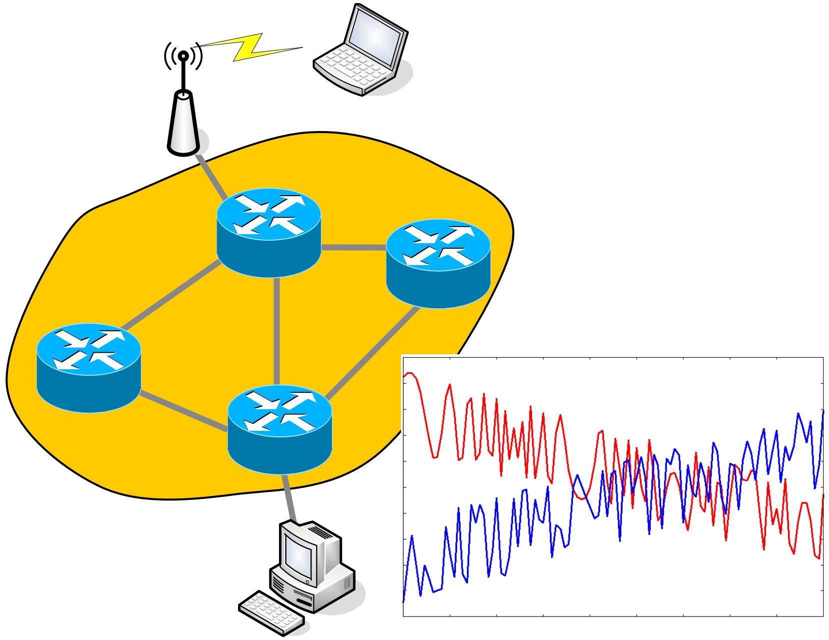
Chapter 13

Design of Experiments (DoE)



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Introduction to DoE

Design of Experiments

- Example: Study the performance of a system in respect to particular parameters
  - System: routing algorithm for a MANET
  - Parameters:
    - Number of nodes: N = {10, 20, 50, 100, 1000, 10000}
    - Mobility: M = {1 m/s, 3 m/s, 5 m/s, 10 m/s}
    - Packet size: P = {64 byte, 256 byte, 512 byte, 1024 byte}
    - Number of parallel flows: F = {1, 3, 5, 7, 10}
  - Parameter space: N x M x P x F = 6 x 4 x 4 x 5 = 480
- At least 10 replications, better 30 replications
  - 4800 to 14400 simulation runs
- Question: how to perform the experiments to understand the effects of the parameters?

Design of Experiments

- Answer: Design of Experiments (DoE)

- The goal is to obtain

maximum information

with the

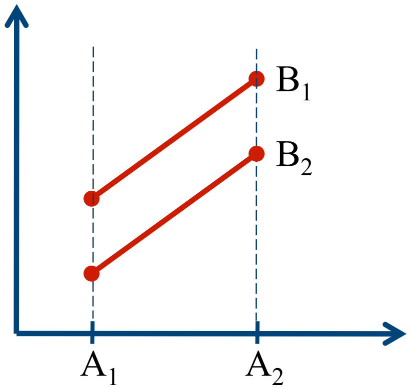
minimum number of experiments

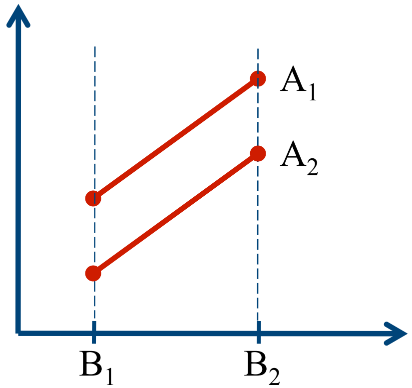
Terminology

Term	**Definition**
Response variable	The outcome of an experiment
Factor	Each variable that affects the response variable and has several alternatives
Level	The values that a factor can assume
Primary Factor	The factors whose effects need to be quantified
Secondary Factor	Factors that impact the performance but whose impact we are not interested in quantifying
Replication	Repetition of all or some experiments
Experimental Unit	Any entity that is used for the experiment
Interaction	Two factors A and B interact if the effect of one depends upon the level of the other

