k < m} \left( \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^i \binom{k}{2i+1} s_1^{2i+1} \cdot c_1^{k-(2i+1)} = s_k \right) \\
&\wedge \qquad c_0 = 1 \\
&\wedge \qquad s_0 = 0
\end{aligned}$$

Since $m$ is constant, all inequalities in this system have constant size and the number of inequalities only depends on $m$. This concludes the proof of the lemma. □

# 3 Stretchability and $k$-Sphericity

This chapter introduces the $\exists\mathbb{R}$-complete problems that will be used in the reductions in chapter 4 and chapter 5.

## 3.1 Stretchability of pseudolines

By a *combinatorial description* of an *arrangement of pseudolines* we mean some way to describe the induced decomposition of $\mathbb{R}^2$ into faces.

The original problem of Stretchability is defined as follows: Given an arrangement $\mathcal{L}$ of pseudolines and its combinatorial description $\mathcal{D}(\mathcal{L})$, is there an arrangement of lines having the same description? The question if an arrangement of simple pseudolines is stretchable is called Simple-Stretchability. Both variants of the problem are complete for $\exists\mathbb{R}$ [9, 8].

There are some equivalent ways of giving a combinatorial description $\mathcal{D}(\mathcal{L})$ of an arrangement $\mathcal{L}$ of pseudolines. We will now introduce the two descriptions that will be used in this thesis.

The first description as defined by Matousek [8] describes the arrangement by the order in which the lines intersect each other. We will call this description *order description*. Here it is assumed that no two lines are parallel and no line is vertical.

Let $g$ be a vertical line that lies to the left of all intersection points. The lines are numbered $l_1, \ldots, l_n$ in the order in which they intersect $g$ from top to bottom. If there is a realization with lines, this ordering corresponds to the ascending order of the slopes. For each line $l_i$ there is a list $O^i$ of the following form:

$$O^i = (o_1^i, \ldots, o_k^i) \qquad o_j^i \subseteq \{1, \ldots, n\}$$

$$\bigcup_{j=1}^{k} o_j^i = \{1, \ldots, n\} \qquad o_j^i \cap o_{j'}^{i'} = \emptyset \text{ for } j \neq j'.$$

The order of the indices in $O^i$ indicates the order in which the $l_j$ cross $l_i$ when going from left to right. The lists $O^i$ form the combinatorial description of the arrangement of pseudolines. If the arrangement is simple, each $o_j^i$ is a singleton. An example for this description can be found in Figure 3.1(b).

The second description, as introduced by Kang and Müller [6] makes use of oriented (pseudo)lines and can be generalized to $k$-hyperplanes in $\mathbb{R}^k$. In this thesis we will denote this description by *oriented description*.

Each oriented pseudoline $l_i$ partitions $\mathbb{R}^2$ into a "positive" half-space $l_i^+$, a "negative" half-space $l_i^-$ and the points on $l_i$.

The sign vector $\sigma(p)$ of a given point $p$ in $\mathbb{R}^2$ with regard to an oriented arrangement of pseudolines $\mathcal{L} = (l_1, ..., l_n)$ is defined as follows:

$$(\sigma(p))_i = \begin{cases} - & \text{if } p \in l_i^- \\ 0 & \text{if } p \in l_i \\ + & \text{if } p \in l_i^+ \end{cases}$$

The combinatorial description $\mathcal{D}(\mathcal{L})$ of a set $\mathcal{L}$ of hyperplanes is defined as:

$$\mathcal{D}(\mathcal{L}) := \{(\sigma(p)) \mid p \in \mathbb{R}^2\}$$

This set is finite and the size of $\mathcal{D}(\mathcal{L})$ is the number of faces of the arrangement $\mathcal{L}$, since the sign vectors of all points in one face is the same, and thus the sign vector of each face is contained once. If a sign vector $\sigma$ contains exactly one zero, all points with $\sigma(p) = \sigma$ lie on the same pseudoline of the arrangement. If the sign vector contains more than one zero, it corresponds to the intersection of two or more pseudolines.

This type of description can also be used to describe the partition of $\mathbb{R}^k$ into faces. Two arrangements are called isomorphic if they have the same set of sign vectors. It can be shown that it suffices to consider the sign vectors without any 0 component to determine whether two arrangements are isomorphic [6]. This corresponds to the description of the $k$-faces. An example for the description can be found in Figure 3.1(a).

A problem related to Stretchability which arises from the oriented description is the recognition of oriented hyperplane arrangements. In order to formally define the related problem that will be used during this thesis, some additional definitions are needed:

**Definition 5.** *($k$-realizable) Let $k > 1$ and $S \subseteq \{-, +\}^n$ with $(-, \ldots, -), (+, \ldots, +) \in S$. Then $S$ is called $k$-realizable if there exists an oriented $k$-hyperplane arrangement $\mathcal{H} = (h_1, \ldots, h_n)$ with $S \subseteq \mathcal{D}(\mathcal{H})$.*

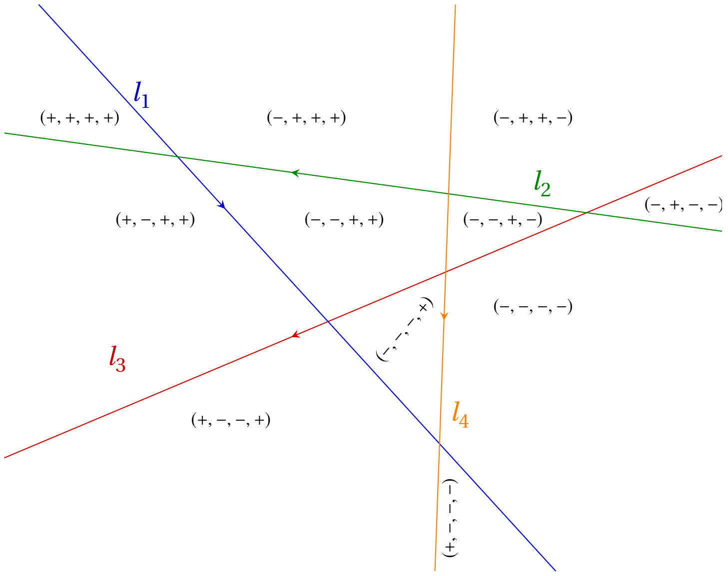
Based on this we can define the corresponding decision problem:

**Definition 6.** *($k$-Realizability) Given a set $S \subseteq \{-, +\}^n$, decide whether $S$ is $k$-realizable.*

It can be shown that the form of the hyperplane arrangement realizing the oriented description can be further specified. The following rather technical lemma will be needed in the proofs of Theorem 2, Theorem 3, and Theorem 4.

**Lemma 2.** *(Lemma 12 [6]) Let $\varepsilon > 0$, $K > 0$ be given and let $\mathcal{H}$ be an oriented hyperplane arrangement with $(-, ..., -), (+, ..., +) \in \mathcal{D}(\mathcal{H})$. Then there exists an oriented hyperplane arrangement $\mathcal{H}'$ with the following properties.*

1. *$\mathcal{D}(\mathcal{H}') = \mathcal{D}(\mathcal{H})$*
2. *The origin $0$ belongs to the $(-, \ldots, -)$ cell of $\mathcal{H}'$*




(a) Four oriented lines with sign vectors

$$O^1 = (2, 3, 4)$$
$$O^2 = (1, 4, 3)$$
$$O^3 = (1, 4, 2)$$
$$O^4 = (1, 3, 2)$$

(b) Order description of the unoriented line arrangement corresponding to (a)

Figure 3.1: Example for the combinatorial descriptions

3. *For each oriented hyperplane $h'_i \in \mathcal{H}'$ we can write:*

$$h'_i = \{z \mid v_i^T z = c_i\} \qquad h_i'^{-} = \{z \mid v_i^T z < c_i\} \qquad h_i'^{+}\{z \mid v_i^T z > c_i\}$$

*where $v \in \mathbb{R}^k$, $||v_i - e_1|| < \varepsilon$, $e_1 = (1, 0, \ldots, 0)$ and*

4. *Every cell of $\mathcal{H}'$ intersects with the ball $B((K, 0, \ldots, 0)^T, \varepsilon)$.*

*Proof.* (Sketch) This theorem can be proved by applying a series of affine transformations. The full proof was given by Kang and Müller [6]. A visual idea of how the arrangement looks like can be found in Figure 3.2.

**Lemma 3** (Theorem 10[6])**.** *Let $k > 1$ be given, then $k$-*Realizability* is complete for $\exists\mathbb{R}$.*

*Proof.* Kang and Müller (Theorem 10 [6]) reduced Stretchability to $k$-Realizability. Now it remains to show that the problem lies in $\exists\mathbb{R}$.

To show that the problem lies in $\exists\mathbb{R}$, given a subset $S = \{\sigma_1, \ldots \sigma_m\}$ of an oriented description, we construct the following sentence in the existential theory of the reals:

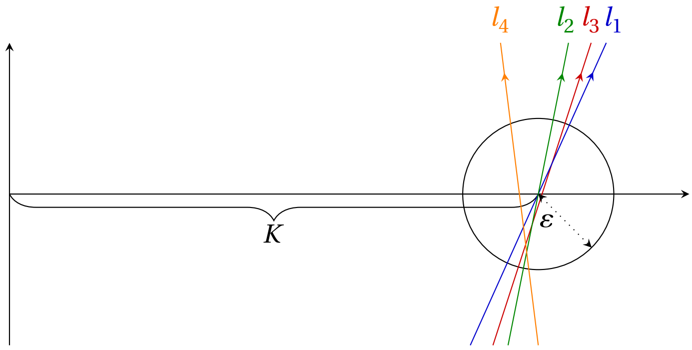



Figure 3.2: Line arrangement from Figure 3.1 with conditions from Lemma 2

$$\exists p_{11}, p_{12}, \ldots, p_{1k}, \ldots, p_{m1}, \ldots p_{mk},$$
$$v_{11}, \ldots v_{1k}, \ldots, v_{n1}, \ldots, v_{nk},$$
$$c_1, \ldots, c_k :$$
$$\bigwedge_{1 \le i \le m, 1 \le j \le k} \tau_{ij}(p_{i1} \cdot v_{j1} + p_{i2} \cdot v_{j2} + \ldots + p_{ik} \cdot v_{jk}) < \tau_{ij} \cdot c_j$$
$$\tau_{ij} = \begin{cases} 1 & (\sigma_i)_j = - \\ -1 & (\sigma_i)_j = + \end{cases}$$

Each $p_i = (p_{i1}, \ldots p_{ik})$ satisfies the condition for $\sigma_i$. The $v_j = (v_{i1}, \ldots, v_{ik})$ and $c_j$ describe the hyperplanes $h_j$ where $v_j$ is the vector that is orthogonal to $h_j$. In order for the sentence to be true, $p_i$ has to lie on the correct side of each hyperplane. □

## 3.2 $k$-SPHERICITY

**Definition 7** ($k$-sphere graph)**.** *For a given $k \in \mathbb{N}$, an undirected graph $G$ is a $k$-sphere graph if there are points $p_1, \ldots, p_n \in \mathbb{R}^k$, one for each vertex, such that $||p_i - p_j|| \le 1$ if and only if $\{p_i, p_j\} \in E$.*

The $k$-sphere graphs can be seen as the generalization of *unit disk graphs* to higher dimensions.

**Definition 8.** *($k$-SPHERICITY) Let $k \in \mathbb{N}$ and a graph $G = (V, E)$, $|V| = n$ be given. Then $k$-SPHERICITY is the question whether $G$ is a $k$-sphere graph.*

The proofs in chapter 4 reuse many ideas of the original proof by Kang and Müller [6]. We will rephrase the original proof here. In chapter 4 we will then only describe the parts of the proof that differ.

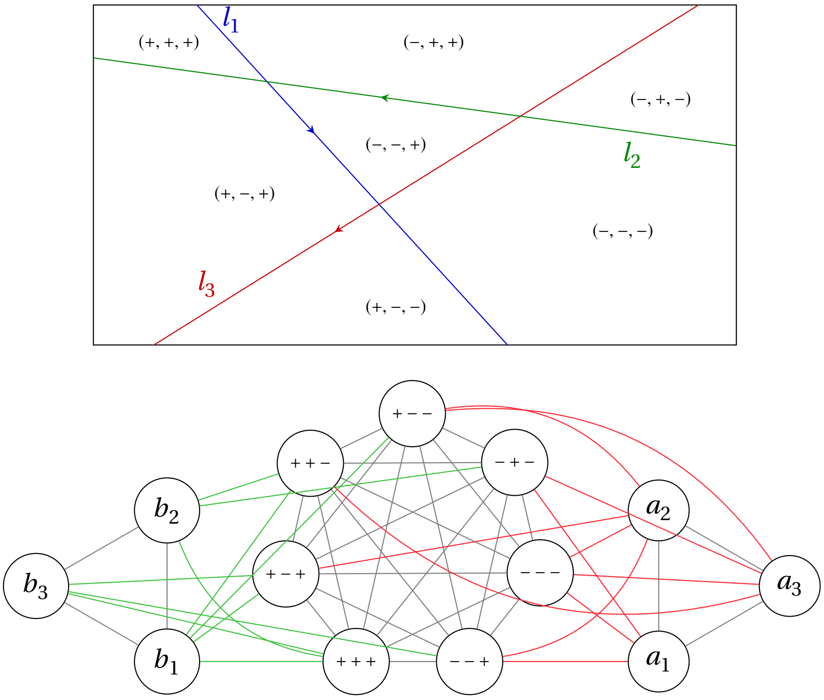



Figure 3.3: The graph $G_S$ for a line arrangement with three lines

**Theorem 2.** *[6] Let $k \in \mathbb{N}$ be fixed. Then $k$-SPHERICITY is complete for $\exists\mathbb{R}$.*

*Proof.* As in the proof of Lemma 3 we first show that $k$-SPHERICITY lies in $\exists\mathbb{R}$. This part of the proof was not explicitly given by Kang and Müller. Let $G_S$ be the given graph. We construct the following sentence of the existential theory of the reals:

$$\exists v_{11}, \ldots, v_{1k}, \ldots v_{n1}, \ldots v_{nk}$$
$$\bigwedge_{\{v_i,v_j\}\in E(G_S)} ((v_{i1} - v_{j1})^2 + \cdots + (v_{ik} - v_{jk})^2) \leq 1 \wedge$$
$$\bigwedge_{\{v_i,v_j\}\notin E(G_S)} -((v_{i1} - v_{j1})^2 + \cdots + (v_{ik} - v_{jk})^2) < -1$$

This sentence directly models the pairwise distance conditions as given by the $k$-sphere graph $G$.

Now we show that $k$-SPHERICITY is $\exists\mathbb{R}$-hard. This is accomplished by the reduction from $k$-REALIZABILITY due to Kang and Müller [6]. Given a set $S \subseteq \{0, 1\}^n$ we aim to construct a graph $G_S = (V_S, E_S)$ such that $G_S$ is a $k$-sphere graph, if and only if there is a $k$-hyperplane arrangement with $S \subseteq \mathcal{D}(\mathcal{H})$.

The Graph $G_S$ has $2 \cdot n + |S|$ vertices and is defined as follows:

$$\begin{aligned} A &= \{a_1, ..., a_n\} \\ B &= \{b_1, ..., b_n\} \\ C &= \{c_\sigma \mid \sigma \in S\} \\ V_S &= A \cup B \cup C \end{aligned}$$

The following edges are in $G_S$.

$$\begin{aligned} \{a_i, a_j\}, \{b_i, b_j\} &\in E_S & 1 \le i < j \le n \\ \{a_i, b_j\} &\notin E_S & 1 \le i, j \le n \\ \{c_\sigma, c_\tau\} &\in E_S & \sigma \neq \tau \in S \\ \{a_i, c_\sigma\} &\in E_S & \sigma_i = - \\ \{b_i, c_\sigma\} &\in E_S & \sigma_i = + \end{aligned}$$

It can easily be seen that this transformation can be carried out in a time that is polynomial in $|S|$. Now it remains to be shown that the constructed graph $G_S$ is indeed a $k$-sphere graph if and only if $S \subseteq \mathcal{D}(\mathcal{H})$ for some $k$-hyperplane arrangement $\mathcal{H}$.

