Automating Gödel's Ontological Proof of God's Existence with Higher-order Automated Theorem Provers

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If controversies were to arise, there would be no more need of disputations between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . .:

Let us calculate.

(Translation by Russell)

Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo . . . dicere: calculemus.

(Leibniz, 1684)

Required: characteristica universalis and calculus ratiocinator
Our Contribution: Towards a Computational Metaphysics

Ontological argument for the existence of God

We focused on Gödel’s modern version in higher-order modal logic

Automation with provers for higher-order classical logic (HOL)
  - confirmation of known results
  - detection of some novel results
  - systematic variation of the logic settings
  - exploited HOL as a universal metalogic (characteristica universalis)
Anselm’s notion of God (Proslogion, 1078):
“God is that, than which nothing greater can be conceived.”

Gödel’s notion of God:
“A God-like being possesses all ‘positive’ properties.”

To show by logical reasoning:

“God exists.”

$\exists x G(x)$
Anselm’s notion of God (Proslogion, 1078):
“God is that, than which nothing greater can be conceived.”

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“A God-like being possesses all ‘positive’ properties.”

To show by logical reasoning:

“God exists.”

\[ \exists x G(x) \]
Anselm’s notion of God (Proslogion, 1078):
“God is that, than which nothing greater can be conceived.”

Gödel’s notion of God:
“A God-like being possesses all ‘positive’ properties.”

To show by logical reasoning:

“Necessarily God exists.”
\[
\Box \exists x G(x)
\]
Gödel’s Manuscript: 1930’s, 1941, 1946-1955, 1970

Ontologischer Beweis

Feb. 10, 1970

(1) \( P(q) \) \ interpret as positive \( \iff q \in P \)

At 1.

\[ P(q) \implies P(q(x)) \]

\[ P(q) \implies P(q(x)) \]

(2) \( G(x) \equiv (\forall \psi \exists y [\psi(y) \iff \psi(x)]) \) (Gödel)

(3) \( x = x \) is positive

(4) \[ x \neq y \] is negative

(5) \[ P(q) \implies P(q(x)) \] because it follows from the nature of the property

At 2.

\[ P(q) \implies NP(q) \]

\[ P(q) \implies N \sim P(q) \]

TA.

\[ G(x) \implies G E m \]

DF.

\[ E(x) = \forall \phi \exists x. \phi(x) \] necessary Existence

AX.

\[ P(E) \]

Th.

\[ G(x) \implies N(\exists y) G(y) \]

\[ (\exists y) G(y) \implies N(\exists y) G(y) \]

\[ M(\exists y) G(y) \implies MN(\exists y) G(y) \]

\[ M(\exists y) G(y) \implies N(\exists y) G(y) \]

\[ M = \text{positive} \]

any two elements of \( x \) are nec. equivalent

exclusive in \( x \) and for any member of humanity

\[ x \neq y \] positive

hence \( x = y \) positive
Scott’s Version of Gödel’s Axioms, Definitions and Theorems

Axiom A1  Either a property or its negation is positive, but not both:
\[ \forall \phi [P(\neg \phi) \leftrightarrow \neg P(\phi)] \]

Axiom A2  A property necessarily implied by a positive property is positive:
\[ \forall \phi \forall \psi [(P(\phi) \land \square \forall x [\phi(x) \to \psi(x)]) \to P(\psi)] \]

Thm. T1  Positive properties are possibly exemplified:
\[ \forall \phi [P(\phi) \to \Diamond \exists x \phi(x)] \]

Def. D1  A God-like being possesses all positive properties:
\[ G(x) \leftrightarrow \forall \phi [P(\phi) \to \phi(x)] \]

Axiom A3  The property of being God-like is positive:

Cor. C  Possibly, God exists:
\[ \Diamond \exists x G(x) \]

Axiom A4  Positive properties are necessarily positive:
\[ \forall \phi [P(\phi) \to \Box P(\phi)] \]

Def. D2  An essence of an individual is a property possessed by it and necessarily implying any of its properties:
\[ \phi \text{ ess } x \leftrightarrow \phi(x) \land \forall \psi(\psi(x) \to \Box \forall y (\phi(y) \to \psi(y))) \]

Thm. T2  Being God-like is an essence of any God-like being:
\[ \forall x [G(x) \to G \text{ ess } x] \]

Def. D3  Necessary existence of an individual is the necessary exemplification of all its essences:
\[ E(x) \leftrightarrow \forall \phi [\phi \text{ ess } x \to \Box \exists y \phi(y)] \]

Axiom A5  Necessary existence is a positive property:

Thm. T3  Necessarily, God exists:
\[ \Box \exists x G(x) \]
Scott’s Version of Gödel’s Axioms, Definitions and Theorems

**Axiom A1** Either a property or its negation is positive, but not both:  
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\( \forall \phi \forall \psi [(P(\phi) \land \Box \forall x[\phi(x) \to \psi(x)]) \to P(\psi)] \)