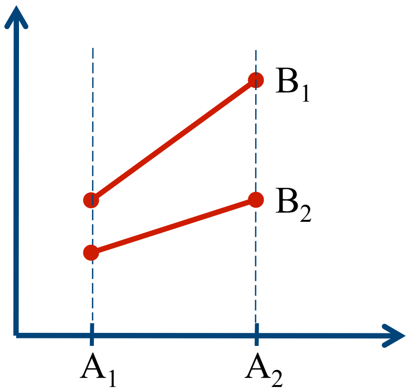
Interaction of factors

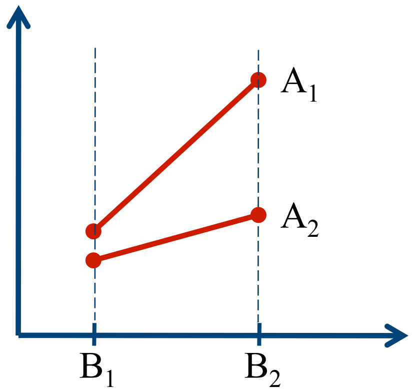
No Interaction





Interaction





Design

- Design: An experimental design consists of specifying the number of experiments, the factor level combinations for each experiment, and the number of replications.

- In planning an experiment, you have to decide
  1. what measurement to make (the response)
  2. what conditions to study
  3. what experimental material to use (the units)

- Example
  1. Measure goodput and overhead of a routing protocol
  2. Network with n nodes in chain
  3. Routing protocol, type of nodes, type of links, traffic

Types of experimental designs

Simple design

- Simple design
  - Start with a configuration and **vary one factor** at a time
  - Given k factors and the i-th factor having n_i levels
  - The required number of experiments

$$n = 1 + \sum_{i=1}^{k} (n_i - 1)$$

  - Example:
    - $k=3$, $\{n_1=3, n_2=4, n_3=2\}$
    - $n = 1 + (2 + 3 + 1) = 7$

Full factorial design

- Full factorial design
  - Use all possible combinations at all levels of all factors
  - Given k factors and the i-th factor having n_i levels
  - The required number of experiments

$$n = \prod_{i=1}^{k} n_i$$

  - Example:
    - $k=3$, $\{n_1=3, n_2=4, n_3=2\}$
    - $n = 3\times4\times2 = 24$

Fractional factorial design

- Fractional factorial design
  - When full factorial design results in a huge number of experiments, it may be not possible to run all
  - Use subsets of levels of factors and the possible combinations of these
  - Given k factors and the i-th factor having n_i levels, and selected subsets of levels $m_i \le n_i$.
  - The required number of experiments

$$n = \prod_{i=1}^{k} m_i$$

  - Example:
    - $k=3$, $\{n_1=3, n_2=4, n_3=2\}$, but use $\{m_1=2, m_2=2, m_3=1\}$
    - $n = 2\times2\times1 = 4$

Types of experimental designs

- Comparison of the design types

Design Type	Factors	Number of experiments
Simple design	k=3, $\{n_1=3, n_2=4, n_3=2\}$	7
Full factorial design		24
Fractional factorial design	Use subset $\{m_1=2, m_2=2, m_3=1\}$	4

