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REACTION-DIFFUSION EQUATIONS WITH SPATIALLY DISTRIBUTED HYSTERESIS*

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Abstract. This paper deals with reaction-diffusion equations involving a hysteretic discontinuity in the source term, which is defined at each spatial point. In particular, such problems describe chemical reactions and biological processes in which diffusive and nondiffusive substances interact according to hysteresis law. We find sufficient conditions that guarantee the existence and uniqueness of solutions as well as their continuous dependence on initial data.

Key words. spatially distributed hysteresis, reaction-diffusion equation, well-posedness

AMS subject classifications. 35K57, 35K45, 47J40

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1. Introduction.

This paper deals with reaction-diffusion equations involving a hysteretic discontinuity which is defined at each spatial point. In particular, these problems describe chemical reactions and biological processes in which diffusive and nondiffusive substances interact according to hysteresis law. We illustrate this by a model describing a growth of a colony of bacteria (Salmonella typhimurium) on a petri plate (see [7, 8]). Let $Q \subset \mathbb{R}^n$ be a bounded domain and $B(x,t)$ denote the density of nondiffusing bacteria in Q, while $u_1(x,t)$ and $u_2(x,t)$ denote the concentrations of diffusing buffer (pH level) and histidine (nutrient) in Q, respectively. These three unknown functions satisfy the following equations in Q:

$$\begin{cases} \dfrac{\partial B}{\partial t} = avB, \\ \dfrac{\partial u_1}{\partial t} = D_1 \Delta u_1 - a_1 v B, \\ \dfrac{\partial u_2}{\partial t} = D_2 \Delta u_2 - a_2 v B \end{cases} \tag{1.1}$$

supplemented by initial and no-flux (Neumann) boundary conditions. In these equations $D_1, D_2, a, a_1, a_2 > 0$ are given constants. The function $v = v(x,t)$ corresponds to the growth rate of bacteria and is defined by hysteresis law. In the simplest case, $v(x,t)$ takes value 1 if $u_1(x,t)$ and $u_2(x,t)$ are large enough and value 0 if $u_1(x,t)$ and $u_2(x,t)$ are small enough. More precisely, one defines two curves Γ_{on} and Γ_{off} on the plane (u_1, u_2), which divide the first quadrant into three parts M_{on}, M_{off}, and $M_{\text{on-off}}$. Now $v(x,t) = 1$ whenever $(u_1(x,t), u_2(x,t)) \in M_{\text{on}}$ and $v(x,t) = 0$ whenever $(u_1(x,t), u_2(x,t)) \in M_{\text{off}}$, while $v(x,t)$ takes either value 1 or 0 in $M_{\text{on-off}}$ depending

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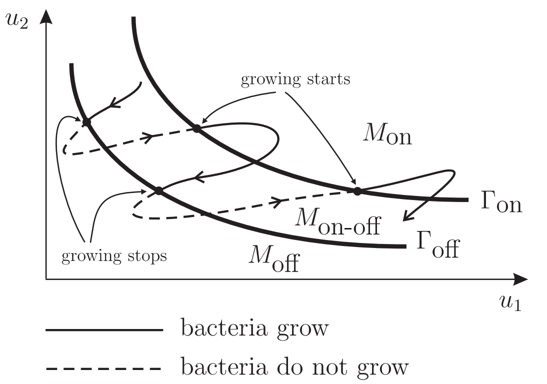

FIG. 1.1. *Regions of different behavior of hysteresis $\mathcal{H}$.*

FIG. 1.2. *Density of bacteria after growth has stopped.*

on whether the trajectory $(u_1(x,t), u_2(x,t))$ entered the region $M_{\text{on-off}}$ through Γ_{on} or Γ_{off}. This hysteretic behavior is depicted in Figure 1.1; see more details in [7, 8].

In the above example, hysteresis $v(x,t)$ may switch at different spatial points at different time moments. This allows one to divide the spatial domain Q into connected subdomains: in each of these subdomains, hysteresis $v(x,t)$ takes the same value (1 or 0) and thus defines the spatial topology of itself. The boundaries between these subdomains are free boundaries whose motion depends both on the reaction-diffusion equations and hysteresis. The interplay between those two leads to formation of spatio-temporal patterns. First numerical simulations exhibiting such patterns have been carried out in [7,8] for the case where Q is a disc in $\mathbb{R}^2$. The appearing pattern corresponds to concentric rings which are formed by $B(x,t)$ as $t \to \infty$ (see Figure 1.2).