**Thm. T1** Positive properties are possibly exemplified:  
\( \forall \phi [P(\phi) \to \Diamond \exists x \phi(x)] \)

**Def. D1** A *God-like* being possesses all positive properties:  
\( G(x) \leftrightarrow \forall \phi [P(\phi) \to \phi(x)] \)

**Axiom A3** The property of being God-like is positive:  
\( P(G) \)

**Cor. C** Possibly, God exists:  
\( \Diamond \exists x G(x) \)

**Axiom A4** Positive properties are necessarily positive:  
\( \forall \phi [P(\phi) \to \Box P(\phi)] \)

**Def. D2** An *essence* of an individual is a property possessed by it and necessarily implying any of its properties:  
\( \phi \text{ ess } x \leftrightarrow \phi(x) \land \forall \psi(\psi(x) \to \Box \forall y[\phi(y) \to \psi(y)]) \)

**Thm. T2** Being God-like is an essence of any God-like being:  
\( \forall x [G(x) \to G \text{ ess } x] \)

**Def. D3** *Necessary existence* of an individual is the necessary exemplification of all its essences:  
\( E(x) \leftrightarrow \forall \phi [\phi \text{ ess } x \to \Box \exists y \phi(y)] \)

**Axiom A5** Necessary existence is a positive property:  
\( P(E) \)

**Thm. T3** Necessarily, God exists:  
\( \Box \exists x G(x) \)

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Difference to Gödel (who omits this conjunct)
Axiom A1  Either a property or its negation is positive, but not both: 
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\[ P(E) \]

Thm. T3  Necessarily, God exists:
\[ \Box \exists x G(x) \]

second-order quantifiers
Proof Overview

**D1:** \( G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)] \)

**D2:** \( \varphi \text{ ess } x \equiv \varphi(x) \land \forall \psi. (\psi(x) \rightarrow \Box \forall x. (\varphi(x) \rightarrow \psi(x))) \)

**D3:** \( NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \Box \exists y. \varphi(y)] \)

\[\begin{align*}
\text{A3} & \quad P(G) \\
\text{A2} & \quad \forall \varphi. \forall \psi. [\neg P(\varphi) \land \Box \forall x. [\varphi(x) \rightarrow \psi(x)]] \rightarrow \neg P(\psi)] \\
\text{A1a} & \quad \forall \varphi. [\neg P(\varphi) \rightarrow \neg P(\varphi)] \\
\text{T1:} & \quad \forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)] \\
\text{C:} & \quad \Diamond \exists z. G(z)
\end{align*}\]

\[\begin{align*}
\text{A1b} & \quad \forall \varphi. [\neg P(\varphi) \rightarrow \neg P(\varphi)] \\
\text{A4} & \quad \forall \varphi. [P(\varphi) \rightarrow \Box \neg P(\varphi)] \\
\text{A5} & \quad \neg P(NE) \\
\text{T2:} & \quad \forall y. [G(y) \rightarrow G \text{ ess } y] \\
\text{L1:} & \quad \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
\text{C:} & \quad \Diamond \exists z. G(z) \\
\text{L2:} & \quad \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
\text{T3:} & \quad \Box \exists x. G(x)
\end{align*}\]
How to automate Higher-Order Modal Logic?
Challenge: No provers for Higher-order Modal Logic (HOML)

Our solution: Embedding in Higher-order Classical Logic (HOL)
Then use existing HOL theorem provers for reasoning in HOML

Previous empirical findings:

Embedding of First-order Modal Logic in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]
[Benzmüller, LPAR, 2013]
Embedding HOML in HOL

**HOML**

\[ \varphi, \psi ::= \ldots | \neg \varphi | \varphi \land \psi | \varphi \rightarrow \psi | \Box \varphi | \Diamond \varphi | \forall x \varphi | \exists x \varphi | \forall P \varphi \]

- Kripke style semantics (possible world semantics)

**HOL**

\[ s, t ::= C | x | \lambda xs | s \ s | \neg s | s \lor t | \forall x t \]

- Church’s simple type theory

  [Church, 1940], [Henkin, 1950]

- various theorem provers exist

  interactive: Isabelle/HOL, HOL4, Hol Light, Coq/HOL, PVS, \ldots

  automated: TPS, LEO-II, Satallax, Nitpick, Isabelle/HOL, \ldots
Embedding HOML in HOL

HOML  \( \varphi, \psi \) ::=  \ldots \mid \neg \varphi \mid \varphi \land \psi \mid \varphi \rightarrow \psi \mid \Box \varphi \mid \Diamond \varphi \mid \forall x \varphi \mid \exists x \varphi \mid \forall P \varphi

HOL  \( s, t \) ::=  C \mid x \mid \lambda x s \mid st \mid \neg s \mid s \lor t \mid \forall x t