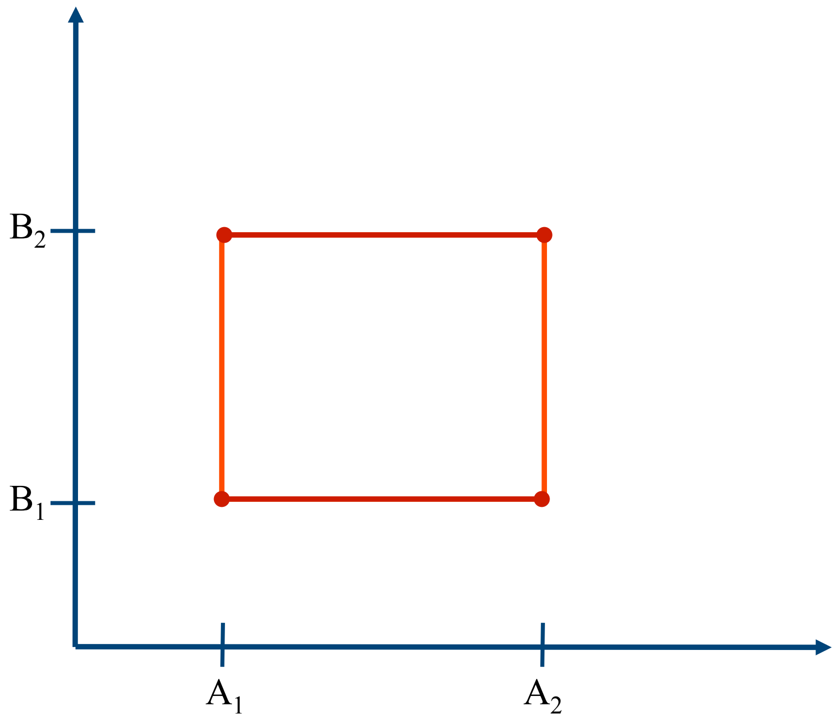
2^k Factorial Designs

2^k Factorial Designs

- A 2^k factorial design is used to determine the effect of k factors
  - Each factor has two levels

- Advantages
  - It is easy to analyze
  - Helps to identify important factors
    ➡reduce the number of factors
  - Often effect of a factor is unidirectional, i.e., performance increase or decrease
  - Begin by experimenting at the minimum and maximum level of a factor
    ➡ two levels

2^k Factorial Designs
Example for $k=2$



2^k Factorial Designs
Example for $k=2$

- Study impact of memory and cache on performance of a workstation
- Memory size, two levels
- Cache size, two levels

- Performance of workstation as regression model

$$y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B$$

Factor 1

Factor 2

		Memory Size	
		4 MB	16 MB
Cache Size	1	-1,-1	1,-1
	2	-1,1	1,1

$$x_A = \begin{cases} -1 & \text{if 4MB memory} \\ 1 & \text{if 16MB memory} \end{cases}$$

$$x_B = \begin{cases} -1 & \text{if 1kb cache} \\ 1 & \text{if 2kb cache} \end{cases}$$

2^k Factorial Designs
Example for k=2

- Regression model

$$y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B$$

- Substitute the results into the model

$$y_1 = q_0 - q_A - q_B + q_{AB}$$
$$y_2 = q_0 + q_A - q_B - q_{AB}$$
$$y_3 = q_0 - q_A + q_B - q_{AB}$$
$$y_4 = q_0 + q_A + q_B + q_{AB}$$

Experiment	A	B	y	AB
1	-1	-1	y_1	1
2	1	-1	y_2	-1
3	-1	1	y_3	-1
4	1	1	y_4	1

- Solve equantions for q_i

$$\left.\begin{aligned} q_0 &= \tfrac{1}{4}(y_1 + y_2 + y_3 + y_4) \\ q_A &= \tfrac{1}{4}(-y_1 + y_2 - y_3 + y_4) \\ q_B &= \tfrac{1}{4}(-y_1 - y_2 + y_3 + y_4) \\ q_{AB} &= \tfrac{1}{4}(y_1 - y_2 - y_3 + y_4) \end{aligned}\right\} \quad y = 40 + 20x_A + 10x_B + 5x_A x_B$$