If $G_S$ is realizable in $\mathbb{R}^k$, then there exist points

$$\begin{aligned} & p(a_1), \ldots, p(a_n), \\ & p(b_1), \ldots, p(b_n) \\ \text{and } & p(c_\sigma), \sigma \in S \end{aligned}$$

in $\mathbb{R}^k$ realizing $G_S$.

Now we aim to define an arrangement of hyperplanes $\mathcal{H} = (h_1, \ldots, h_n)$ with $S \subseteq \mathcal{D}(\mathcal{H})$. The hyperplanes are defined as follows:

$$\begin{aligned} h_i &= \{z \in \mathbb{R}^k \mid ||z - p(a_i)|| = ||z - p(b_i)||\} \\ h_i^- &= \{z \in R^k \mid ||z - p(a_i)|| < ||z - p(b_i)||\} \\ h_i^+ &= \{z \in R^k \mid ||z - p(a_i)|| > ||z - p(b_i)||\} \end{aligned} \tag{3.1}$$

Let $\sigma \in S$. For $i \in \mathbb{N}$ with $\sigma_i = -$ we have $p(c_\sigma) \in B(p(a_i), 1) \setminus B(p(b_i), 1)$ by the construction of the graph. This particularly implies that $p(c_\sigma)$ lies closer to $p(a_i)$ than to $p(b_i)$. Since $h_i$ is equidistant to $p(a_i)$ and $p(b_i)$, this implies that $p(c_\sigma) \in h_i^-$. The analogous case holds for $i \in \mathbb{N}$ with $\sigma_i = +$. Thus for all $\sigma \in S$ the point $p(c_\sigma)$ has sign vector $\sigma$ with regard to $\mathcal{H}'$ and $S \subseteq \mathcal{D}(\mathcal{H}')$ holds as desired.

Now we consider the other direction of the proof of the construction's correctness. Given a $k$-hyperplane arrangement with $S \subseteq \mathcal{D}(\mathcal{H})$, we aim to construct a set of points whose

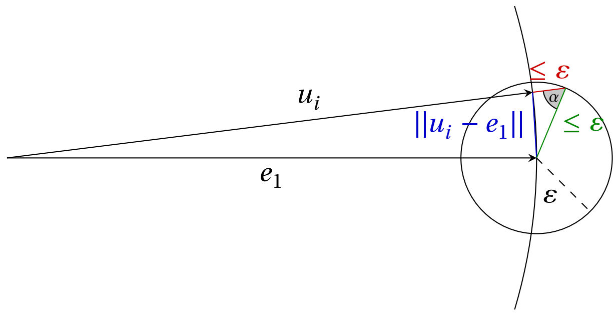



Figure 3.4: Illustration of the bound for $||u_i - e_1||$. The red line is $||v_i - u_i||$ and the green line is $||v_i - e_1||$.

$k$-sphere graph is isomorphic to $G_S$. By Lemma 2 we can assume that $\mathcal{H}$ has the form as specified there. We set $\varepsilon = \frac{1}{1000}$ and $K = 1000$. The ball $B((K, 0, \ldots, 0)^T, \varepsilon)$ which intersects with all $k$-facets is denoted with $B$. Then by the afore mentioned lemma there are vectors $v_i$ that are orthogonal to $h_i$ with $||v_i - e_1|| \leq \varepsilon$. In the following proof we will need normed vectors and thus set $u_i = \frac{v_i}{||v_i||}$.

Moreover, from $||v_i - e_1|| \leq \varepsilon$ it follows that $||u_i - e_1|| \leq 2\varepsilon$. To see this we first consider $||v_i||$ and apply the inverse triangle inequality.

$$\begin{aligned} | \, ||v_i|| - ||e_1|| \, | &\leq ||v_i - e_1|| \\ &\leq \varepsilon \\ \implies 1 - \varepsilon \leq &||v_i|| \leq 1 + \varepsilon \\ &\implies ||u_i - v_i|| \leq \varepsilon \end{aligned}$$

The three vectors we consider are $u_i$, $v_i$ and $e_1$. Since $u_i$ is $v_i$ scaled we basically consider the directions of two vectors. If $u_i = e_1$ we are done. Otherwise we can consider the situation in $\mathbb{R}^2$ since the affine span of two nonparallel vectors is a plane.

Now we consider the situation as it can be seen in Figure 3.4. We want to determine an upper bound for $||u_i - e_1||$. Let $\alpha$ be the angle as indicated in the figure. It can be seen that $0 \leq \alpha \leq \frac{\pi}{2}$ and thus $0 \leq \cos(\alpha) \leq 1$. Now we can apply the law of cosines to determine the upper bound:

$$\begin{aligned} ||u_i - e_1||^2 &= ||u_i - v_i||^2 + ||v_i - e_1||^2 - 2||u_i - v_i|| \cdot ||v_i - e_1|| \cdot \cos(\alpha) \\ &\overset{0 \leq \cos(\alpha) \leq 1}{\leq} ||u_i - v_i||^2 + ||v_i - e_1||^2 \\ &\leq 2\varepsilon^2 \\ \implies ||u_i - e_1|| &\leq \sqrt{2}\varepsilon \leq 2\varepsilon \end{aligned}$$

With this knowledge we can now give a description of the points that realize $G_S$. First, we construct points $w(x)$, $x \in V$. These $w(x)$ will be the base for the points $p(x)$. For each $\sigma$ we choose a point $w(c_\sigma) \in B$ with $\sigma(w(c_\sigma)) = \sigma$. It can be shown that such a point exists for each $\sigma$ [6]. Now we select one point $p_i \in h_i \cap B$ in each hyperplane. Based on these $p_i$ the $w(a_i)$ and $w(b_i)$ will be constructed. For a given $r \geq 1 \in \mathbb{R}$ we set:

$$
\begin{aligned}
w^+_{i,r} &= p_i + r \cdot u_i & \qquad B^+_{i,r} &= B(w^+_{i,r}, r)\\
w^-_{i,r} &= p_i - r \cdot u_i & \qquad B^+_{i,r} &= B(w^+_{i,r}, r)
\end{aligned}
$$

With increasing $r$ each of the $B^+_{i,r}$ covers its $h^+_i$ and each $B^-_{i,r}$ covers its $h^-_i$. Now set $r$ large enough, such that all $w(c_\sigma)$ are inside of the corresponding balls and we set $w(a_i) = w^-_{i,r}$ and $w(b_i) = w^+_{i,r}$. Finally, scale all constructed points by a factor of $\frac{1}{r}$ to $p(x) = \frac{w(x)}{r}$.

It remains to be shown that the $k$-sphere graph of these points is indeed the graph $G_S$. The edges $\{a_i, c_\sigma\}$, $\{b_i, c_\sigma\}$ are in the graph by the choice of $r$.

Now consider the edges $\{a_i, a_j\}$ and analogously $\{b_i, b_j\}$:

$$
\begin{aligned}
||v(a_i) - v(a_j)|| &= \frac{||(p_i + ru_i) - (p_j + ru_j)||}{r}\\
&= \frac{||(p_i - p_j) + r(u_i - u_j)||}{r}\\
&\leq \frac{2\varepsilon}{r} + ||u_i - u_j||\\
&\leq \frac{2\varepsilon}{r} + ||(u_i - e_1) - (u_j - e_1)||\\
&\leq \frac{2\varepsilon}{r} + 4\varepsilon\\
&<^* 1
\end{aligned}
$$

Next consider the $c_\sigma$. Since the $w(c_\sigma)$ were chosen to lie inside $B$, we have:

$$
\begin{aligned}
||v(c_\sigma) - v(c_\tau)|| &= \frac{||w(c_\sigma) - w(c_\tau)||}{r}\\
&\leq \frac{2\varepsilon}{r}\\
&<^* 1
\end{aligned}
$$

Finally, we show that there are no additional edges. Edges $\{a_i, c_\sigma\}$, $\sigma_i = +$ and $\{b_i, c_\sigma\}$, $\sigma_i = -$ cannot exist by construction. So we only consider edges $\{a_i, b_i\}$:

$$
\begin{aligned}
\|v(a_i) - v(b_i)\| &= \frac{\|(p_i - p_j) + r(u_i + u_j)\|}{r} \\
&= \frac{\| - r(u_i + u_j) - (p_j - p_i)\|}{r} \\
&\geq \left| \|u_i + u_j\| - \frac{\|p_j - p_i\|}{r} \right| \\
&\geq \| - u_i - u_j\| - \frac{2\varepsilon}{r} \\
&\geq \| - u_i - u_i\| - \|u_j - u_i\| - \frac{2\varepsilon}{r} \\
&\geq 2 - 4\varepsilon - \frac{2\varepsilon}{r} \\
&>^* 1
\end{aligned}
$$

The inequalities marked with a $*$ hold for all $r > \frac{2\varepsilon}{1-4\varepsilon}$. Since $\varepsilon = \frac{1}{1000}$ and $r \geq 1$, this condition is satisfied by construction.

Thus all edges of the $k$-sphere graph for the newly constructed points are exactly the same as the edges of $G_S$. This concludes the proof that $k$-SPHERICITY is $\exists\mathbb{R}$-complete. □

# 4 Generalized transmission graphs of convex sets

This chapter covers the recognition of transmission graphs of $k$-balls and regular $m$-gons.

## 4.1 Transmission graphs of $k$-balls

We will now consider the complexity of recognizing generalized transmission graph of arrangements of $k$-balls. Let $B$ be such a $k$-ball, then we denote by $p(B)$ the center of the ball and by $r(B)$ the radius of $B$.

**Theorem 3.** *The recognition of generalized transmission graphs of $k$-balls is complete for $\exists\mathbb{R}$.*

*Proof.* First, we show that the recognition of generalized transmission graphs of $k$-balls lies in $\exists\mathbb{R}$. Let $G = (V, E), V = \{v_1, ..., v_n\}$ be a directed graph. We construct the following sentence:

$$
\begin{aligned}
&\exists v_{11}, v_{12}, \ldots, v_{1k}, \ldots v_{n1}, \ldots v_{nk},\\
&\quad r_1, \ldots, r_n\\
&\quad \bigwedge_{(v_i,v_j)\in E} (v_{i1} - v_{j1})^2 + \cdots + (v_{ik} - v_{jk})^2 \le r_i^2\\
&\quad \bigwedge_{(v_i,v_j)\notin E} -((v_{i1} - v_{j1})^2 + \cdots + (v_{ik} - v_{jk})^2) < -r_i^2
\end{aligned}
$$

This construction can be carried out in polynomial time and results in a sentence of polynomial size. The sentence is true if and only if $G$ is realizable as an arrangement of $k$-balls since the inclusion constraint is directly stated by the inequalities.

Now we show that the recognition of generalized transmission graphs of $k$-balls is hard for $\exists\mathbb{R}$. The proof uses a reduction from $k$−Realizability. This reduction closely follows the reduction given in the proof of Theorem 2.

Given the set $S \subseteq \{0, 1\}^n$ of sign vectors with $(-, \ldots, -), (+, \ldots, +) \in S$, a directed graph $G_S = (V_S, E_S)$ is constructed with the property that there is a hyperplane arrangement $\mathcal{H}$ with $\mathcal{D}(\mathcal{H}) \subseteq S$ if and only if $G_S$ can be realized by $k$-balls. The graph is basically the same graph as in the proof of Theorem 2 but each undirected edge is represented by two directed edges. $G_S$ consists of $2n + |S|$ vertices and is defined as follows:

$$
\begin{aligned}
A &= \{a_1, \ldots, a_n\} \\
B &= \{b_1, \ldots, b_n\} \\
C &= \{c_\sigma \mid \sigma \in S\} \\
V_S &= A \cup B \cup C
\end{aligned}
$$

The graph has the following edges:

$$
\begin{aligned}
(a_i, a_j), (b_i, b_j) &\in E_S && 1 \le i, j \le n, i \ne j \\
(a_i, b_j), (b_i, a_j) &\notin E_S && 1 \le i, j \le n \\
(c_\sigma, c_\tau) &\in E_S && \sigma \ne \tau \in S \\
(a_i, c_\sigma), (c_\sigma, a_i) &\in E_S && \sigma_i = - \\
(b_i, c_\sigma), (c_\sigma, b_i) &\in E_S && \sigma_i = +
\end{aligned}
$$

It is easy to see that this construction can be carried out in polynomial time. Now it is left to show the correctness of this construction.

First, we consider a given arrangements of $k$-balls with various radii that realize the graph. The ball associated with a vertex $u \in V$ is denoted by $u$. For each pair $p(a_i)$, $p(b_i)$ one hyperplane $h_i$ separating these points is constructed.

There are two cases to be considered, in the first case the $k$-balls $a_i$ and $b_i$ do not intersect or touch in one point. The hyperplane $h_i$ is chosen to separate $a_i$ and $b_i$. Such a separating hyperplane exists by the hyperplane separation theorem (Theorem 1).

In the second case the $k$-balls intersect. The intersection of the $(k-1)$-spheres surrounding the balls form a $(k-2)$ sphere $s$. The hyperplane $h_i$ is set to the affine span of $s$. It is oriented in such a way that $p(a_i) \in h_i^-$ and $p(b_i) \in h_i^+$. This is well defined since $p(a_i) \notin B(p(b_i), r(b_i))$ and $p(b_i) \notin B(p(a_i), r(a_i))$ by the definition of $G_S$. See Figure 4.1 for an example of this construction in $\mathbb{R}^2$.

Now it has to be shown for $\mathcal{H} = (h_1, ..., h_n)$ that $S \subseteq \mathcal{D}(\mathcal{H})$. Fix a $\sigma \in S$ with $\sigma_i = -$ for $i \in \mathbb{N}$. By the construction of the graph, we have $p(c_\sigma) \in a_i \setminus b_i$. This area is located in $h_i^-$ by construction. The analogous case holds for $i$ with $\sigma_i = +$. Thus $p(c_\sigma)$ has signature $\sigma$ and it holds that $S \subseteq \mathcal{D}(\mathcal{H})$.

Now it has to be shown that given a hyperplane arrangement $\mathcal{H}$ with $S \subseteq \mathcal{D}(\mathcal{H})$, there are $2n + |S|$ points $p(u)$ in $\mathbb{R}^k$ whose generalized transmission graph is $G_S$. Since $G_S$, as defined above, has two directed edges for each undirected edge of the graph from the proof of Theorem 2, the construction and the argumentation for the correctness from that proof can be reused. This yields an arrangement on $k$-spheres where all radii are 1. □

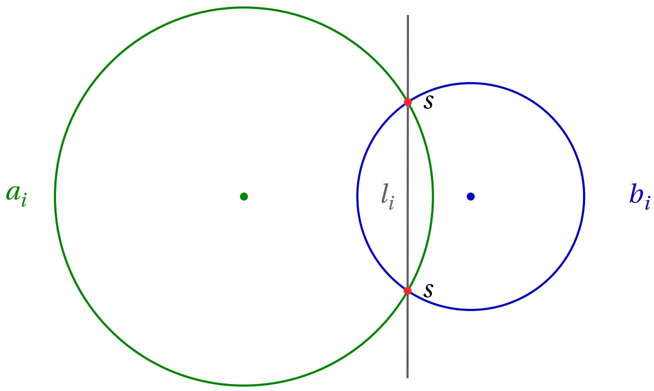



Figure 4.1: Construction of the hyperplanes in the intersecting case

## 4.2 Transmission graphs of regular polygons

In this section we consider the transmission graphs of regular convex polygons. The polygons are restricted to have $m \geq 6$ vertices. The exact form of a polygon $x$ is given by the center points $p(x)$, the radius $r(x)$ of the inscribed circle of $x$ and a vector $u(x)$ that is orthogonal to one of the edges. By $P_m(p, r, u)$ we denote the regular polygon $x$ with $m$ vertices, center point $p$, radius $r$ of the inscribed circle and the vector $u$ orthogonal to one edge. The directionality of $u$ points towards $p$. The parameters are depicted in Figure 4.2.