HOML in HOL:  HOML formulas \( \varphi \) are mapped to HOL predicates \( \varphi_{\mu \to o} \)

\[
\begin{align*}
\neg &= \lambda \varphi_{\mu \to o} \lambda w_{\mu} \neg \varphi w \\
\land &= \lambda \varphi_{\mu \to o} \lambda \psi_{\mu \to o} \lambda w_{\mu} (\varphi w \land \psi w) \\
\rightarrow &= \lambda \varphi_{\mu \to o} \lambda \psi_{\mu \to o} \lambda w_{\mu} (\neg \varphi w \lor \psi w) \\
\forall &= \lambda h_{\gamma \to (\mu \to o)} \lambda w_{\mu} \forall d_{\gamma} hdw \\
\exists &= \lambda h_{\gamma \to (\mu \to o)} \lambda w_{\mu} \exists d_{\gamma} hdw \\
\Box &= \lambda \varphi_{\mu \to o} \lambda w_{\mu} \forall u_{\mu} (\neg rwu \lor \varphi u) \\
\Diamond &= \lambda \varphi_{\mu \to o} \lambda w_{\mu} \exists u_{\mu} (rwu \land \varphi u) \\
\text{valid} &= \lambda \varphi_{\mu \to o} \forall w_{\mu}. \varphi w
\end{align*}
\]

The equations in \textbf{Ax} are given as axioms to the HOL provers!
Embedding HOML in HOL

HOML \( \varphi, \psi \) ::= \ldots | \neg \varphi | \varphi \land \psi | \varphi \rightarrow \psi | \Box \varphi | \Diamond \varphi | \forall x \varphi | \exists x \varphi | \forall P \varphi

HOL \( s, t \) ::= C | x | \lambda x s | s t | \neg s | s \lor t | \forall x t

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\forall &= \lambda h_{\gamma \rightarrow (\mu \rightarrow o)} \lambda w_{\mu} \forall \gamma d_{\gamma} h d w \\
\exists &= \lambda h_{\gamma \rightarrow (\mu \rightarrow o)} \lambda w_{\mu} \exists \gamma d_{\gamma} h d w
\end{align*}\]

\[\begin{align*}
\Box &= \lambda \varphi_{\mu \rightarrow o} \lambda w_{\mu} \forall u_{\mu} (\neg r w u \lor \varphi u) \\
\Diamond &= \lambda \varphi_{\mu \rightarrow o} \lambda w_{\mu} \exists u_{\mu} (r w u \land \varphi u) \\
\text{valid} &= \lambda \varphi_{\mu \rightarrow o} \forall w_{\mu} \cdot \varphi w
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HOL \hspace{1cm} s, t ::= C | x | \lambda xs | s \cdot t | \neg s | s \lor t | \forall x t

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\end{align*}
\]

The equations in \textit{Ax} are given as axioms to the HOL provers!
Embedding HOML in HOL

Example

**HOML formula**

*HOML formula in HOL*

expansion, $\beta\eta$-conversion

expansion, $\beta\eta$-conversion

expansion, $\beta\eta$-conversion

**Expansion:** user or prover may flexibly choose expansion depth

What are we doing?

In order to prove that $\varphi$ is valid in HOML,

$\rightarrow$ we instead prove that $\text{valid } \varphi_{\mu\to\circ}$ can be derived from $\text{Ax}$ in HOL.

This can be done with interactive or automated HOL theorem provers.

For the experts: soundness and completeness wrt Henkin semantics

\[ \Diamond \exists x G(x) \]

\[ \text{valid } (\Diamond \exists x G(x))_{\mu\to\circ} \]

\[ \forall w_{\mu} (\Diamond \exists x G(x))_{\mu\to\circ} w \]

\[ \forall w_{\mu} \exists u_{\mu} (rwu \land (\exists x G(x))_{\mu\to\circ} u) \]

\[ \forall w_{\mu} \exists u_{\mu} (rwu \land \exists x Gxu) \]
Embedding HOML in HOL

Example

HOML formula

\[ \Diamond \exists x G(x) \]

HOML formula in HOL

\[ \text{valid} (\Diamond \exists x G(x)) \]

expansion, \( \beta \eta \)-conversion

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What are we doing?

In order to prove that \( \varphi \) is valid in HOML,

\[ \rightarrow \text{we instead prove that valid} \varphi \text{ can be derived from Ax in HOL.} \]

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Christoph Benzmüller and Bruno Woltzenlogel Paleo

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What are we doing?

In order to prove that $\varphi$ is valid in HOML, we instead prove that valid $\varphi_{\mu \rightarrow o}$ can be derived from Ax in HOL.

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- HOML formula
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Expansion: user or prover may flexibly choose expansion depth

What are we doing?