2^k Factorial Designs
Example for $k=2$: Sign table method

- Sign table contains the effect of factors

I	A	B	AB	y
1	-1	-1	1	15
1	1	-1	-1	45
1	-1	1	-1	25
1	1	1	1	75
$\sum I \cdot y =$ 160	$\sum A \cdot y =$ 80	$\sum B \cdot y =$ 40	$\sum AB \cdot y =$ 20	Total
$\frac{\sum I \cdot y}{4} =$ 40	$\frac{\sum A \cdot y}{4} =$ 20	$\frac{\sum B \cdot y}{4} =$ 10	$\frac{\sum AB \cdot y}{4} =$ 5	Total/4

Result

2^k Factorial Designs
Example for $k=2$: Allocation of variation

- Determine the importance of a factor
  - Calculate the variance

$$s_y^2 = \frac{\sum_{i=1}^{2^2} (y_i - \bar{y})^2}{2^2 - 1}$$

  - Sum of squares total (SST): Total variation of y

$$y = SST = \sum_{i=1}^{2^2} (y_i - \bar{y})^2$$

  - For 2^2 design, the variation is given by

$$SST = \underbrace{2^2 q_A^2}_{SSA} + \underbrace{2^2 q_B^2}_{SSB} + \underbrace{2^2 q_{AB}^2}_{SSAB}$$

2^k Factorial Designs
Example for $k=2$: Allocation of variation

- For 2^2 design, the variation is given by

$$SST = \underbrace{2^2 q_A^2}_{SSA} + \underbrace{2^2 q_B^2}_{SSB} + \underbrace{2^2 q_{AB}^2}_{SSAB}$$

  - SSA: explained by factor A
  - SSB: explained by factor B
  - SSAB: explained by factor AB

- Variation
  - Fraction of variation explained by A: SSA/SST
  - Fraction of variation explained by B: SSB/SST
  - Fraction of variation explained by AB: SSAB/SST

2^k Factorial Designs
The General Case

- In the general case there are k factors, each factor has two levels
- A total of 2^k experiments are required
- Analysis produces 2^k effects (results)
  - k main effects
  - $\binom{k}{2}$ two-factor interactions
  - $\binom{k}{3}$ three-factor interactions
  - …

- Sign table method is used!

2^k Factorial Designs
The General Case

- Sign table, example for k=3

I	A_1	A_2	A_3	A_1A_2	A_1A_3	A_2A_3	$A_1A_2A_3$	y
+	-	-	-	+	+	+	-	y_1
+	+	-	-	-	-	+	+	y_2
+	-	+	-	-	+	-	+	y_3
+	+	+	-	+	+	-	-	y_4
+	-	-	+	+	+	-	+	y_5
+	+	-	+	-	-	-	-	y_6
+	-	+	+	-	-	+	-	y_7
+	+	+	+	+	+	+	+	y_8

2^k Factorial Designs
The General Case

- Sign table

I	A_1	A_2	A_3	…	A_1A_2	A_1A_3	…	$A_1A_2A_3$	…	y
1	-1									y_1
1	1									y_2
1	-1									y_3
…	…									…
SumI SumI/2^k										Total Total/2^k

$2^k r$ Factorial Design with Replications

$2^k r$ Factorial Design with Replications

- Problem with 2^k factorial design is that it does not provide the estimation of experimental errors, since no repetitions
- Solution: Repeat an experiment r times ➡ replication
- If each of the 2^k experiments is repeated r times
  ➡ $2^k r$ factorial design with replications
- Extended model

$$y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e$$

Experimental error

$2^k r$ Factorial Design with Replications

- For analysis, the same method is used, except for y, the mean of the replications is used.