In Theorem 4 we will show that the recognition problem of generalized transmission of regular polygons for $m \geq 6$ is complete for $\exists\mathbb{R}$. There are no known results for $m < 6$ since the result of Lemma 4 suggests that the proof in Theorem 4 cannot be easily modified to match the case $m < 6$.

In order to show the main result of this section we need the following lemma. It will allow us to uniquely define the line arrangement when given an arrangement of polygons.

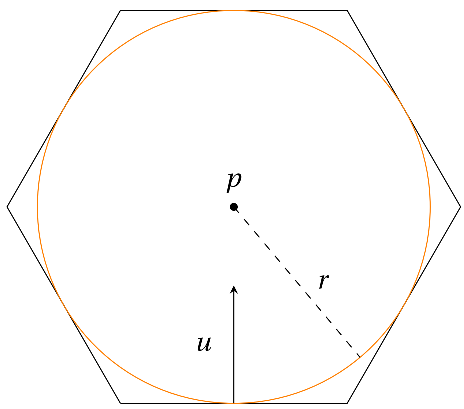



Figure 4.2: Parameters of the polygon

**Lemma 4.** *Let $m \geq 6$ be fixed and let $P_1$ and $P_2$ be two regular convex $m$-gons. Moreover assume that $P_1 \cap P_2 \neq \emptyset$, while $p(P_1) \notin P_2$ and $p(P_2) \notin P_1$.*
*Then $I = \partial(P_1) \cap \partial(P_2)$ has one of the following shapes:*

1. *$I$ consists of one connected component with infinitely many collinear points*
2. *$I$ consists of one touching point, or*
3. *$I$ consists of two crossing points*

*Proof.* In the following we will distinguish between *inner* and *outer* touching points. A touching point of $P_1$ and $P_2$ is called an outer touching point, if the polygonal chains of one polygon incident to the point both lie outside the other polygon. Otherwise the touching point is called an inner touching point. We divide the proof into two cases:

#### Case one: $I$ contains infinitely many points

If there are infinitely many intersection points, then at least one edge $d \in P_1$ and one edge $e \in P_2$, with $d$ and $e$ lying on the same line $l$, are involved in $I$. If $P_1$ and $P_2$ lie on different sides of $l$ there is only one component, since the polygons are convex.

Now it is left to show that $P_1$ and $P_2$ cannot lie on the same side of $l$. If $I$ consists of one component and $P_1$ and $P_2$ lie on the the same side of $l$, then we either have $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$. This is not possible since it implies $p(P_1) \in P_2$ or $p(P_2) \in P_1$.

Assume that $I$ consists of two components. If the second component is an inner touching point or consists of infinitely many points, one polygon would still be included in the other and the contradiction from above holds. Hence there has to be at least one crossing point.

W.l.o.g. let $P_1$ and $P_2$ be located such that $d$ and $e$ are horizontal edges. Moreover let $r_1$ and $r_2$ be the radii of the inscribed circles of $P_1$ and $P_2$ and assume that $r_1 \geq r_2$. Denote by $C_1$ the inscribed circle of $P_1$ and by $l_1$ the length of the edges in $P_1$. Let $P_2 \setminus P_1$ be located to the right of $P_1$.

Now define $x_1$ as the rightmost endpoint of $d$ and $x_2$ as the leftmost endpoint of $e$. Furthermore denote by $h$ the horizontal line through $p(P_1)$ and by $a$ the bisector of the angle at $x_2$. Note that the rightmost endpoint of $e$ cannot lie on $d$ since otherwise the perpendicular bisector of $d$ and thus the center of $P_2$ would be fully included in $P_1$.

Since $P_2$ is smaller than $P_1$, the center $p(P_2)$ lies below $h$. Since $a$ is the bisector of an angle, the center point lies on $a$. Now we consider the line segment $b$ through $x_2$ and the rightmost point of $h \cap C_1$. This line segment is completely included in $P_1$. If the angle $\beta$ between $b$ and the horizontal line through $d$ is smaller than the angle $\alpha$ between $a$ and the same horizontal line, then $p(P_2)$ is always included in $P_1$. The parameters of this situation are depicted in Figure 4.3.

We will now show that $\beta < \alpha$ holds for all regular polygons with $m \geq 7$ vertices. The angle $\alpha$ is fixed at $\frac{(m-2)\pi}{2m}$, since $P_1$ and $P_2$ are regular convex polygons. Our goal is now to find an upper bound for $\beta$. Consider the situation in the yellow rectangle, which is depicted in more detail in Figure 4.4. As long as $x_2$ lies strictly to the left of the vertical line , it can be seen that $\beta < \beta'$.

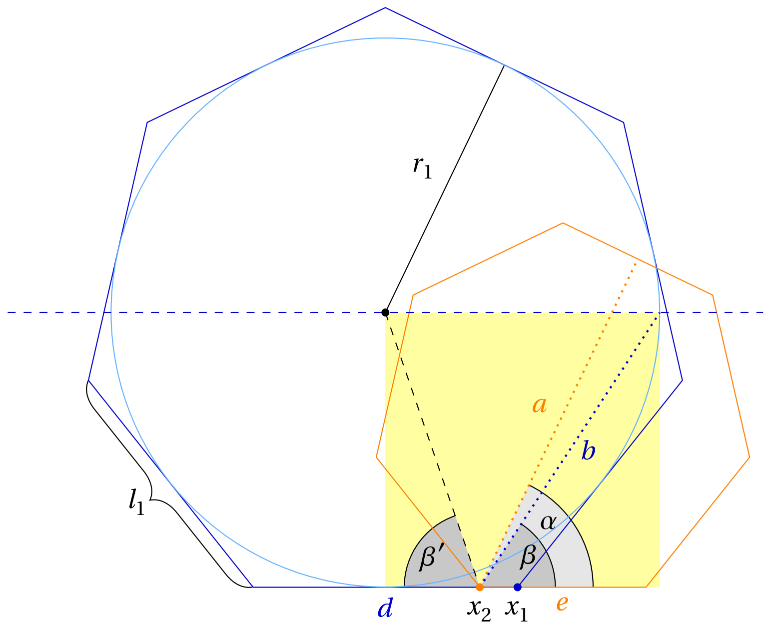



Figure 4.3: Situation for $P_1$ and $P_2$

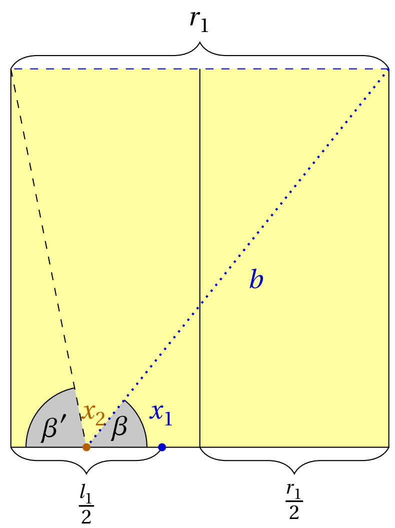



Figure 4.4: If $l_1$ is smaller than $\frac{r_1}{2}$ it holds that $\beta \leq \beta'$.

When $x_2$ is moved to the right, $\beta$ is getting larger and $\beta'$ is getting smaller. The largest possible value for $\beta$ (and accordingly the smallest possible value for $\beta'$) is reached, if $x_2 = x_1$. In this case, from the properties of regular convex polygons, the following is known:

$$\begin{aligned} \beta' &= \frac{\pi}{2} - \frac{\pi}{m} \\ &= \frac{(m-2)\pi}{2m} \end{aligned} \tag{4.1}$$

We already established that $\beta < \beta'$, if $x_2$ lies to the left of the vertical line. If $x_1$ also lies to the left of the vertical line, (4.1) implies that $\beta < \frac{(m-2)\pi}{2m}$. It can be seen that $x_1$ lies to the left of the vertical line, if $l_1 < r_1$. When combining this upper bound with the known value for $\alpha$ we get:

$$\beta < \frac{(m-2)\pi}{2m} = \alpha$$

Now we show that $r_1 < l_1$ holds for $m \geq 7$. From the properties of a regular polygon we know that:

$$\begin{aligned} l_1 &= 2 \cdot r_1 \cdot \tan\left(\frac{\pi}{m}\right) \\ \frac{l_1}{r_1} &= 2\tan\left(\frac{\pi}{m}\right) \end{aligned}$$

If $r_1 < l_1$ this implies:

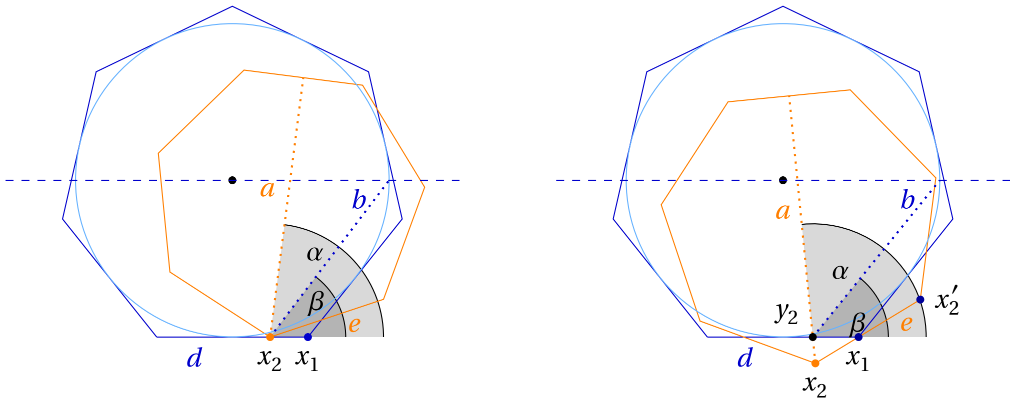
$$2\tan\left(\frac{\pi}{m}\right) < 1$$

This holds for $m \geq 7$. Thus up to now we have proved that for $m \geq 7$ there is at most one component in $I$.

Now we consider the special case $m = 6$. Here $b$ is replaced by the other edge of $P_1$ at $x_1$. In this case and the following inequalities hold:

$$\begin{aligned} \beta &\leq \pi - \frac{(m-2)\pi}{m} \\ &= \frac{2\pi}{6} \\ \alpha &= \frac{(m-2)\pi}{2m} \\ &= \frac{2\pi}{6} \end{aligned}$$

Equality in the first case only holds if $x_1 = x_2$ and then there are exactly two intersection point. This concludes the proof of the form of $I$ if there are infinitely many intersection points.




(a) The vertex creating the touching point belongs to the smaller polygon.
(b) The vertex creating the touching point belongs to the larger polygon.

Figure 4.5: Situations for one touching point

**Case two: $I$ contains finitely many points**

Our goal in this case is to show that $|I| > 2$ is not possible. This is done by first reducing all cases with an inner touching point to the case with infinitely many intersection points and then reducing the case with more than two crossing points to a case with an inner intersection point.

If there is an inner touching point there have to be at least two crossing points. Otherwise, one polygon would be completely included in the other polygon.

Note that at an inner touching point there is one edge of one polygon and one vertex of the other polygon involved. As in the first case, assume that $r_1 \geq r_2$.

Now there are two cases to consider. In the first case, one vertex $x_2$ of $P_2$ touches one edge $d$ of $P_1$. Let the situation rotated and mirrored in such a way that $d$ is horizontal and that $x_2$ is located right of the vertical line through $p(P_1)$. Let $e$ be the edge incident to $x_2$ in counter-clockwise direction. As in case one, let $b$ be the line segment connecting $x_2$ with the rightmost point of $h \cap C_1$ and let $a$ be the bisecting line segment at $x_2$. A schematic of the situation with all parameters can be found in Figure 4.5(a).

Now we rotate $P_2$ around $x_2$ in clockwise direction until $e$ and $d$ are collinear. This is exactly the situation as in the first case and we already saw that this case is not possible. As $\alpha$ is getting smaller when rotating $P_2$ we cannot move $p(P_2)$ into $P_1$ with this rotation.

In the second case, one vertex $x_1$ of $P_1$ induces the touching point. W.l.o.g. let $x_1$ be the right endpoint of $d$ and $x_2$ be the endpoint in clockwise direction of the edge of $P_2$ that is touched by $x_1$. Furthermore, let $y_2$ be the intersection point of $a$ and $d$.The situation can be seen in Figure 4.5(b). We rotate $P_2$ in clockwise direction around $x_1$ until $d$ and $e$ are collinear. This creates the same situation as in the case with infinitely many intersection points.

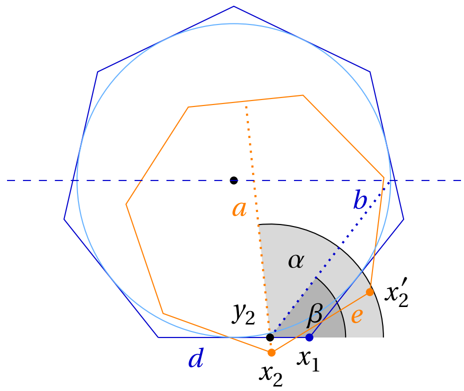



Figure 4.6: Situation with for intersection points

Now we have to show that this rotation is not moving the center of $P_2$ into $P_1$. Let $x_2'$ be the other vertex of $e_1$. The center of $P_2$ lies on the bisector through $x_2'$, while all points on $a$ that lie outside of $P_1$ before the rotation lie under the horizontal line through $x_2'$.

As for $m \geq 6$ the bisector is steeper than the horizontal line, the center point can not be moved into the polygon by the rotation. That the bisector is steeper can be seen as follows: The angle between the horizontal line through $x_2'$ and $e$ is smaller than $\frac{2\pi}{m}$. The angle between $e$ and the bisector is $\frac{(m-2)\pi}{2}$. This leads to

$$\begin{aligned} \frac{2\pi}{m} &\leq \frac{(m-2)\pi}{2m} \\ \Leftrightarrow\ m &\geq 6 \end{aligned}$$

which shows the claim.

Now we consider a situation with four crossing points. Let $d$ be one of the edges of $P_1$ that contains an intersection point. Let $e$ be the rightmost edge that crosses $d$ and let $x_2$ be the vertex of $e$ that lies beneath $d$. Let $a$ be the bisector at $x_2$ and let $y_2$ be the intersection point of $a$ and $d$.

Since there are four intersection points, the boundary of $P_2$ is subdivided into four polygonal chains. Two of these lie inside of $P_1$ and two lie outside of $P_1$. Let the outer polygonal chain not containing $x_2$ be called $Q_2$ and let it have at least one vertex to the right of $x_2$.

Let $x_2'$ be the lowest vertex of $Q_2$. Since $P_2$ is the smaller of the polygons, $x_2'$ is the second vertex of $e$. This also implies the existence of $y_2$. This situation can be seen in Figure 4.6.