In order to prove that \(\varphi\) is valid in HOML,
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For the experts: soundness and completeness wrt Henkin semantics
Example

HOML formula

HOML formula in HOL

expansion, βη-conversion

expansion, βη-conversion

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Expansion: user or prover may flexibly choose expansion depth

What are we doing?

In order to prove that \( \varphi \) is valid in HOML,

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For the experts: soundness and completeness wrt Henkin semantics

Embedding HOML in HOL

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For the experts: soundness and completeness wrt Henkin semantics
Embedding HOML in HOL

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HOML formula
\( \diamond \exists \, G(x) \)

HOML formula in HOL
valid \( (\diamond \exists \, G(x))_{\mu \rightarrow o} \)

expansion, \( \beta \eta \)-conversion
\( \forall \, w_{\mu \rightarrow o} \quad (\exists \, G(x))_{\mu \rightarrow o} \quad w \)

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\( \rightarrow \) we instead prove that valid \( \varphi_{\mu \rightarrow o} \) can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

For the experts: soundness and completeness wrt Henkin semantics
Automated Theorem Provers and Model Finders for HOL

- TPS ... (Peter Andrews)
- LEO-I/LEO-II (myself)
- Isabelle (Nipkow/Paulson/Blanchette)
- Satallax (Brown)
- Nitpick (Blanchette)
- agsyHOL (Lindblatt)

- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
  - can be called remotely via SystemOnTPTP at Miami
  - they significantly gained in strength over the last years
  - they can be bundled into a combined prover HOL-P

Exploit HOL with Henkin semantics as metalogic
Automate other logics (& combinations) via semantic embeddings
— HOL-P becomes a Universal Reasoner —
Provers are called remotely in Miami — no local installation needed!

Download our experiments from https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/THF
corollary C: "[\forall (\square G)]"

sledgehammer [provers = remote_leo2] by (metis A3 T1)

text {"Axiom @text "A4" is added: $\forall \phi \ P(\phi) \to \Box \ P(\phi)$
(Positive properties are necessarily positive). "}\

axiomatization where A4: "[\forall (\lambda \psi. \ P \to \Box (P \psi))]"

text {"Symbol @text "ess" for 'Essence' is introduced and defined as
$\forall \phi (\psi(x) \to \Box \psi(y) \to \chi(x) \to \Box \chi(y))$ (An emphasised essence)
of an individual is a property possessed by it
and necessarily implying any of its properties). "}\

definition ess :: "(\mu \to \sigma) \to \mu \to \sigma" (infixr "ess" 85) where
"ess x = \lambda \phi. \ \Box \ (\lambda \phi \ x \to \Box (\lambda \phi \ y \to \Box y))"

text {"Next, Sledgehammer and Metis prove theorem @text "T2": $\forall x [G(x) \to \Box \ ess(G)\{x\}]$
(Being God-like is an essence of any God-like being). "}\

theorem T2: "[\forall (\lambda x. \ G \ x \to \Box \ ess \ x)]]"
sledgehammer [provers = remote_leo2] by (metis A1b A4 G_def ess_def)

text {"Symbol @text "NE". for 'Necessary Existence', is introduced and
defined as $\exists \ (\lambda \phi \ y \ \phi(y))$ (Necessary
existence of an individual is the necessary exemplification of all its essences). "}\

definition NE :: "\mu \to \sigma" where "NE = (\lambda x. \ (\lambda \phi. \ ess x \to \Box (\lambda \phi)))"

Sledgehammering...

See verifiable Isabelle/HOL journal article at:
http://afp.sourceforge.net/entries/GoedelGod.shtml
See verifiable Coq document at: https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/Coq
“God is dead.”
- Nietzsche, 1883