I	A	B	AB	y	$\bar{y}$
1	-1	-1	1	(15,18,12)	15
1	1	-1	-1	(45,48,51)	48
1	-1	1	-1	(25,28,19)	24
1	1	1	1	(75,75,81)	77
164 **41**	**86** **21.5**	**38** **9.5**	**20** **5**		**Total** **Total/4**

$2^k r$ Factorial Design with Replications

Effect					Estimated Response	Measured Responses			Errors		
i	I	A	B	AB	$\hat{y}_i$	y_{i1}	y_{i2}	y_{i3}	e_{i1}	e_{i2}	e_{i3}
1	1	-1	-1	1	15	15	18	12	0	3	-3
2	1	1	-1	-1	48	45	48	51	-3	0	3
3	1	-1	1	-1	24	25	28	19	1	4	-5
4	1	1	1	1	77	75	75	81	-2	-2	4

- Experimental error is given: $e_{ij} = y_{ij} - \hat{y}_i$
- Sum of squared errors (SSE) and the standard deviation of errors:

$$SSE = \sum_{i=1}^{2^2} \sum_{j=1}^{r} e_{ij}^2 \qquad s_e = \sqrt{\frac{SSE}{2^2(r-1)}}$$

$2^k r$ Factorial Design with Replications

- Total Sum of Squares (SST)

$$SST = \sum_i \sum_j (y_{ij} - \bar{y}_{\bullet\bullet})^2$$

- SST can be divided into parts

$$\underbrace{\sum_i \sum_j (y_{ij} - \bar{y}_{\bullet\bullet})^2}_{SST} = \underbrace{2^2 r q_A^2}_{SSA} + \underbrace{2^2 r q_B^2}_{SSB} + \underbrace{2^2 r q_{AB}^2}_{SSAB} + \underbrace{\sum_{i,j} e_{ij}^2}_{SSE}$$

- Standard deviation of errors

$$s_e = \sqrt{\frac{SSE}{2^k (r-1)}}$$

- Confidence interval for the effects

$$q_i \pm t_{\frac{\alpha}{2};2^2(r-1)} \frac{s_e}{\sqrt{2^2 r}}$$

2^{k-p} Fractional Factorial Design

2^{k-p} Fractional Factorial Design

- When the number of factors is large, a full factorial design requires a large number of experiments

- In that case fractional factorial design can be used
  - Requires fewer experiments, e.g., 2^{k-1} requires half of the experiments as a full factorial design

2^{k-p} Fractional Factorial Design

- Preparing the sign table
  - Choose k-p factors and prepare a complete sign table.
    ➡Sign table with 2^{k-p} rows and 2^{k-p} columns
  - The first column will be marked I and consists of all 1s
  - The next k-p columns will be marked with the k-p factors that were chosen
  - The remaining columns are simply products of these factors

$2^{k\text{-}p}$ Fractional Factorial Design

- Sign table, example for $k=7, p=4 \Rightarrow 2^{7-4}=2^3$

k-p chosen factors (F_1, F_2, F_3); *products of chosen factors* (F_1F_2, F_1F_3, F_2F_3, $F_1F_2F_3$)

I	F_1	F_2	F_3	F_1F_2	F_1F_3	F_2F_3	$F_1F_2F_3$
+	-	-	-	+	+	+	-
+	+	-	-	-	-	+	+
+	-	+	-	-	+	-	+
+	+	+	-	+	+	-	-
+	-	-	+	+	+	-	+
+	+	-	+	-	-	-	-
+	-	+	+	-	-	+	-
+	+	+	+	+	+	+	+

$2^{k\text{-}p}$ *rows*

$2^{k\text{-}p}$ *columns*

$2^{k\text{-}p}$ Fractional Factorial Design

- Confounding
  - with fractional factorial design some of the effects can not be determined
  - only combined effects of several factors can be computed
- A fractional factorial design is not unique

- Design resolution
  - The resolution of a design is measured by the **order of effects** that are confounded
  - The **order of effect** is the number of factors included in it
    I = ABC order of 3 ➡Resolution R_{III}
    I = ABCD order of 4 ➡Resolution R_{IV}
  - A design of higher resolution is considered a better design.

Summary

- Design of experiments provides a method for planned experiments
- Goal: Obtain maximum information with minimum experiments
- Basic techniques
  - Factorial design
  - Factorial design with replications
  - Fractional factorial design