Now $P_2$ is rotated around $y_2$ in counter clockwise direction until $x_1$ touches $e$. The angle $\alpha$ between the bisector and the horizontal line is only getting smaller. Additionally, the part of $a$ that lies outside of $P_1$ stays unchanged, thus the center point of $P_2$ cannot be rotated into the polygon. We have transformed the situation with four intersection points into one with an inner touching point. As we already showed that no inner touching point is possible, this concludes the proof that for $m \geq 6$ the form of $I$ is as stated in the lemma.

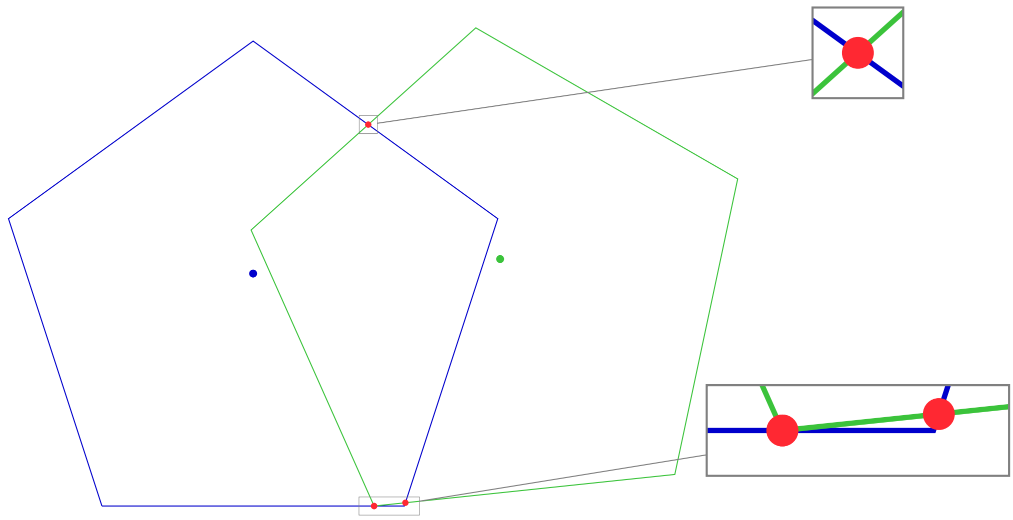


Figure 4.7: Two regular 5-gons with 3 intersection points. A fourth intersection point can be created by moving the touching point down, the situation with one infinitely large component and a crossing point can be created by rotating the green polygon.

It can be seen that the bound of $m \geq 6$ is sharp. The counterexample for $m = 5$ can be found in Figure 4.7. □

With the help of this lemma we can now show the following theorem:

**Theorem 4.** *For any fixed $m$, the recognition problem of generalized transmission graphs of regular $m$-gons with $m \geq 6$ is complete for $\exists\mathbb{R}$.*

*Proof.* First, we show that the recognition problem lies in $\exists\mathbb{R}$. Given a directed graph $G = (V, E), V = \{v_1, \dots v_n\}$, we want to construct a sentence in the existential theory of the reals that is true if and only if $G$ is realizable as a generalized transmission graph of regular $m$-gons.

Each convex polygon can be defined as the intersection of finitely many halfplanes. In order to check whether a given point $p$ lies inside a polygon $P$, it suffices to check if $p$ lies in all halfplanes that define the polygon. In our construction we are given a vector that is orthogonal to one edge. The orthogonal vectors for the other edges can be calculated by rotating $v$ by $\frac{2\pi k}{m}$, $1 \leq k < m$.

In our sentence this rotation is executed by explicitly applying the rotation matrix. In order to do this, we need constants of the form $(\sin(\frac{2\pi k}{m}))$ and $(\cos(\frac{2\pi k}{m}))$. By Lemma 1 these constants can be used in our sentence. The inclusion of $p$ in the halfplane of a line with orthogonal vector $v$ and a point $x$ on the line can be expressed as follows:

$$\begin{aligned} & v \cdot (p - x) \geq 0 \\ \Leftrightarrow \quad & v \cdot p \geq v \cdot x \end{aligned} \tag{4.2}$$

With this knowledge we can now construct the following sentence:

$$
\begin{aligned}
&\exists p_{1x}, p_{1y}, p_{2x}, p_{2y}, \dots p_{nx}, p_{ny},\\
&\ u_{1x}, u_{1y}, u_{2x}, u_{2y}, \dots u_{nx}, u_{ny},\\
&\ r_1, \dots r_n\\
&\ \bigwedge_{(1\le i\le n)} \qquad u_{ix}^2 + u_{iy}^2 = r_i^2\\
&\ \bigwedge_{((v_i,v_j)\in E_S)} \bigwedge_{(0\le k<m)} \left(u_{ix}\cdot\cos\left(\frac{2\pi k}{m}\right) - u_{iy}\cdot\sin\left(\frac{2\pi k}{m}\right)\right)\cdot p_{jx}\\
&\qquad + \left(u_{ix}\cdot\sin\left(\frac{2\pi k}{m}\right) + u_{iy}\cdot\cos\left(\frac{2\pi k}{m}\right)\right)\cdot p_{jy}\\
&\qquad \ge \left(u_{ix}\cdot\cos\left(\frac{2\pi k}{m}\right) - u_{iy}\cdot\sin\left(\frac{2\pi k}{m}\right)\right)\\
&\qquad \cdot\left(p_{ix} - \left(u_{ix}\cdot\cos\left(\frac{2\pi k}{m}\right)\right) - u_{iy}\cdot\sin\left(\frac{2\pi k}{m}\right)\right)\\
&\qquad + \left(u_{ix}\cdot\sin\left(\frac{2\pi k}{m}\right) + u_{iy}\cdot\cos\left(\frac{2\pi k}{m}\right)\right)\\
&\qquad \cdot\left(p_{iy} - \left(u_{ix}\cdot\sin\left(\frac{2\pi k}{m}\right)\right) + u_{iy}\cdot\cos\left(\frac{2\pi k}{m}\right)\right)\\
&\ \bigwedge_{((v_i,v_j)\notin E_S)} \bigvee_{(0\le k<m)} \left(u_{ix}\cdot\cos\left(\frac{2\pi k}{m}\right) - u_{iy}\cdot\sin\left(\frac{2\pi k}{m}\right)\right)\cdot p_{jx}\\
&\qquad + \left(u_{ix}\cdot\sin\left(\frac{2\pi k}{m}\right) + u_{iy}\cdot\cos\left(\frac{2\pi k}{m}\right)\right)\cdot p_{jy}\\
&\qquad \le \left(u_{ix}\cdot\cos\left(\frac{2\pi k}{m}\right) - u_{iy}\cdot\sin\left(\frac{2\pi k}{m}\right)\right)\\
&\qquad \cdot\left(p_{ix} - \left(u_{ix}\cdot\cos\left(\frac{2\pi k}{m}\right)\right) - u_{iy}\cdot\sin\left(\frac{2\pi k}{m}\right)\right)\\
&\qquad + \left(u_{ix}\cdot\sin\left(\frac{2\pi k}{m}\right) + u_{iy}\cdot\cos\left(\frac{2\pi k}{m}\right)\right)\\
&\qquad \cdot\left(p_{iy} - \left(u_{ix}\cdot\sin\left(\frac{2\pi k}{m}\right)\right) + u_{iy}\cdot\cos\left(\frac{2\pi k}{m}\right)\right)
\end{aligned}
$$

The first $\wedge$-expression ensures that the length of the orthogonal vector $u_i = (u_{ix}, u_{iy})$ is exactly $r_i$. The second set of $\wedge$-expressions ensures that (4.2) is satisfied by rotating $u_i$ and explicitly calculating the dot product. Finally the third $\wedge$ expression enforces that no other center points are included in any polygon.

Now we show that the recognition problem is $\exists\mathbb{R}$-hard. The proof is carried by a reduction from 2-Realizability. We can reuse the reduction from the proof of Theorem 3 and define

$G_S = (V_S, E_S)$ in the same way:

$$
\begin{aligned}
A &= \{a_1, \ldots, a_n\} \\
B &= \{b_1, \ldots, b_n\} \\
C &= \{c_\sigma \mid \sigma \in S\} \\
V_S &= (A \cup B \cup C)
\end{aligned}
$$

$$
\begin{aligned}
(a_i, a_j), (b_i, b_j) &\in E_S && 1 \le i, j \le n, i \ne j \\
(a_i, b_j), (b_i, a_j) &\notin E_S && 1 \le i, j \le n \\
(c_\sigma, c_\tau) &\in E_S && \sigma \ne \tau \in S \\
(a_i, c_\sigma), (c_\sigma, a_i) &\in E_S && \sigma_i = - \\
(b_i, c_\sigma), (c_\sigma, b_i) &\in E_S && \sigma_i = +
\end{aligned}
$$

First, we show that given a realization of $G_S$ with regular $m$-gons we can construct lines $\mathcal{L} = (l_1, \ldots l_n)$ with $S \subseteq \mathcal{D}(\mathcal{L})$. To construct $l_i$ we consider the polygons $a_i$ and $b_i$.

There are three cases to consider. In the first case, $a_i$ and $b_i$ do not intersect. Then, by the hyperplane separation theorem (Theorem 1) there is a line separating the polygons.

If $a_i$ and $b_i$ touch in one point or in one edge each, $l_i$ is constructed through the touching point or is the line defined by the edge containing the touching point.

If the boundaries of the polygons intersect in more than one but finitely many points, we know from Lemma 4 that they have a crossing intersection in exactly two points. Then $l_i$ is defined by these two intersection points.

With Lemma 4 we can now construct the $l_i$ through the intersection points. The $l_i$ satisfy the description of $S$ by the same argument as in the proof of Theorem 3.

Now we consider the second direction of the reduction. Given an arrangement of lines realizing the condition of Lemma 2 with $\varepsilon = \frac{1}{1000}$ and $K = 1000$, we call the disk containing all intersection points $D$. Let $u_i = \frac{v_i}{||v_i||}$. As in the proof of Theorem 3 we choose points $w(c_\sigma) \in D$ for each $\sigma \in S$ and points $p_i \in l_i \cap D$ on each line. Analogously to the proof of Theorem 3 we define:

$$
\begin{aligned}
w^+_{i,r} &= p_i + r \cdot u_i & w^-_{i,r} &= p_i - r \cdot u_i \\
P^+_{i,r} &= P(w^+_{i,r}, r, u_i) & P^-_{i,r} &= P(w^-_{i,r}, r, -u_i)
\end{aligned}
$$

This definition ensures that one edge of each $P_{i,r}$ touches $l_i$. Now choose $r$ large enough such that all $w(c_\sigma)$ are inside the inscribed circles of the polygons. Then set $w(a_i) = w^-_{i,r}$, $w(b_i) = w^+_{i,r}$ and $v(p) = \frac{w(p)}{r}$. This scales all polygons to have an inscribed circle of 1. The radii of the circumscribed circles are $\frac{1}{\cos \frac{\pi}{m}} \le \sqrt{2}$.

The proof of Theorem 2 gave the following results:

$$||v(c_\sigma) - v(c_\tau)|| < 1 \tag{4.3}$$

$$||v(a_i) - v(a_j)||, ||v(b_i), v(b_j)|| < 1 \tag{4.4}$$

$$||v(a_i) - v(b_j)|| \geq 2 - 4\varepsilon - \frac{2\varepsilon}{r} \tag{4.5}$$

Now (4.3) and (4.4) show that the points lie within the inscribed circle and thus lie within the polygons. Since for $\varepsilon = \frac{1}{1000}$ (4.5) is larger than $\sqrt{2}$, it follows that the points lie outside the circumscribed circle and thus outside the polygon. This concludes the proof. □

# 5 Generalized Transmission graphs of line segments and circular sectors

This chapter examines the complexity of recognizing generalized transmission graphs of line segments and circular sectors in $\mathbb{R}^2$. Circular sectors form a possible model for sector antennas with limited range and a given directionality.

## 5.1 Line segments

In order to define the generalized transmission graph of an arrangement of line segments, one point per line segment has to be singled out. This is reflected in the definition of a line segment as used in this section.

A line segment $l$ is characterized by an endpoint $p(l)$, a direction $u(l)$ and a length $r(l)$. When considering the generalized transmission graph of an arrangement $l_1, \ldots, l_n$ of line segments, there is an edge $(l_i, l_k)$ if $p(l_i) \in l_k$.

In this section we show that the recognition of generalized transmission graphs for line segments is complete for $\exists\mathbb{R}$. The $\exists\mathbb{R}$-hardness part of this proof is done by a reduction from SIMPLE-STRETCHABILITY.

**Theorem 5.** *The recognition of generalized transmission graphs of line segments is complete for* $\exists\mathbb{R}$.

*Proof.* First, we show that the problem lies in $\exists\mathbb{R}$. This can be achieved with the following construction Given a directed graph $G_L = (V_L, E_L)$, $|V_L| = n$, we can express its realizability with the following sentence:

$$
\begin{aligned}
&\exists x_1, \ldots, x_n, y_1, \ldots, y_n \\
&\quad r_1, \ldots, r_n \\
&\quad d_{ik} && 1 \leq i \neq k \leq n \\
&\quad u_{1x}, u_{1y}, \ldots, u_{nx}, u_{ny}
\end{aligned}
$$

$$
\begin{aligned}
& & u_{ix}^2 + u_{iy}^2 &= 1 & 1 \le i \le n \\
& & (x_i - x_j)^2 + (y_i - y_j)^2 &= d_{ik}^2 & 1 \le i \le k \le n \\
\wedge & & d_{ik} &\le r_i & (i,k) \in E_L \\
\wedge & & x_k &= x_i + d_{ik} \cdot u_{ix} & (i,k) \in E_L \\
\wedge & & y_k &= y_i + d_{ik} \cdot u_{iy} & (i,k) \in E_L \\
\wedge & & \big(d_{ik} &\ge r_i & (i,k) \notin E_L \\
\vee & & x_k &\ne x_i + d_{ik} \cdot u_{ix} & (i,k) \notin E_L \\
\vee & & y_k &\ne y_i + d_{ik} \cdot u_{iy}\big) & (i,k) \notin E_L
\end{aligned}
$$

The first inequality enforces that the vectors describing the directions have length one. The $d_{ik}$ describe the distance between $p_i = (x_i, y_i)$ and $p_k = (x_k, y_k)$. The remaining equations and inequalities enforce that if there is and edge $(i,k) \in E_L$, the distance $d_{ik}$ is smaller than the radius $r_i$ and the direction of $u_i = (u_{i1}, u_{i2})$ corresponds to the direction of $p_i - p_j$. The last three expressions enforce that $p_k$ does not lie on the line segment defined by $p_i$, $r_i$ and $u_i$ if there is no edge $(i,k)$ in the graph.

Now we show that the problem is hard for $\exists\mathbb{R}$. This will be done by a reduction from SIMPLE-STRETCHABILITY using the order description. Given a description $\mathcal{D}(\mathcal{L}) = O$, $O = (O^1, \ldots, O^n)$ of a simple line arrangement, we construct a Graph $G_L = (V_L, E_L)$ with $\mathcal{D}(\mathcal{L})$ realizable as a line arrangement if and only if $G_L$ is the generalized transmission graph of an arrangement of line segments.

We set $V_L = A \cup B \cup C$ with

$$
\begin{aligned}
A &= \{a_{\{i,k\}} & &\mid 1 \le i \ne k \le n\} \\
B &= \{b_k^i & &\mid 1 \le i \le n, 1 \le k \le n-1\} \\
C &= \{c_i & &\mid 1 \le i \le n\}
\end{aligned}
$$

where the $c_i$ are numbered in the same order as given by $\mathcal{D}(\mathcal{L})$. The $\{\ \}$ in the naming of the $a_{\{i,k\}}$ is to be read in such a way that $a_{\{i,k\}} = a_{\{k,i\}}$.