“Nietzsche is dead.”
- God, 1900

Main Findings
Main Findings

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<tr>
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<td>A1(\approx), A2</td>
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<td>—/—</td>
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<tr>
<td>$\forall \phi \mu \rightarrow \sigma, \forall \psi \mu \rightarrow \sigma, (p(\mu \rightarrow \sigma) \rightarrow (\phi \land \forall X \mu \rightarrow (\phi X \supset \psi X)) \supset p \psi)$</td>
<td>A1, A2</td>
<td>K</td>
<td>THM</td>
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<td>0.0/5.2</td>
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<tr>
<td>$\forall \phi \mu \rightarrow \sigma, p(\mu \rightarrow \sigma) \rightarrow \phi \land \exists X \mu \rightarrow (\phi X)$</td>
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<td>K</td>
<td>THM</td>
<td>0.0/0.0</td>
<td>0.0/0.0</td>
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</tr>
<tr>
<td>$g_{\mu \rightarrow \sigma} = \lambda X \mu \rightarrow \forall \phi \mu \rightarrow \sigma, p(\mu \rightarrow \sigma) \rightarrow (\phi \supset \phi X)$</td>
<td>A1, D1, A3</td>
<td>K</td>
<td>THM</td>
<td>0.0/0.0</td>
<td>5.2/31.3</td>
<td>—/—</td>
</tr>
<tr>
<td>$P(\mu \rightarrow \sigma) \rightarrow g_{\mu \rightarrow \sigma}$</td>
<td>A1, A2, D1, A3</td>
<td>K</td>
<td>THM</td>
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<td>—/—</td>
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<td>$\forall \phi \mu \rightarrow \sigma, p(\mu \rightarrow \sigma) \rightarrow (\phi \supset (p \phi))$</td>
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<td>0.0/0.0</td>
<td>—/—</td>
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<tr>
<td>$\exists (\mu \rightarrow \sigma) = \lambda X \mu \rightarrow (\phi X \land \forall \psi \mu \rightarrow \sigma, (\psi X \supset \forall Y \mu \rightarrow (\phi Y \supset \psi Y))$</td>
<td>D2</td>
<td>K</td>
<td>THM</td>
<td>19.1/18.3</td>
<td>0.0/0.0</td>
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<tr>
<td>$\forall X \mu \rightarrow (g_{\mu \rightarrow \sigma} \supset (\exists (\mu \rightarrow \sigma) \rightarrow \phi X))$</td>
<td>A1, D1, A4, D2</td>
<td>K</td>
<td>THM</td>
<td>12.9/14.0</td>
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<td>$NE_{\mu \rightarrow \sigma} = \lambda X \mu \rightarrow (\phi X \land \forall \psi \mu \rightarrow \sigma, (\psi Y \supset \forall Y \mu \rightarrow (\phi Y \supset \psi Y))$</td>
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<td>THM</td>
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<td>THM</td>
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<td>0.1/5.3</td>
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</tr>
<tr>
<td>$\forall \exists X \mu \rightarrow (g_{\mu \rightarrow \sigma} \supset X)$</td>
<td>A1, C, T2, D3, A5</td>
<td>K</td>
<td>CSA</td>
<td>3.8/6.2</td>
<td>—/—</td>
<td>—/—</td>
</tr>
<tr>
<td>$Fg \rightarrow \exists s_{\sigma}$</td>
<td>A1, A2, D1, A3, A4, D2, D3, A5</td>
<td>K</td>
<td>CSA</td>
<td>8.2/7.5</td>
<td>—/—</td>
<td>—/—</td>
</tr>
<tr>
<td>$\forall \exists X \mu \rightarrow (g_{\mu \rightarrow \sigma} \supset (s_{\sigma} \equiv (p(\mu \rightarrow \sigma) \rightarrow (\phi X)) \supset (p \phi))$</td>
<td>T3</td>
<td>K</td>
<td>THM</td>
<td>17.9/—</td>
<td>3.3/3.2</td>
<td>—/—</td>
</tr>
<tr>
<td>$MC \rightarrow \forall \exists s_{\sigma}$</td>
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<td>K</td>
<td>THM</td>
<td>0.0/0.0</td>
<td>0.0/0.0</td>
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</tr>
<tr>
<td>$FG \rightarrow \forall \exists X \mu \rightarrow (\phi X \equiv (p(\mu \rightarrow \sigma) \rightarrow (\phi X)) \equiv (p \phi))$</td>
<td>A1, D1</td>
<td>K</td>
<td>THM</td>
<td>16.5/—</td>
<td>0.0/5.4</td>
<td>—/—</td>
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<tr>
<td>$MT \rightarrow \forall \exists X \mu \rightarrow (\phi X \equiv (p(\mu \rightarrow \sigma) \rightarrow (\phi X)) \equiv (p \phi))$</td>
<td>D2, FG</td>
<td>K</td>
<td>THM</td>
<td>0.0/3.3</td>
<td>—/—</td>
<td>—/—</td>
</tr>
<tr>
<td>$CO \rightarrow \emptyset$ (no goal, check for consistency)</td>
<td>A1, A2, D1, A3, A4, D2, D3, A5</td>
<td>K</td>
<td>SAT</td>
<td>—/—</td>
<td>—/—</td>
<td>7.3/7.4</td>
</tr>
<tr>
<td>$D2' \rightarrow \forall \exists (\mu \rightarrow \sigma) \rightarrow \lambda X \mu \rightarrow \lambda \psi \mu \rightarrow \sigma, (\psi Y \supset \forall Y \mu \rightarrow (\phi Y \supset \psi Y))$</td>
<td>A1(\approx), A2, D2', D3, A5</td>
<td>K</td>
<td>UNS</td>
<td>7.5/7.8</td>
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<td>K</td>
<td>UNS</td>
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### Main Findings