Before defining the edges, we describe the function of the different classes of vertices. The endpoints of the line segments associated with the vertices $a_{\{i,k\}}$ will enforce that there is an intersection of the line segments associated with $c_i$ and $c_k$. The endpoints of the line segments associated with the vertices $b_k^i$ will be placed between the $a_{i,k}$ on $c_i$ and thus enforce the order of the intersection. Now we define the edges accordingly:

$$
\begin{aligned}
(c_i, a_{\{i,k\}}) &\in E_L && 1 \le i \ne k \le n \\
(c_i, b^i_k) &\in E_L && 1 \le i \ne k \le n \\
(b^i_k, c_i) &\in E_L && 1 \le i \ne k \le n \\
(b^i_{o^i_k}, b^i_{o^i_l}) &\in E_L && 1 \le i \le n, 1 \le l < k \le n-1 \\
(b^i_{o^i_k}, a_{\{i,o^i_l\}}) &\in E_L && 1 \le i \le n, 1 \le l \le k \le n-1
\end{aligned}
$$

It can be seen that this construction can be computed in polynomial time. Now we have to show that $G_L$ is a transmission graph of line segments if and only if $\mathcal{D}(\mathcal{L})$ is realizable as a line arrangement.

First, we consider a line arrangement $\mathcal{L} = (l_1, \ldots, l_n)$ that realizes $\mathcal{D}(\mathcal{L})$. We show that then there exists an arrangement $C$ of lines segments realizing $G_L$.

Since $\mathcal{L}$ is a simple line arrangement, there is a disk $D$ containing all intersection points. We chose $D$ large enough such that the distance of $\partial(D)$ to each intersection point is positive.

Since the lines are ordered by slope, the circular order of intersections of $l_1, \ldots l_n$ with $\partial(D)$ is $l_1, \ldots, l_n, l_1 \ldots, l_n$. There is no vertical line in $\mathcal{L}$. Thus we can add a virtual vertical line $l'$ dividing the intersection points along $D$ into a "left" set $D_l = \{q^l_i \mid 1 \le i \le n\}$ and a "right" set $D_r = \{q^r_i \mid 1 \le i \le n\}$.

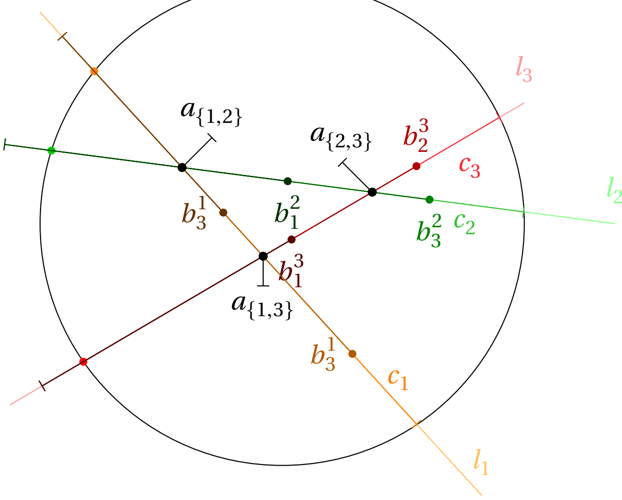
Choose $l_i \cap D$ as the representative for $c_i$ and $q^l_i$ as its endpoint $p(c_i)$. Now set $u(c_i)$ to $\overrightarrow{q^l_i q^r_i}$. The $a_{\{i,k\}}$ are constructed such that $p(a_{\{i,k\}})$ lies on the intersection point of $l_i$ and $l_k$. The direction and length are chosen in such a way that no other points are contained in $a_{\{i,k\}}$.

The line segments $b^i_{o^i_k}$ are placed along the line segment such that $p(b_{o^i_k})$ lies between $l_i \cap l_{o^i_k}$ and $l_i \cap l_{o^i_{k+1}}$ for $1 \le k \le n-2$. Furthermore place $p(b^i_{o_{n-1}})$ to the right of $l_i \cap l^i_{o_{n-1}}$. The directions of all $b^i_k$ are set to $-u(c_i)$ and the radii are set to $r(b^i_k) = \mathrm{dist}(p(b^i_k), p(c_i)) + \varepsilon$ for $\varepsilon > 0$. This ensures that $p(b^i_k)$ lies on $c_i$ and that $p(c_i)$ lies on $b^i_k$. For an example of this construction see Figure 5.1.

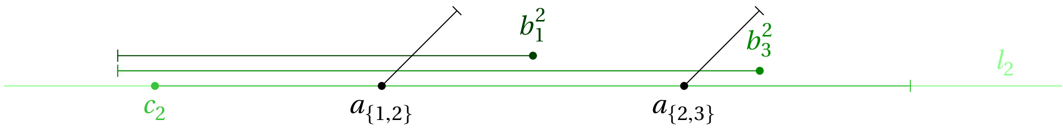
Now we have to show that the generalized transmission graph of $C$ is indeed $G_L$. It can easily be seen that the sets of vertices are the same and thus only the edges have to be considered in detail. We divide the considerations of the edges by their form. Given an edge $e = (u, v) \in E_L$ we will say that $e$ starts at $u$ and ends at $v$.

First, we consider all edges starting at $c_i$. These edges are all present, since the endpoints of $b^i_k$ are all placed on $c_i$ and the endpoints of $a_{\{i,k\}}$ are all placed on $c_i$ and on $c_k$. Next we consider the edges that end at $c_i$. The only edges of $G_L$ of this form are the edges $(b^i_k, c_i)$ and these are present since the direction of the $b^i_k$ is the inverse direction of the direction of $c_i$ and the radii are chosen large enough such that $p(c_i)$ lies on $b^i_k$.

Last we have to consider the edges starting at $b^i_k$. Note that each line segment with a starting point with a larger distance from $p(c_i)$ on $c_i$ overlaps with all line segments with starting points that lie closer to $p(c_i)$. This creates all edges starting at $b^i_k$.




(a) Complete line segment construction for three lines




(b) Closeup of $c_2$. The line segments $b_1^2$ and $b_3^2$ are shifted upwards to show their positioning.

Figure 5.1: Construction of the line segments

It remains to be shown that there are no additional edges. Since only $a_{\{i,k\}}$ and $b_k^i$ are placed on $c_i$ there are no additional edges starting at $c_i$. The parameters of the $a_{\{i,k\}}$ are chosen in a way such that they contain no endpoints. Hence there are no edges starting at $a_{\{i,k\}}$. As already argued in the paragraph above, all points that lie on $b_k^i$ have matching edges in $E_L$. Thus there are only the edges of $G_L$ in the generalized transmission graph of $C$.

Now consider an arrangement $C$ of line segments realizing $G_L$. We show that we can construct lines $l_i'$ that realize $\mathcal{D}(\mathcal{L})$. Let $\mathcal{L}' = (l_1', \ldots, l_n')$ be the arrangement of lines defined by the lines $l_i'$ through $c_i$. We want to show that $\mathcal{D}(\mathcal{L}') = \mathcal{D}(\mathcal{L})$.

We first consider the role of the line segments $a_{\{i,k\}}$. Since $p(a_{\{i,k\}})$ lies on $c_i$ and $c_k$, we have $p(a_{\{i,k\}}) = l_i \cap l_k$ and therefore $l_i'$ and $l_k'$ intersect in $p(a_{\{i,k\}})$. This ensures that all pairs of lines have an intersection point that is also the endpoint of one of the $a_{\{i,k\}}$.

Next we have to show that the ordering of the intersections along each line $l_i'$ is in the order as given by $\mathcal{D}(\mathcal{L})$. This is ensured by the line segments $b_k^i$ as follows: By the definition of $E_L$, namely by the edges of the form $(c_i, b_k^i)$ and $(b_k^i, c_i)$ it is ensured that $b_k^i$ lies on the same line as $c_i$. The definition also enforces the order of the $p(a_{\{i,k\}})$ and $p(b_k^i)$ along the line. Since $p(a_{\{i,o_k\}})$ lies on $b_{o_{k+1}}^i$ but not on $b_{o_k}^i$ it has to lie between the corresponding endpoints. This enforces the correct order of the intersections. □

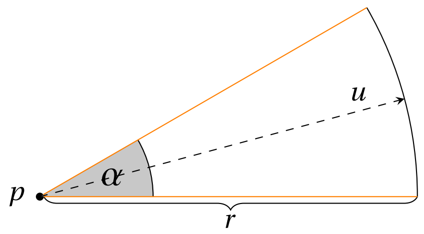
## 5.2 Circular sectors

In this section we consider the complexity of the recognition problem of generalized transmission graphs of circular sectors. This section is structured as follows: In subsection 5.2.1 we state some general definitions, observations and lemmas about circular sectors and define the restricted recognition problem named SECTOR that we will consider. In subsection 5.2.2 we show that SECTOR is complete for $\exists\mathbb{R}$.

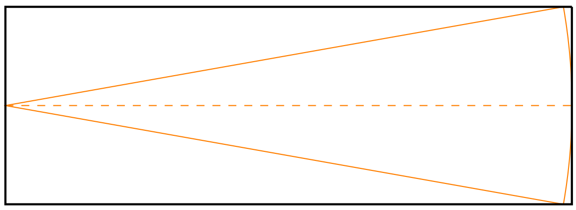
### 5.2.1 Definitions and Observations

A circular sector $c$ is defined by a point $p(c)$, a radius $r(c)$, an angle $\alpha(c)$ and a direction $u(c)$. The direction is given as a vector in $\mathbb{R}^2$ and indicates the direction of the bisector. We will call the bounding line segments the *outer* line segments of $c$ and $p(c)$ the *apex* of the circular sector.

Thereafter we will only consider circular sectors with $\alpha \leq \frac{\pi}{2}$. Let $B(c)$ be the minimal rectangle with two sides parallel to $u(c)$ that contains $c$. The parameters and concepts are depicted Figure 5.2.




(a) Circular sector with $p$, $r$, $u$ and $\alpha$. The outer line segments are marked orange.



(b) A circular sector $c$ with $B(c)$

Figure 5.2: Circular sector $c$ with its parameters

If $x$ and $y$ are circular sectors with $p(x) \in y$ and $p(y) \in x$ we will call $x$ and $y$ a *mutual couple* of circular sectors. By $\gamma(u(x), u(y))$ we will denote the counter-clockwise angle between the vectors $u(x)$ and $u(y)$.

**Observation 1.** *Let $x$ and $y$ be a mutual couple of circular sectors, then*

$$|\pi - \gamma(u(x), u(y))| \leq \frac{\max(\alpha(x), \alpha(y))}{2}.$$

*A visualization can be seen in Figure 5.3.*

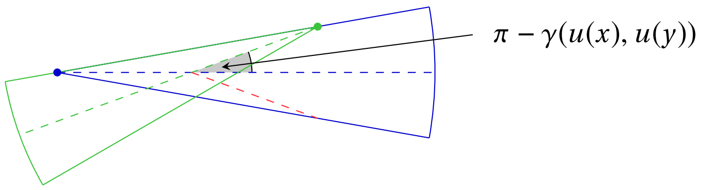



Figure 5.3: Extreme position of $y$ relative to $x$. This induces the largest angle. The symmetrical case is indicated by the red line.

**Observation 2.** *Let $x$ and $y$ be circular sectors where the bisectors intersect with an acute angle of*

$$\beta > \frac{\max(\alpha(x), \alpha(y))}{2}.$$

*Then the acute angle between the outer line segments of $x$ and the bisector of $y$ is at least*

$$\beta - \frac{\max(\alpha(x), \alpha(y))}{2}.$$

*This observation is visualized in Figure 5.4.*

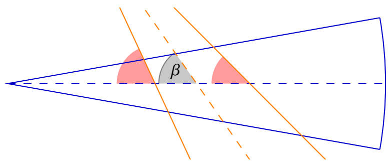



Figure 5.4: The red angles are at least $\left(\beta - \frac{\max(\alpha(x),\alpha(y))}{2}\right)$.

**Lemma 5.** *Let $l$ be a circular sector and let $a_1, \ldots, a_n$ be circular sectors with*

$$\begin{aligned} p(a_i) &\in l & 1 \leq i \leq n \\ p(a_i) &\in a_j & 1 \leq i < j \leq n \\ p(l) &\in a_j & 1 \leq j \leq n. \end{aligned}$$

*Then the projection of the $p(a_i)$ onto the directed line $\ell$ defined by $u(l)$ has the order*

$$\begin{aligned} O &= o_1, \ldots, o_n \\ &= a_1, \ldots, a_n. \end{aligned}$$

*Proof.* Each $a_i$ forms a mutual couple with $l$ and with Observation 1 we get

$$|\pi - \gamma(u(a_i), u(l))| \leq \frac{\pi}{4}. \tag{5.1}$$

Assume that the order of the projection differs from $O$. Let $O' = o'_1, \ldots, o'_n$ be the actual order of the projection of the $p(a_i)$ onto $\ell$. Let $j$ be the first index with $o'_j \neq o_j$ and let $o'_j = a_k$. Then there is an $o'_i$, $i > j$ with $o'_i = a_{k-1}$. By the definition of the circular sectors, $p(a_{k-1})$ has to be included in $a_k$ while still having a projection onto $\ell$ to the right of the projection of $p_k$. This is only possible, if

$$\begin{aligned} |\pi - \gamma(u(a_j), \ell)| &> \frac{\pi}{2} - \frac{\alpha(a_k)}{2} \\ &\geq \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

This is a contradiction to (5.1) and consequently the order of the projection is as claimed. This constraint is illustrated in Figure 5.5. □

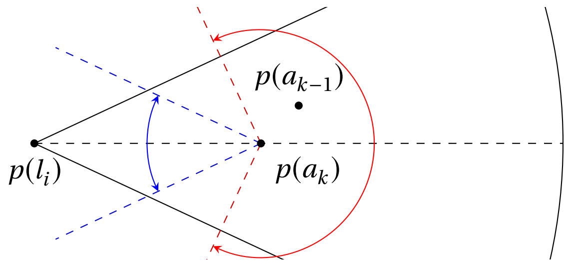



Figure 5.5: By definition, $a_k$ and $l_i$ form a mutual couple. This enforces $u(a_k)$ to lie in the blue range. The apex of $a_{k-1}$ is projected to the right of $p(a_k)$. This enforces $u(a_k)$ to be in the red range. These ranges do not overlap.

Let us consider an arrangement $C$ of circular sectors. We will give two definitions that further determine the form of $C$.

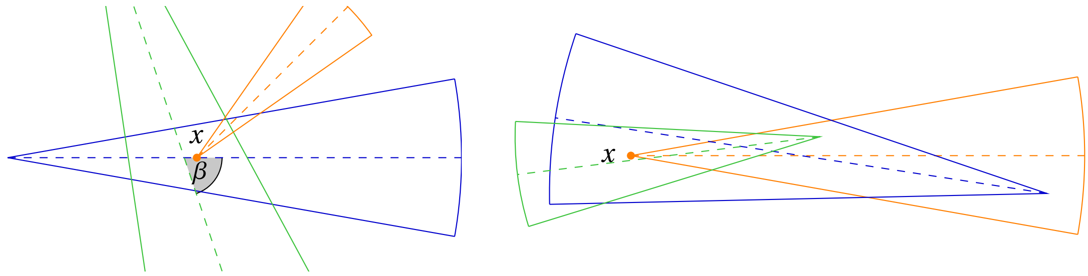
**Definition 9.** *An arrangement $C$ of circular sectors is called* equiangular *if $\alpha(c) = \alpha(c')$ for all circular sectors $c, c' \in C$.*

**Definition 10.** *Let $c, c'$ be two circular sectors of $C$ and assume that $d \in C$ is a circular sector with $p(d) \in c$ and $p(d) \in c'$, such that $c$ and $c'$ do not form both a mutual couple with the same circular sector. Moreover let $\beta_{min}$ be the smallest acute angle between the bisector of any pair $c, c'$ with this property. We will call the arrangement* wide spread *if*

$$\beta_{min} \geq \frac{3 \cdot \max_{c \in C}(\alpha(c))}{2}$$

*The different situations that are described above can be seen in Figure 5.6.*

We will call the recognition problem of the generalized transmission graph of equiangular, wide spread circular sectors Sector. The following lemma will be needed in the proof of the $\exists\mathbb{R}$-completeness of Sector.




(a) Two circular sectors with a common apex but no common mutual couple. If the arrangement is wide spread, $\beta$ is larger than $\frac{3\alpha}{2}$.

(b) The green and blue circular sectors have an apex in common. Since both form a mutual couple with the orange circular sector the angle is not restricted.

Figure 5.6: Depiction of the constraints from Definition 10

**Lemma 6.** *Let $c, c'$ be two circular sectors of an equiangular, wide spread arrangement of circular sectors. Assume there exists an apex of a third circular sector included in both $c$ and $c'$ while there is no common circular sector forming a mutual couple with $c$ and $c'$. Furthermore let there be points $p_l, p_r \in c'$ such that $p_l$ lies to the left of $l_c$ and $p_r$ lies to the right of $l_c$. Let $p_c$ be a point in $c$. Then the following holds:*

$$|\pi - \gamma\left(\overrightarrow{p_l p_c}, u(c')\right)| \geq \alpha$$
$$|\pi - \gamma\left(\overrightarrow{p_c p_r}, u(c')\right)| \geq \alpha$$

*Proof.* Since $c$ and $c'$ are from an equiangular, wide spread arrangement and both include a common apex but do not share a mutual couple we know by definition that $|\pi - \gamma(u(c), u(c'))| \geq \frac{3\alpha}{2}$ and by Observation 2 that the acute angle between the outer line segments and $u(c')$ is at least $\alpha$. This implies that for any vector $v$ along this outer line segment we have $|\pi - \gamma(v, u(c'))| \geq \alpha$.

The directed line segment from $p_l$ to $p_c$ either crosses an outer line segment of $c'$ or has an acute angle with $u(c')$ of at least $\alpha$. If the acute angle is at least $\alpha$, this directly implies the statement of this lemma. If the directed line segment crosses an outer line segment it has to cross from the left to the right region of this line segment. Thus the value $|\pi - \gamma(\overrightarrow{p_l p_c}, u(c'))|$ can only get larger and the first statement of the lemma holds. An analogous argument holds for the line segment from $p_c$ to $p_r$. This argument is visualized in Figure 5.7. □

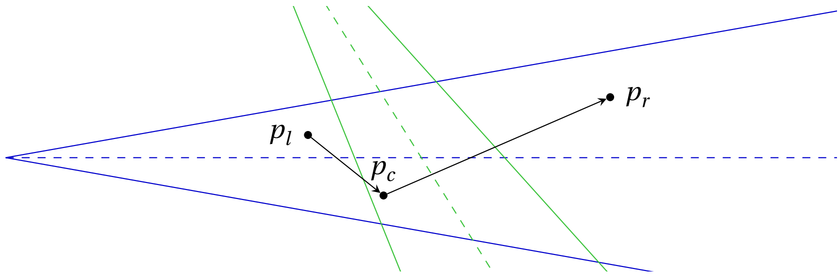



Figure 5.7: The directed line segments cross the outer line segments from left to right.

### 5.2.2 Sector is complete for $\exists\mathbb{R}$

The goal of this section is to prove

**Theorem 6.** *Sector is complete for* $\exists\mathbb{R}$.

First, we show that the recognition problem for general arrangements of circular sectors lies in $\exists\mathbb{R}$.

**Lemma 7.** *The recognition problem of generalized transmission graphs of arrangements of circular sectors, with* $\alpha(c) \leq \frac{\pi}{2}$ *for all circular sectors* $c$ *of the arrangement, lies in* $\exists\mathbb{R}$.

*Proof.* We prove this lemma by giving a sentence in the existential theory of the reals which is true if and only if there is a realization of the graph $G_L = (V_L, E_L)$ by circular sectors.

$$
\begin{aligned}
&\exists x_1, y_1, \ldots, x_n, y_n \\
&\quad u_{1x}, u_{1y}, \ldots u_{nx}, u_{ny} \\
&\quad c\alpha_1, \ldots, c\alpha_n \\
&\quad \lambda_{ik} && 1 \leq i \neq k \leq n \\
&\quad d_{ikx}, d_{iky} && 1 \leq i \neq k \leq n \\
&\quad r_1, \ldots, r_n :
\end{aligned}
$$

$$
\begin{array}{llr}
 & (c\alpha_i \geq 0) & \\
\wedge & \left(c\alpha_i \leq \cos\left(\frac{\pi}{4}\right)\right) & \\
\wedge & u_{ix}^2 + u_{iy}^2 = 1 & \\
\wedge & d_{ikx}^2 + d_{iky}^2 = 1 & \\
\wedge & \lambda_{ik} \cdot d_{ikx} = (x_i - x_k) & \\
\wedge & \lambda_{ik} \cdot d_{iky} = (y_i - y_k) & \\
\wedge & (x_i - x_k)^2 + (y_i - y_k)^2 \leq r_i & (i,k) \in E_L \\
\wedge & -u_{ix} \cdot d_{ikx} + u_{iy} \cdot d_{iky} \leq -c\alpha_i & (i,k) \in E_L \\
\wedge & \big((x_i - x_k)^2 + (y_i - y_k)^2 \geq r_i & (i,k) \notin E_L \\
\vee & -u_{ix} \cdot d_{ikx} + u_{iy} \cdot d_{iky} \geq -c\alpha_i\big) & (i,k) \notin E_L
\end{array}
$$

The variables of this construction have the following meanings: Each point $p_i = (x_i, y_i)$ corresponds to an apex of a circular sector. The $u_i = (u_{ix}, u_{iy})$ are the vectors giving the direction of the circular sectors. The radius of the $i$-th circular sectors is given by $r_i$.

If $\alpha_i$ is the angle of the $i$-th circular sector in a realization of $G_L$, then $c\alpha_i$ has the value $\frac{\cos \alpha_i}{2}$. The $\cos(\frac{\pi}{4})$ constant can be used by Lemma 1. The vectors $\lambda_{ik} \cdot d_{ik} = \lambda_{ik} \cdot (d_{ikx}, d_{iky})$ will give the direction of the vector $p_i - p_k$. The vectors $d_{ik}$ have unit size. The last four inequalities enforce that $p_k$ is included in the circular sector defined by $p_i$, $u_i$, $\alpha_i$ and $r_i$, if and only if there is an edge $(i, k) \in E_L$. The first of these inequalities is satisfied, if $p_k$ lies inside the disk with radius $r_i$ centered at $p_i$. The second inequality uses the definition of the dot product to enforce that the angle between $d_{ik}$ and $u_i$ is smaller than $\frac{\alpha_i}{2}$. The last two inequalities enforce that at least on of these conditions is not satisfied, if $(i, k) \notin E_L$. □

Next we want to show the $\exists\mathbb{R}$-hardness of Sector. This endeavour will be carried by a reduction from Simple-Stretchability using the order description.

The main idea of the $\exists\mathbb{R}$-hardness proof is derived from the construction in the proof of Theorem 5. When considering line segments, the existence of an edge between two line segments always ensures that one point lies on the line that is defined by the direction of the second line segment. The correctness argument in Theorem 5 is based on this fact. The latter situation is not necessarily given when considering circular sectors and thus the order of the intersection is generally not enforced by this construction. This is the reason that the construction is extended to enforce a placement of the apexes that ensures the right intersection order.

In the case for the line segments there was one "large" line segment for each line. In order to disallow undesirable intersections we will have three "large" circular sector for each line in the following construction. The circular sectors corresponding to $l_i$ will be denoted by $c_{i1}$, $c_{i2}$ and $c_{i3}$. When considering an arbitrary the circular sector, it will be denoted by $c_{im}$. If a two circular sectors is considered, they will be written in the form $c_{im}, c_{km'}$ where $i \neq k$ and $m, m' \in \{1, 2, 3\}$.

In Construction 1, some circular sectors have designators of the form $a^{im}_{km'}$ and $b^{im}_{km'}$. In most cases the upper index is $im$ and the lower index differs. For better readability, the indices are marked bold ($a^{\mathbf{km'}}_{\mathbf{im}}$), if $im$ is the lower index.

The rest of this section is structured as follows: In Construction 1 we give the polynomial time reduction from Simple-Stretchability to Sector. Then in Lemma 8 and Lemma 9 we will show the correctness of this construction. All intermediate results from this section will then be combined in the proof of Theorem 6.

**Construction 1.** Given a description $\mathcal{D}(\mathcal{L}) = O^1, \ldots, O^n$ of a simple line arrangement without vertical lines, we construct a graph $G_L = (V_L, E_L)$ with the property that $G_L \in$ Sector if and only if $\mathcal{D}(\mathcal{L})$ is realizable as a simple line arrangement. For this construction let

$1 \leq i, k, l \leq n$, $1 \leq m, m', m'' \leq 3$. The set of vertices is defined as follows:

$$\begin{aligned} V_L = & \{c_{im}\} \\ & \cup \{a^{im}_{km'} \mid i \neq k\} \\ & \cup \{b^{im}_{km'} \mid i \neq k\} \end{aligned}$$

For the vertices $a^{im}_{km'}$ and $b^{im}_{km'}$ the upper index will denote the $c_{im}$ with whom $a^{im}_{km'}$ and $b^{im}_{km'}$ form a mutual couple. The lower index indicates a relation to $c_{km'}$.

The $c_{i2}$ will later define the lines of the arrangement. The circular sectors $a^{im}_{km'}$ and $a^{\mathbf{km'}}_{\mathbf{im}}$ have a similar role as the $a_{\{i,k\}}$ in the construction for the line segments. They will be enforcing the intersection of $c_{im}$ and $c_{km'}$. The $b^{im}_{km'}$ are similar to the $b^i_k$ in the proof for Theorem 5. They will enforce a local order of the circular sectors. The $c_{i1}$ and $c_{i3}$ are needed to enforce the correct order of the line intersections.

We divide the edges of the graph into categories. The first category $E_I$ contains the edges that enforce an intersection of two circular sectors $c_{im}$ and $c_{km'}$ for $k < l$.

$$\begin{aligned} E_I = & \{(c_{im}, a^{im}_{km'}) \mid i \neq k\} \\ & \cup \{(c_{im}, a^{km'}_{im}) \mid i \neq k\} \end{aligned}$$

The edges $E_C$ enforce that each $a^{im}_{km'}$ and each $b^{im}_{km'}$ forms a mutual couple with $c_{im}$.

$$\begin{aligned} E_C = & \{(a^{im}_{km'}, c_{im}) \mid i \neq k\} \\ & \cup \{(c_{im}, b^{im}_{km'}) \mid i \neq k\} \\ & \cup \{(b^{im}_{km'}, c_{im}) \mid i \neq k\} \end{aligned}$$

The edges of $E_{GO}$ will enforce the order of the projection of the apexes of $a^{im}_{o_k m'}$, $a^{im}_{o_l m''}$, $b^{im}_{o_k m'}$ and $b^{im}_{o_l m''}$ for $k > l$ onto the bisector of $c_{im}$. They are chosen such that $p(a^{im}_{o_k m'})$ will be projected closer to $p(c_{im})$ than $p(a^{im}_{o_l m''})$ for $k < l$. Also included in $E_{GO}$ are edges that enforce that all $p(a^{\mathbf{o_k m'}}_{\mathbf{im}})$ are included in the circular sectors $a^{im}_{o_l m''}$ and $b^{im}_{o_l m''}$.

$$\begin{aligned} E_{GO} = & \{(a^{im}_{o_k m'}, a^{im}_{o_l m''}) \mid i \neq k, k > l\} \\ & \cup \{(a^{im}_{o_k m'}, a^{\mathbf{o_l m''}}_{\mathbf{im}}) \mid i \neq k, k > l\} \\ & \cup \{(a^{im}_{o_k m'}, b^{im}_{o_l m''}) \mid i \neq k, k > l\} \\ & \cup \{(b^{im}_{o_k m'}, a^{im}_{o_l m''}) \mid i \neq k, k > l\} \\ & \cup \{(b^{im}_{o_k m'}, a^{\mathbf{o_l m''}}_{\mathbf{im}}) \mid i \neq k, k > l\} \end{aligned}$$

The last two categories of edges will enforce the projection order of the apexes of $a^{im}_{o_k1}$, $a^{im}_{o_k2}$, $a^{im}_{o_k3}$ and $b^{im}_{o_k1}$, $b^{im}_{o_k2}$, $b^{im}_{o_k3}$ onto the bisector of $c_{im}$. This order is $a^{im}_{o_k1}$, $b^{im}_{o_k1}$, $a^{im}_{o_k2}$, $b^{im}_{o_k2}$, $a^{im}_{o_k3}$, $b^{im}_{o_k3}$ if $o_k > i$ and the inverse order otherwise. The edges for the first case are $E_{LOI}$ and the edges for the second case are $E_{LOD}$.

$$\begin{aligned}
E_{LOI} = \quad & \{(a^{im}_{o_k m'}, a^{im}_{o_k m''}) \mid i \neq k, m'' < m', o_k > i\} \\
& \cup \{(a^{im}_{o_k m'}, a^{\mathbf{o_k m''}}_{\mathbf{im}}) \mid i \neq k, m'' < m', o_k > i\} \\
& \cup \{(a^{im}_{o_k m'}, b^{im}_{o_k m''}) \mid i \neq k, m'' < m', o_k > i\} \\
& \cup \{(b^{im}_{o_k m'}, b^{im}_{o_k m''}) \mid i \neq k, m'' < m', o_k > i\} \\
& \cup \{(b^{im}_{o_k m'}, a^{im}_{o_k m''}) \mid i \neq k, m'' \leq m', o_k > i\} \\
& \cup \{(b^{im}_{o_k m'}, a^{\mathbf{o_k m''}}_{\mathbf{im}}) \mid i \neq k, m'' \leq m', o_k > i\}
\end{aligned}$$

$$\begin{aligned}
E_{LOD} = \quad & \{(a^{im}_{o_k m'}, a^{im}_{o_k m''}) \mid i \neq k, m'' > m', o_k < i\} \\
& \cup \{(a^{im}_{o_k m'}, a^{\mathbf{o_k m''}}_{\mathbf{im}}) \mid i \neq k, m'' > m', o_k < i\} \\
& \cup \{(a^{im}_{o_k m'}, b^{im}_{o_k m''}) \mid i \neq k, m'' > m', o_k < i\} \\
& \cup \{(b^{im}_{o_k m'}, b^{im}_{o_k m''}) \mid i \neq k, m'' > m', o_k < i\} \\
& \cup \{(b^{im}_{o_k m'}, a^{im}_{o_k m''}) \mid i \neq k, m'' \geq m', o_k < i\} \\
& \cup \{(b^{im}_{o_k m'}, a^{\mathbf{o_k m''}}_{\mathbf{im}}) \mid i \neq k, m'' \geq m', o_k < i\}
\end{aligned}$$

The set of all edges is defined as

$$E_L = E_I \cup E_C \cup E_{GO} \cup E_{LOI} \cup E_{LOD}$$

Clearly, this construction can be carried out in polynomial time. For a possible realization of this graph see Figure 5.8 and Figure 5.9.