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<tr>
<td>A1 [\forall \mu \phi \rightarrow \forall \mu (\lambda X \phi X) \equiv \phi ]</td>
<td>A1(\rightarrow), A2</td>
<td>K</td>
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<td>0.1/0.1</td>
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<tr>
<td>A2 [\forall \mu \phi \rightarrow \forall \mu (p_{(\mu \rightarrow \sigma)} \phi \land \forall X \phi X) \rightarrow \psi \psi ]</td>
<td>A1, A2</td>
<td>K</td>
<td>THM</td>
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<td>0.0/5.2</td>
<td>—/—</td>
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<tr>
<td>T1 [\forall \mu \phi \rightarrow \forall \mu (p_{(\mu \rightarrow \sigma)} \phi \rightarrow \exists ! X \phi X) ]</td>
<td>T1, D1, A3</td>
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<td>THM</td>
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<td>0.0/0.0</td>
<td>—/—</td>
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<td>D1 [g_{\mu \rightarrow \sigma} = \lambda X \forall \phi \rightarrow (\lambda X \phi X) ]</td>
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<td>K</td>
<td>THM</td>
<td>0.0/0.0</td>
<td>5.2/31.3</td>
<td>—/—</td>
</tr>
<tr>
<td>A3 [p_{(\mu \rightarrow \sigma)} \rightarrow \forall \mu \phi ]</td>
<td>T1, D1, A3</td>
<td>K</td>
<td>THM</td>
<td>0.0/0.0</td>
<td>0.0/0.0</td>
<td>—/—</td>
</tr>
<tr>
<td>C [\forall \exists X \rightarrow \exists X \phi X ]</td>
<td>A1, A2, D1, A3</td>
<td>K</td>
<td>THM</td>
<td>0.0/0.0</td>
<td>0.0/0.0</td>
<td>—/—</td>
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<tr>
<td>A4 [\forall \phi \rightarrow \forall \phi \phi \oplus \phi ]</td>
<td>A1, A2, D1, A3</td>
<td>K</td>
<td>THM</td>
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<td>0.0/0.0</td>
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<tr>
<td>D2 [\text{ess}_{(\mu \rightarrow \sigma)} \rightarrow \forall \mu \phi \rightarrow \lambda X \phi X \land \forall \psi \rightarrow \phi \psi ]</td>
<td>A1, D1, A4, D2</td>
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<tr>
<td>T2 [\forall \phi \rightarrow \forall \phi (\lambda X \phi X \rightarrow \forall X \phi X) ]</td>
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<tr>
<td>A5 [p_{(\mu \rightarrow \sigma)} \rightarrow \exists ! Y \phi X \oplus \exists ! Y \phi Y ]</td>
<td>A1, D1, C1, D2, A3, A5</td>
<td>K</td>
<td>CSA</td>
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<td>—/—</td>
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</tr>
<tr>
<td>T3 [\forall \exists X \rightarrow \exists X \phi X ]</td>
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<td>CSA</td>
<td>—/—</td>
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<tr>
<td>MC [s_{\sigma} \rightarrow \exists ! s_{\sigma} ]</td>
<td>A1, A2, D1, A3, A4, D2, D3, A5</td>
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<td>CSA</td>
<td>—/—</td>
<td>—/—</td>
<td>—/—</td>
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<td>FG [\forall \phi \rightarrow \forall \phi (g_{(\mu \rightarrow \sigma)} \phi X \rightarrow (\exists (p_{(\mu \rightarrow \sigma)}) \phi \rightarrow \exists (\phi X)) ]</td>
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<td>MT [\forall X \forall Y \rightarrow (g_{(\mu \rightarrow \sigma)} \phi X \rightarrow (g_{(\mu \rightarrow \sigma)} \phi Y \rightarrow Y)) ]</td>
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### Main Findings

**Automating Scott’s proof script**

**T1**: "Positive properties are possibly exemplified" proved by LEO-II and Satallax

- **in logic**: K
- **from axioms**:
  - A1 and A2
- **for domain conditions**:
  - constant domains

### HOL encoding

<table>
<thead>
<tr>
<th>A1</th>
<th>[\forall \phi_{\mu \to \sigma}, p_{\mu \to \sigma} \to A_{\mu} \forall (\phi_X) \equiv \forall (\neg \phi)]</th>
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<tbody>
<tr>
<td>A2</td>
<td>[\forall \phi_{\mu \to \sigma}, \psi_{\mu \to \sigma}, p_{\mu \to \sigma} \to \exists X_{\mu} \neg (\phi_X) \equiv \neg (\psi)]</td>
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<tr>
<td>T1</td>
<td>[\forall \phi_{\mu \to \sigma}, p_{\mu \to \sigma} \to \phi \equiv \exists X_{\mu} \phi X]</td>
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<tr>
<th>A3</th>
<th>[p_{\mu \to \sigma} \to A_{\mu} ]</th>
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<tr>
<td>C</td>
<td>[\exists X_{\mu}, \exists X_{\mu} \land \psi_X \equiv \exists Y_{\mu} \psi Y]</td>
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<tr>
<th>D1</th>
<th>[g_{\mu \to \sigma} \equiv A_{\mu} \land \psi_{\mu \to \sigma} \land p_{\mu \to \sigma} \land \psi_X ]</th>
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<tr>
<th>T2</th>
<th>[\forall X_{\mu} \exists g_{\mu \to \sigma} \exists (\psi_X \equiv \psi_Y)]</th>
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<tr>
<td>A1, D1, A4, D2</td>
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<tr>
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<tr>
<th>D3</th>
<th>[\exists g_{\mu \to \sigma} \equiv A_{\mu} \land \psi_{\mu \to \sigma} \land (\exists Y_{\mu} \psi Y)]</th>
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<tbody>
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<td>A5</td>
<td>[p_{\mu \to \sigma} \to A_{\mu} ]</td>
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<table>
<thead>
<tr>
<th>T3</th>
<th>[\exists X_{\mu}, \exists g_{\mu \to \sigma} X]</th>
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### References

- Christoph Benzmüller and Bruno Woltzenlogel Paleo
- Automating Gödel’s Ontological Proof of God’s Existence
### Main Findings

Automating Scott’s proof script

**T1:** "Positive properties are possibly exemplified" proved by LEO-II and Satallax

- **in logic:** K
- **from axioms:**
  - A1 and A2
  - A1(⊢) and A2
- **for domain conditions:**
  - constant domains
Main Findings

### Automating Scott’s proof script

T1: "Positive properties are possibly exemplified" proved by LEO-II and Satallax

- **in logic:** K
- **from axioms:**
  - A1 and A2
  - A1(\supset) and A2
- **for domain conditions:**
  - constant domains
  - varying domains (individuals)
Main Findings

Automating Scott’s proof script

C: "Possibly, God exists" proved by LEO-II and Satallax

- in logic: K
- from assumptions:
  - T1, D1, A3
  - A1, A2, D1, A3
- for domain conditions:
  - constant domains
  - varying domains (individuals)
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<tr>
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<th>Nitpick</th>
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<tbody>
<tr>
<td>A1</td>
<td>(\forall \phi_{\mu,\sigma} \cdot p_{(u,=\sigma)\rightarrow \sigma}(\lambda X_{\mu} \cdot \neg (\phi X)) \equiv \neg (p\phi))</td>
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<td></td>
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<tr>
<td>A2</td>
<td>(\forall \phi_{\mu,\sigma} \cdot \forall \psi_{\mu,\sigma} \cdot (p_{(u,=\sigma)\rightarrow \sigma} \phi \land \exists \exists X_{\mu,\phi} \phi) \equiv p\psi)</td>
<td></td>
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<tr>
<td>T1</td>
<td>(\forall \phi_{\mu,\sigma} \cdot p_{(u,=\sigma)\rightarrow \sigma} \phi \equiv \exists \exists X_{\mu,\phi} \phi)</td>
<td>A1((\sim)), A2</td>
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<td>THM</td>
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<td>0.0/0.0</td>
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<td>(g_{\mu,\sigma} = \lambda X_{\mu} \cdot \forall \phi_{\mu,\sigma} \cdot p_{(u,=\sigma)\rightarrow \sigma} \phi \equiv \phi)</td>
<td>A1, A2</td>
<td>K</td>
<td>THM</td>
<td>0.1/0.1</td>
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</tr>
<tr>
<td>A3</td>
<td>(p_{(u,=\sigma)\rightarrow \sigma} g_{\mu,\sigma})</td>
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</tr>
<tr>
<td>C</td>
<td>(\exists \exists X_{\mu,\phi} g_{\mu,\sigma} X)</td>
<td>T1, D1, A3</td>
<td>K</td>
<td>THM</td>
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<tr>
<td>A4</td>
<td>(\forall \phi_{\mu,\sigma} \cdot p_{(u,=\sigma)\rightarrow \sigma} \phi \equiv \neg \phi)</td>
<td>A1, A2, D1, A3</td>
<td>K</td>
<td>THM</td>
<td>0.0/0.0</td>
<td>5.2/31.3</td>
</tr>
<tr>
<td>D2</td>
<td>(\forall \phi_{\mu,\sigma} \cdot p_{(u,=\sigma)\rightarrow \sigma} (\exists \exists \phi_{\mu,\sigma} \exists \exists Y_{\mu,\phi} \phi) \equiv (\exists \exists \phi_{\mu,\sigma} \exists \exists Y_{\mu,\phi} \phi))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T2</td>
<td>(\forall \phi_{\mu,\sigma} \cdot g_{\mu,\sigma} \phi \equiv (\exists \exists \phi_{\mu,\sigma} \exists \exists Y_{\mu,\phi} \phi))</td>
<td>A1, D1, A4, D2</td>
<td>K</td>
<td>THM</td>
<td>19.1/18.3</td>
<td>0.0/0.0</td>
</tr>
<tr>
<td>A5</td>
<td>(p_{(u,=\sigma)\rightarrow \sigma} \exists \exists \phi_{\mu,\sigma} \exists \exists Y_{\mu,\phi} \phi)</td>
<td>A1, A2, D1, A3, A4, D2</td>
<td>K</td>
<td>THM</td>
<td>12.9/14.0</td>
<td>0.0/0.0</td>
</tr>
</tbody>
</table>