Now we show that this construction is indeed a reduction. To do so we prove that there exists a line arrangement realizing $\mathcal{D}(\mathcal{L})$ if and only if there is an arrangement of equiangular, wide spread circular sectors realizing $G_L$. Each direction of the proof is shown in a separate lemma.

**Lemma 8.** *If there is a line arrangement $\mathcal{L}' = \{l_1, \dots, l_n\}$ realizing $\mathcal{D}(\mathcal{L})$ then there is an equiangular, wide spread arrangement $C$ of circular sectors realizing $G_L$ as defined in Construction 1.*

*Proof.* Since $\mathcal{L}'$ is simple and has no vertical lines, there is a disk $D$ containing all intersection points of the lines $l_1, \dots, l_n$ where the distance of $\partial(D)$ to each intersection point is positive. Since the lines are ordered by slope, the circular order of the intersections with $D$ is $l_1, \ldots, l_n, l_1 \ldots, l_n$. There is no vertical line in $\mathcal{L}'$, hence we can add a virtual vertical line $l'$ dividing the intersection points along $D$ into a "left" set $D_l = \{q_i^l \mid 1 \leq i \leq n\}$ and a "right" set $D_r = \{q_i^r \mid 1 \leq i \leq n\}$.

We now show how a realization of $G_L$ with circular sectors can be constructed from this. The constructed circular sectors are denoted by $a_{km'}^{im}$, $b_{km'}^{im}$ or $c_{im}$. By $l_x$ we will denote the directed line through any constructed circular sector $x$.

Let $\alpha_{\min}$ be the smallest acute angle between any two lines of $\mathcal{L}'$. The angle $\alpha$ for the constructed arrangement will be set depending on $\alpha_{\min}$ and the placement of the constructed circular sectors $c_{im}$.

In the first step we place the circular sectors $c_{i2}$. We set $p(c_{i2}) = q_i^l$, $r(c_{i2}) = \text{dist}(q_i^l, q_i^r)$ and $u(c_{i2}) = \overrightarrow{q_i^l q_i^r}$. This basically sets the bisecting line segment in $c_{i2}$ to $D \cap l_i$.

We place $p(c_{i1})$ next to $p(c_{i2})$ onto the boundary of $D$. The distance between $p(c_{i1})$ and $p(c_{i2})$ on $\partial(D)$ is some small $\tau > 0$ and the point is rotated in clockwise direction. The point $p(c_{i3})$ is placed in the same way but rotated in counter-clockwise direction. The directions of $c_{i1}$ and $c_{i3}$ are set to the direction of $c_{i2}$. The radii for $c_{i1}$ and $c_{i3}$ are chosen to be the length of the line segments $l_{i1} \cap D$ and $l_{i3} \cap D$.

The distance $\tau$ is chosen small enough such that no intersection of any two lines lies between $l_{i1}$ and $l_{i3}$. Let $\beta$ be the largest angle such that if the angle of all $c_{im}$ is set to $\beta$ there is always at least one point in $c_{im}$ between the bounding boxes $B(c_x)$ and $B(c_y)$. Here $c_x$ and $c_y$ are of the form that there is no line intersecting $l_{im}$ between $l_x$ and $l_y$. Since $\mathcal{L}$ is a simple line arrangement this is always possible.

The angle $\alpha$ for the construction is now set to $\min\left(\frac{2\alpha_{\min}}{3}, \beta\right)$. This first part of the construction is illustrated in Figure 5.8.

Now we place the remaining circular sectors. Their placement can be seen in Figure 5.9. The points $p(a_{km'}^{im})$ all lie on $l_{im}$ with a distance of $\delta$ to the left of the intersection point of $l_{im}$ and $l_{km'}$. By "to the left" it is meant that the point lies closer to $p(c_{im})$ on the line $l_{im}$ than the intersection point.

The distance $\delta$ is chosen small enough such that $p(a_{km'}^{im})$ lies inside of all $a_{\mathbf{im}}^{\mathbf{km}'}$ that have a larger distance to $p(c_{\mathbf{km}'})$ than $p(a_{km'}^{im})$. The direction of the circular sector $a_{km'}^{im}$ is set to $-u(c_{im})$ and its radius is set to $r(a_{o_k m'}^{im}) = \text{dist}(p(a_{km'}^{im}), p(c_{im})) + \varepsilon$, for $\varepsilon > 0$. This lets $p(c_{im})$ lie on the bisecting line segment of every circular segment $a_{km'}^{im}$.

The directions and radii for the $b_{km'}^{im}$ are chosen in the same way as for the $a_{km'}^{im}$. Apexes of $b_{km'}^{im}$ are placed, such that they lie between the corresponding bounding boxes $B(c_{km'})$. Since $\alpha$ is chosen small enough this is always possible.

Now we have to show that the generalized transmission graph of this arrangement is $G_L$. It can easily be seen that the number of vertices is correct. This means that we have to show that only the edges in $E_L$ are created.

As $\delta$ is chosen small enough that $a_{km'}^{im}$ and $a_{\mathbf{im}}^{\mathbf{km}'}$ lie in $c_{im}$, the edges of $E_I$ are created. Since $b_{km'}^{im}$ and $a_{km'}^{im}$ have the inverse direction of $c_{im}$ and the radii are large enough, $p(c_{im})$ is included in $a_{km'}^{im}$ and in $b_{km'}^{im}$. Hence and all edges in $E_C$ are created.

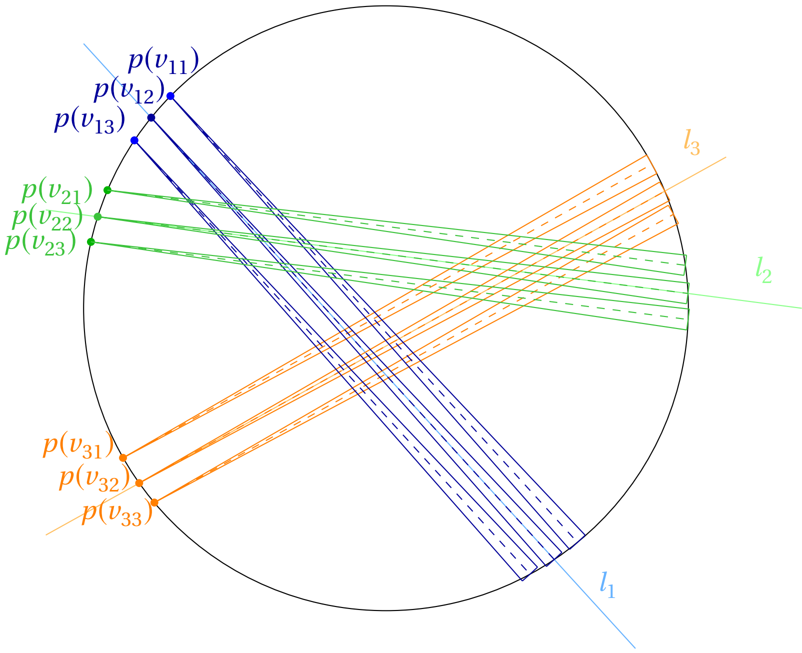



Figure 5.8: Construction of the circular sectors $c_{im}$ based on a given line arrangement

By the choice of the radii and the direction, $a^{im}_{o_k m'}$ includes all apexes of circular sectors that lie on $l_{im}$ and closer to $p(c_m)$ than $p(a^{im}_{o_k m'})$. Furthermore $\delta$ is small enough, such that all $a^{\mathbf{o_l m''}}_{\mathbf{im}}$, $l < k$ are included in $a^{im}_{o_k m'}$. This argumentation implies that edges from $E_{GO}$ are present in the generalized transmission graph of $C$.

The only edges that have not been considered yet, are the edges in $E_{LOI}$ and $E_{LOD}$. For a circular sector $a^{im}_{o_k m'}$ with $o_k > i$, the slope of $l_{o_k}$ is larger than the slope of $l_i$. By the counter-clockwise construction, $l_{o_k 1}$ lies above $l_{o_k 2}$. This implies that the intersection point of $l_{o_k 1}$ and $l_{im}$ lies closer to $p(c_{im})$ than the intersection points with $l_{o_k 2}$ or $l_{o_k 3}$. The presence of the edges can now be seen by the same argument as for the edges of $E_{GO}$. Symmetrical considerations can be made for the edges of $E_{LOD}$.

It remains to be shown that no additional edges are created. Note that all apexes of the circular sectors lie inside of $D$ and that all $a^{im}_{km'} \cap D$ and $b^{im}_{km'} \cap D$ are included in the boxes $B(c_{im})$.

Since only the apexes of $a^{im}_{km'}$, $a^{\mathbf{km'}}_{\mathbf{im}}$ and $b^{im}_{km'}$ lie in $c_{im}$ there are no additional edges starting at $c_{im}$. The rectangles $B(c_{im})$ are disjoint on the boundary of $D$ and all $a^{im}_{km'} \cap D$ and $b^{im}_{km'} \cap D$ lie inside of $B(c_{im})$. This implies that there are no additional edges ending at $c_{im}$.

Now we have to consider additional edges starting at $a^{im}_{km'}$ and $b^{im}_{km'}$. Note that $\alpha \leq \frac{\pi}{2}$ enforces that no circular sector $a^{im}_{km'}$ or $b^{im}_{km'}$ can reach an apex having a larger distance

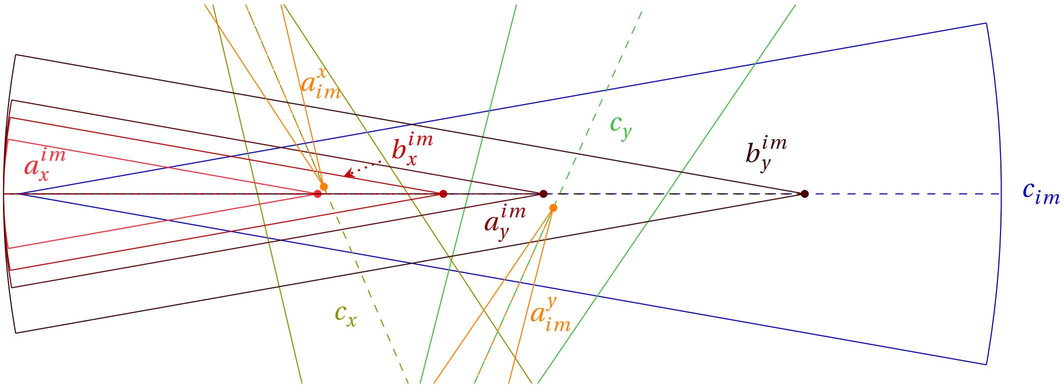



Figure 5.9: Detailed placement inside of one circular sector $c_{im}$

to $p(c_{im})$. Also note that there are edges for all circular sectors with smaller distances in $E_{GO}$, $E_{LOD}$ or $E_{LOI}$. This covers all possible additional edges.

It is left to show that this arrangement is indeed wide spread. All $a^{im}_{km'}$ and $b^{im}_{km'}$ form a mutual couple with $c_{im}$ and thus no angle restriction has to be considered for pairs of $a^{im}_{km'}$ and $b^{im}_{km'}$. The choice of $\alpha \leq \frac{2\alpha_{\min}}{3}$ directly implies the wide spread property for all other circular sectors. □

**Lemma 9.** *If there is an arrangement $C$ of the form*

$$\begin{aligned} C = \; & \{c_{im} \mid 1 \leq i \leq n, , \dots n\} \\ & \cup \{a^{im}_{km'} \mid 1 \leq i \neq j \leq n, 1 \leq m, m' \leq 3\} \\ & \cup \{b^{im}_{km'} \mid 1 \leq i \neq j \leq n, 1 \leq m, m' \leq 3\} \end{aligned}$$

*with $C$ realizing $G_L$ as defined in Construction 1, then there is an arrangement of lines realizing $\mathcal{D}(\mathcal{L})$.*

*Proof.* From $C$ we will construct an arrangement $\mathcal{L}' = (l_1, \dots, l_n)$ of lines such that $\mathcal{D}(\mathcal{L}') = \mathcal{D}(\mathcal{L})$. This is done by setting $l_i$ to the line along $u(c_{i2})$. Now we show that this line arrangement indeed satisfies the description.

All $a^{im}_{km'}$ and $b^{im}_{km'}$ form mutual couples with $c_{im}$. Thus Lemma 5 can be applied to them. It follows that the order of the projections of the apexes of the circular sectors is known. In particular, the order of projections of the $p(a^{i2}_{j2})$ onto $l_i$ is the order given by $\mathcal{D}(\mathcal{L})$ and $p(b^{i2}_{o_j2})$ is projected between $p(a^{i2}_{o_j2})$ and $p(a^{i2}_{o_{j+1}2})$.

Now we have to show that the order of intersections of the lines corresponds to the order of the projections of the $p(a^{i2}_{j2})$. This will be done through a contradiction. We consider two circular sectors $c_{x2}$ and $c_{y2}$. Assume that the order of the projection of the apexes of $a^{i2}_{x2}$ and $a^{i2}_{y2}$ onto $l_i$ is $p(a^{i2}_{x2})$, $p(a^{i2}_{y2})$, while the order of intersection of the lines is $l_y$, $l_x$.

Note that by the definition of the edges of $G_L$, $c_{x2}$ and $c_{y2}$ share the apexes of $a^{y2}_{x2}$ and $a^{x2}_{y2}$ but they share no common mutual couple.

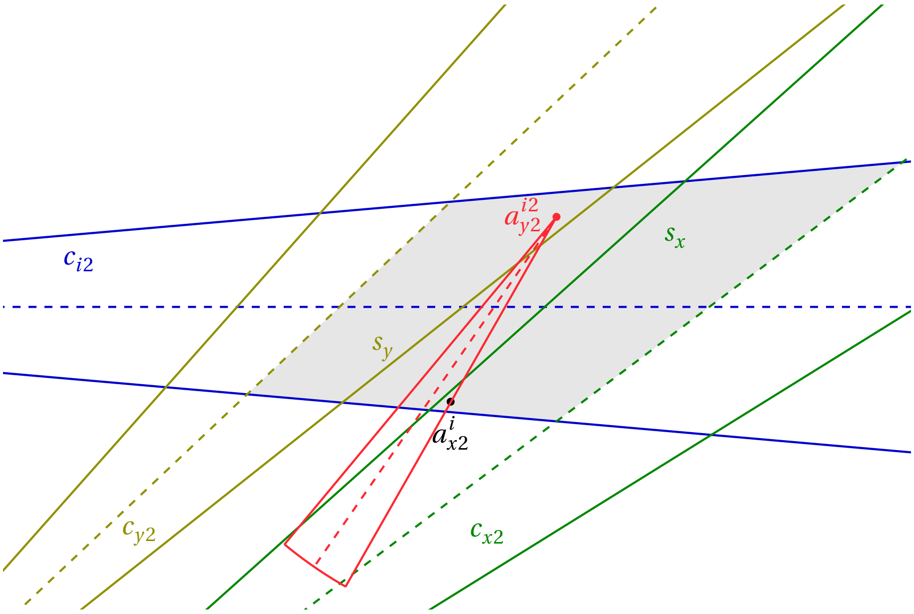



Figure 5.10: The red circular sector cannot reach $p(c_{i2})$.

There are two main cases to consider based on the position of the intersection point $p$ of $l_x$ and $l_y$ relative to $c_{i2}$:

**Case one $p \notin c_{i2}$:**

If $p$ does not lie in $c_{i2}$, then $l_x$ and $l_y$ divide $c_{i2}$ into three parts. Let $s_x$, $s_y$ be the outer line segments of $c_{x2}$ and $c_{y2}$ that lie in the middle part of this decomposition. A schematic of this situation can be seen in Figure 5.10.