**Automating Scott’s proof script**

**T2: "Being God-like is an ess. of any God-like being" proved by LEO-II and Satallax**

- **in logic:** K
- **from assumptions:**
  - A1, D1, A4, D2
  - A1, A2, D1, A3, A4, D2
- **for domain conditions:**
  - constant domains
  - varying domains (individuals)
Main Findings

Automating Scott’s proof script

T3: “Necessarily, God exists” proved by LEO-II and Satallax

- in logic: KB
- from assumptions:
  - D1, C, T2, D3, A5
- for domain conditions:
  - constant domains
  - varying domains (individuals)

For logic K we got a countermodel by Nitpick
### Main Findings

**Automating Scott’s proof script**

**Summary**
- proof verified and automated
- KB is sufficient (criticized logic S5 not needed!)
- proof works for constant and varying domains
- exact dependencies determined experimentally
- ATPs have found alternative proofs (shorter)
### Main Findings

**Consistency check: Gödel vs. Scott**

- **Scott’s assumptions are consistent; shown by Nitpick**
- **Gödel’s assumptions are inconsistent; shown by LEO-II (new philosophical result!)**

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<table>
<thead>
<tr>
<th>HOL encoding</th>
<th>dependencies</th>
<th>logic</th>
<th>status</th>
<th>LEO-II</th>
<th>Satallax</th>
<th>Nitpick</th>
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<tbody>
<tr>
<td>A1</td>
<td></td>
<td>K</td>
<td>THM</td>
<td>0.1/0.1</td>
<td>0.0/0.0</td>
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<td>A2</td>
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<tr>
<td>T1</td>
<td>[\exists \phi \mu, \exists \psi \mu, \exists \phi \psi]</td>
<td>A1(\omega), A2</td>
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<tr>
<td>D1</td>
<td>\forall \mu. \exists \psi \mu, \exists \phi \psi</td>
<td>A1(\omega), A2</td>
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</tbody>
</table>

**CO**

- \( \emptyset \) (no goal, check for consistency)
- **D2’**
- \( \exists \phi \mu, \exists \psi \mu, \exists \phi \psi \)
- **CO’**

---

**Christoph Benzmüller and Bruno Woltzenlogel Paleo**

Automating Gödel’s Ontological Proof of God’s Existence 29
Main Findings

Further Results

- Monotheism holds
- God is flawless
Main Findings

Modal Collapse

\[ \forall \varphi (\varphi \supset \Box \varphi) \]

- proved by LEO-II and Satallax
- for constant and varying domains

Main critique on Gödel’s ontological proof:
- there are no contingent truths
- everything is determined / no free will
- why using modal logic in the first place?

Christoph Benzmüller and Bruno Woltzenlogel Paleo
Automating Gödel’s Ontological Proof of God’s Existence 31
Avoiding the Modal Collapse: Very recent work (not yet published)

Variants of Gödel’s proof that avoid the modal collapse


Future work

- [André Fuhrmann, 2005]
- [Szatkowski, 2011]
- ...
Conclusion

Achievements
- significant contribution towards a **Computational Metaphysics**
- **HOL** very fruitfully exploited as a **universal metalogic**
- systematic study of a **prominent philosophical argument**
- even some **novel results** were found by **HOL-ATPs**
- infrastructure can be adapted for **other logics and logic combinations**

Relevance (wrt foundations and applications)
- Theoretical Philosophy, Artificial Intelligence, Computer Science, Maths

Little related work: only for Anselm’s simpler argument
- first-order ATP **PROVER9**  
  [OppenheimerZalta, 2011]
- interactive proof assistant **PVS**  
  [Rushby, 2013]

Future work
- continuation of systematic study of the ontological argument
- further studies in **Computational Metaphysics**
Germany
- Telepolis & Heise
- Spiegel Online
- FAZ
- Die Welt
- Berliner Morgenpost
- Hamburger Abendpost
-...

Austria
- Die Presse
- Wiener Zeitung
- ORF
-...

Italy
- Repubblica
- Ilsussidario
-...

India
- DNA India
- Delhi Daily News
- India Today
-...

US
- ABC News
-...

International
- Spiegel International
- Yahoo Finance
- United Press Intl.
-...

See links at https://github.com/FormalTheology/GoedelGod/tree/master/Press