From Observation 2 and since $C$ is an equiangular, wide spread arrangement it follows that

$$|\pi - \gamma(s_x, u(c_i))| > \alpha$$
$$\text{and } |\pi - \gamma(s_y, u(c_i))| > \alpha.$$

In order to have an intersection order that differs from the projection order, the circular sector $a_{y2}^{i2}$ has to reach $p(a_{x2}^{i2})$. The latter point is projected to the left of $a_{y2}^{i2}$ but lies right of $s_y$. Thus Lemma 6 can be applied to this situation and for the line segment $d$ from $p(a_{y2}^{i2})$ to $p(a_{x2}^{i2})$ it holds that $|\pi - \gamma(d, u(c_{i2}))| \geq \alpha$. This line segment has to lie inside of $a_{y2}^{i2}$, which is only possible if $|\pi-\gamma(u(a_y^i), u(c_i))| > \frac{\alpha}{2}$. However, this is a contradiction to $|\pi-\gamma(u(a_y^i), u(c_i))| \leq \frac{\alpha}{2}$ which follows from Observation 1.

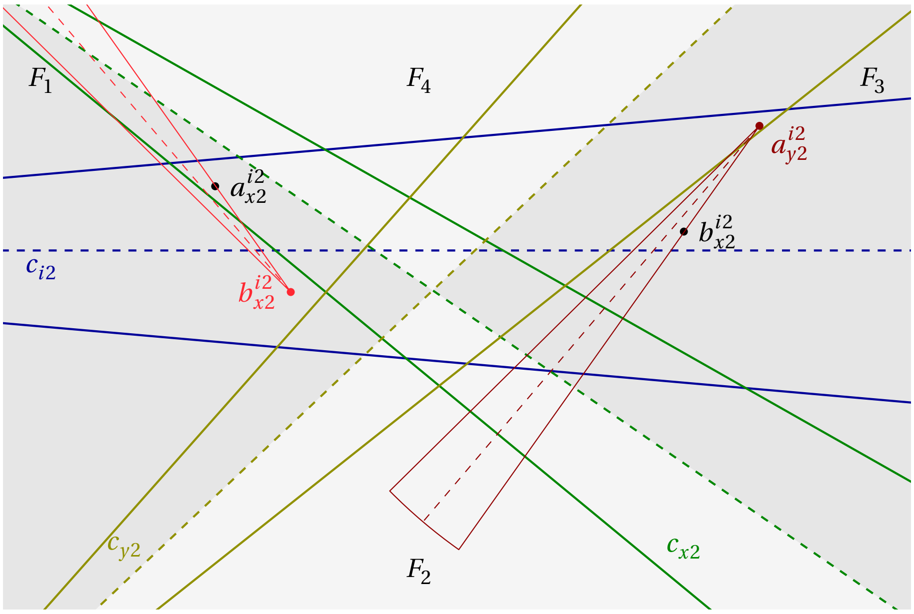



Figure 5.11: Possible placement of $p(b_{x2}^{i2})$ in $F_1$ or $F_3$. The red circular sectors cannot reach $p(c_{i2})$.

**Case two** $p \in c_{i2}$

W.l.o.g. let $u(c_{i2}) = \lambda \cdot (1,0)$, $\lambda > 0$ and let $\mathcal{F} = \{F_1, F_2, F_3, F_4\}$ be the decomposition of the plane into facets induced by $l_x$ and $l_y$. Here $F_1$ is the facet including $p(c_{i2})$, and the facets are named in counter-clockwise order.

We consider the possible placements of $p(b_{x2}^{i2})$ in one of the facets. First, it will be shown that $p(b_{x2}^{i2})$ cannot be placed in $F_1$ or in $F_3$. See Figure 5.11 for a visualization of this situation.

From the form of $E_{GO}$ we know that $p(a_{x2}^{i2})$ has to be projected left of $p(b_{x2}^{i2})$ and $p(a_{x2}^{i2})$ has to lie inside of $b_{x2}^{i2}$. If $p(b_{x2}^{i2})$ lies in $F_1$, the line segment in $b_{x2}^{i2}$ that connects $p(b_{x2}^{i2})$ and $p(a_{x2}^{i2})$ has to cross an outer line segment of $c_{x2}$. Now we can apply Lemma 6 and come to the same contradiction as in case one. If $p(b_{x2}^{i2})$ was lying in $F_3$ the analogues argument holds for $p(b_{x2}^{i2})$ which has to lie inside of $a_{y2}^{i2}$.

This leaves $F_2$ or $F_4$ as possible positions for $b_{x2}^{i2}$. W.l.o.g. let $b_{x2}^{i2}$ be located in $F_4$. We divide the $c_{x2}$ and $c_{y2}$ into two parts, divided by $l_x$ or $l_y$, respectively, and denote the parts containing the line segments that are incident to $F_4$ by $X$ and $Y$. Then, again by applying Lemma 6, it can be seen that $p(a_{x2}^{i2})$ and $p(a_{y2}^{i2})$ are located in $X$ and $Y$. The possible placement is visualized in Figure 5.12.

Note that the argumentation so far yields, that if $p \in c_{i2}$, the intersection order of $l_x$ and $l_y$ with $l_i$ is the same as the order of projection if $l_i$ lies above $p$ and is the inverse order if $l_i$ lies below $p$. The uncertainty of this situation is not desirable. By considering the circular sectors $c_{i1}$ and $c_{i3}$ we will now show that such a situation cannot occur.

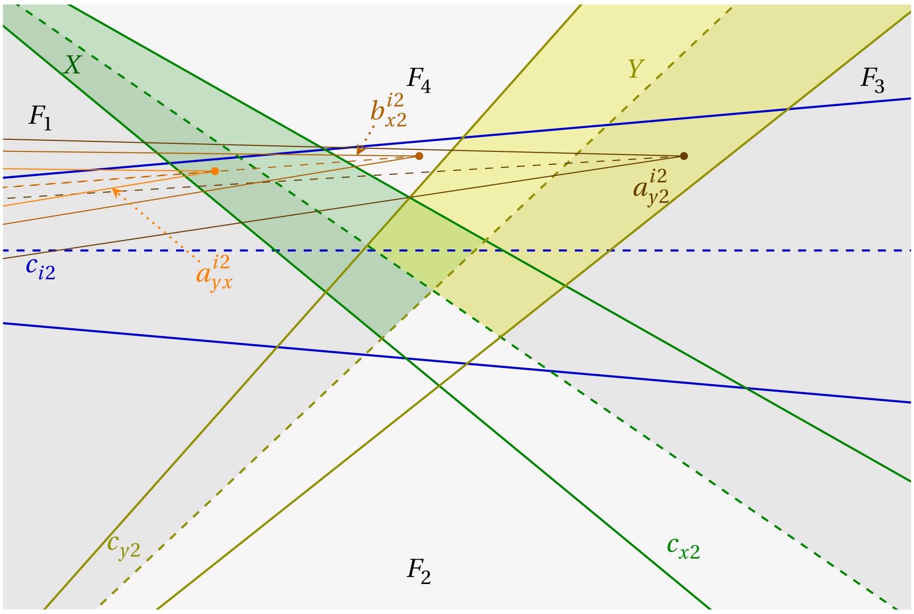



Figure 5.12: If $p(b^{i2}_{x2})$ is located in $F_4$ this placement of the $p(a^{i2}_{x2})$ and $p(a^{i2}_{y2})$ is the only one possible.

First, we show that $c_{i1}$ and $c_{i3}$ cannot contain the intersection point of $l_x$ and $l_y$. W.l.o.g. assume that the intersection point lies in $c_{i1}$. Then $b^{i1}_{x2}$ is included in either $F_2$ or $F_4$.

Consider the case that $b^{i1}_{x2}$ lies in $F_4$. Since $u(c_{i2}) = \lambda \cdot (1,0)$ and one of the outer line segments of $c_{i2}$ has to lie beneath $p$, there is only one outer line segment of $c_{i2}$ that intersects $F_4 \setminus (X \cup Y)$, $X$ and $Y$.There are at most two intersection points of this outer line segment with $\partial(c_{i1})$. This implies that there is no intersection point of $\partial(c_{i2})$ and $\partial(c_{i1})$ in at least one of $X$, $Y$ and $F_4 \setminus (X \cup Y)$. If there is no intersection point then $c_{i1}$ and $c_{i2}$ overlap in this interval. W.l.o.g. let this area be $X$ and let $c_{i1} \cap X$ be fully contained in $c_{i2} \cap X$. Then $p(a^{im}_{x1})$ cannot be placed and consequently this situation is not possible. The argument is depicted in Figure 5.13.

If $p(b^{i1}_{x2})$ was included in $F_2$, then the order of projection of $p(a^{i2}_{y2})$ and $p(a^{i2}_{x2})$ would be the same order as the order of intersections of $l_x$ and $l_y$ with a parallel line to $l_i$ that lies below $l_i$. This order is the inverse order of the order of projection in $c_{i2}$. Since the order of the projection as defined by $E_{GO}$ only depends on $k$ and $i$, the order of projection of $p(a^{i2}_{x2})$ and $p(a^{i2}_{y2})$ has to be the same in all $c_{im}$. This implies that $p(b^{i1}_{y2})$ is not included in $F_2$.

Now we know that $c_{i1}$ and $c_{i3}$ do not contain the intersection point. This implies that the argument from the case $p \notin c_{i2}$ can be applied to them and the order of intersection in $c_{i1}$ and $c_{i3}$ is the same as the order of the projections of $p(a^{i1}_{x2})$ and $p(a^{i1}_{y2})$. This order is the same in all three $c_{im}$ and thus the bisectors of $c_{i1}$ and $c_{i3}$ have to lie on the same side of the intersection point. Furthermore, the points $p(a^{i1}_{x})$ and $p(a^{i3}_{x})$ have to lie in $X$ but outside of $c_{i2}$. This implies that $l_{i1}$ and $l_{i3}$ both intersect $l_x$ and $l_y$ either before $l_i$ or after $l_i$, while $p(b^{i2}_{x2})$ lies in $F_4$.

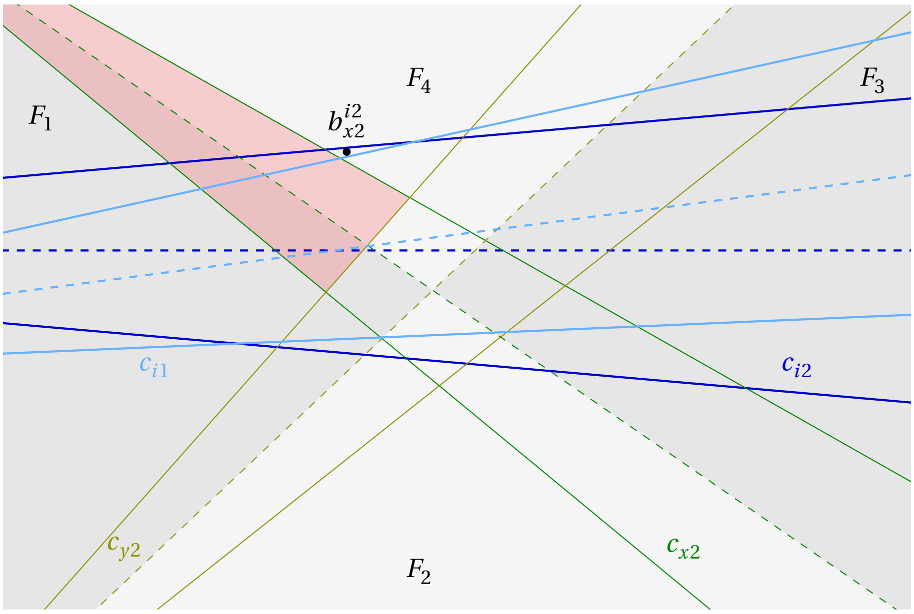



Figure 5.13: If $p(b_{x2}^{i1})$ and $p(b_{x2}^{i2})$ are located in $F_4$ then there is an overlap in $c_{x2}$ or $c_{y2}$.

The edges in $E_{LOI} \cup E_{LOD}$ define that the order of projection onto $l_x$ is $p(a_{x2}^{i1})$, $p(a_{x2}^{i2})$, $p(a_{x2}^{i3})$ (or the inverse), and the analogues statement holds for $l_y$. This order is not possible with $c_{i1}$ and $c_{i3}$ both lying above or below $c_{i2}$, which implies that the intersection point cannot lie in $c_{i2}$.

Since the order of intersection is the same as the order of the projection, if $p \notin c_{i2}$ and a situation with $p \in c_{i2}$ is not possible, we have shown that $\mathcal{D}(\mathcal{L}') = \mathcal{D}(\mathcal{L})$. □

With the tools above we can give the proof of the main result of this section:

**Theorem 6.** *Sector is complete for* $\exists\mathbb{R}$.

*Proof.* The containment of Sector in $\exists\mathbb{R}$ is a direct corollary from Lemma 7. The $\exists\mathbb{R}$-hardness follows from Construction 1, Lemma 8 and Lemma 9. □

# 6 Conclusion

In this thesis we have examined the complexity of recognizing generalized transmission graphs. The classes of objects that we considered were $k$-spheres, regular polygons, line segments and circular sectors.

We showed that the recognition problem for $k$-spheres and line segments is $\exists\mathbb{R}$-complete without further restrictions. For regular polygons the number of vertices was restricted to be at least 6. In the construction we considered, this restriction is necessary in order to uniquely define a separating hyperplane. Thus when reducing from SIMPLE-STRETCHABILITY this seems to be the best possible result. The question whether this bound can be lowered by reducing from a different problem or by considering a different combinatorial description of the line arrangement could be considered in the future.

The form of the arrangement of circular sectors considered in this thesis is even more restricted. First, we only considered circular sectors with $\alpha \leq \frac{\pi}{2}$. For larger angles the projection order cannot be enforced as easily. This order is crucial for the correctness of our proof, hence we assume that when considering larger angles another base problem or another description of the line arrangement has to be considered.

The arrangements were restricted even more to be equiangular and wide spread. We assume that both constraints can be relaxed if not completely omitted and the problem remains $\exists\mathbb{R}$-complete. This assumption is based on the fact that in the proof of Lemma 9 we only considered the intersecting circular sectors that define a line. The other two "large" circular sector corresponding to the line contain apexes with projections between the line defining circular sectors. These additional circular sectors have not been considered yet. We assume that these circular sectors enforce a similar situation to an equiangular, wide spread arrangement without explicitly having to state the constraints.

We introduced arrangements of circular sectors as a possible model for sector antennas. However, this model is not suitable for some real world uses. When considering the application for this recognition problem, we are given a graph and want to place antennas with the direct reachabilities as defined by the graph. If there is no edge between two antennas, one antenna is never allowed to reach the other antenna which often is not necessary. In real world applications one could consider an extended model with three restriction levels for the reachabilities between antennas $u$ and $v$. If $(u, v)$ is included in the first level, $u$ is never allowed to reach $v$. If the tuple is included in the second level, the reachability is allowed but not enforced. Finally the inclusion in the third level enforces reachability. There is a trivial reduction from generalized transmission graphs to this model by classifying none of the edges as level 2. All sentences of the existential theory of the reals that are true if and only if a generalized transmission graph is realizable must have a dependency on the edges of the graph. The same sentence can be used for the extended model by ignoring the tuples from level 2. Hence with minor modifications, our result is applicable in real world situations.

The recognition problems considered in this thesis are the first recognition problems of directed graphs that were proved to be $\exists\mathbb{R}$-complete. It can be assumed that the recognition problem of generalized transmission graphs is complete for $\exists\mathbb{R}$ for other classes of objects. Furthermore there might be other directed graph classes on geometric objects to be considered. Summing up, thesis extends the understanding of $\exists\mathbb{R}$ by considering the first classes of recognition problems of directed graphs and thus opens a new field of research.

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