There is a vast literature on parabolic equations containing regularized hysteresis, in particular the Preisach model (see, e.g., [3,17] and the references therein). However, the case of discontinuous hysteresis has been studied much less. First rigorous results about the existence of solutions of parabolic equations with hysteresis in the source term have been obtained in [1, 10, 18] for multivalued hysteresis. Formal asymptotic expansions of solutions were recently obtained for some special case in [9]. Questions about the uniqueness of solutions and their continuous dependence on initial data as well as a thorough analysis of pattern formation remained open.

All the above questions are closely connected with slow-fast systems where hysteresis is replaced by a "fast" function satisfying an ordinary differential equation with a small parameter at the time derivative and (typically) cubic nonlinearity. Then a possibility of singular perturbation limit is generally an open question. Some results in this direction can be found, e.g., in [4, 13], where the authors study equations of the form $u_t = \Delta\Phi(u)$ with cubic nonlinearity Φ. However, this direction is beyond the scope of our paper.

In this paper, we deal with the following prototype problem:

$$(1.2) \qquad \begin{cases} u_t = u_{xx} + f(u,v), \quad x \in (0,1),\ t>0, \\ u_x|_{x=0} = u_x|_{x=1} = 0, \\ u|_{t=0} = \varphi(x), \\ v(x,t) = \mathcal{H}(u(x,\cdot))(t), \quad x \in (0,1),\ t>0, \end{cases}$$

where $v(x,t)$ represents hysteresis at a point x. Loosely speaking, it is defined as follows. We fix two thresholds α and β, $\alpha < \beta$ (analogues of $\Gamma_{\rm on}$ and $\Gamma_{\rm off}$ in the above example). Further, we fix two functions $H_1(u)$ and $H_2(u)$. Let $x \in (0,1)$ be fixed. If $u(x,t) \le \alpha$, then we set $v(x,t) = H_1(u(x,t))$; if $u(x,t) \ge \beta$, then we set $v(x,t) = H_2(u(x,t))$; if $\alpha < u(x,t) < \beta$, then we set $v(x,t) = H_1(u(x,t))$ or $H_2(u(x,t))$ depending on whether $u(x,t)$ entered the interval (α,β) through α or β, respectively (see Figure 2.1).

We introduce a novel approach for treating, in a unified manner, existence, uniqueness, and continuous dependence of solutions on initial data for problem (1.2). Our approach is based on a so-called *spatial transversality* notion, which we also introduce for the first time in the context of spatially distributed hysteresis. In the one-dimensional case, we say that the initial function is spatially transverse if, roughly speaking, it has a nonvanishing derivative on the free boundaries between the above-mentioned subdomains. Similarly, one can define spatial transversality of a solution (at each time moment). Under the assumption that the initial data are transverse, we prove that a solution exists on a small time interval and can be continued as long as it remains spatially transverse. Moreover, if the solution is unique, it continuously depends on the initial data. For completeness of exposition, we formulate a theorem on uniqueness of a solution and refer the reader to [5] for its proof.

To our knowledge, these are the first results, where the existence and uniqueness of a solution is guaranteed in the sense that $v(x,t) = \mathcal{H}(u(x,\cdot))(t)$ with a single-valued function $\mathcal{H}(u(x,\cdot))(t)$ (rather than $v(x,t) \in \mathcal{H}(u(x,\cdot))(t)$ with a multiple-valued function $\mathcal{H}(u(x,\cdot))(t)$). Furthermore, we believe that our approach is also an appropriate tool for a rigorous analysis of a mechanism of pattern formation described above as it allows one to “track” the free boundaries responsible for the appearance of the concentric rings. This is a subject of future work.

For clarity, all the results of the present paper are proved for a one-dimensional domain and a scalar reaction-diffusion equation. But we note that the developed methods are not based on the maximum principle; therefore, they can be applied to systems of reaction-diffusion equations (see [6]). Furthermore, we believe that our methods can be applied to systems in multidimensional domains, which is also a subject of future work.

The paper is organized as follows. In section 2, we define functional spaces, introduce spatially distributed hysteresis (i.e., defined at every spatial point), and set the prototype problem (1.2). Next, we discuss assumptions concerning the nonlinearity f, the hysteresis operator $\mathcal{H}$, and the transversality of the initial function φ. The most important notions here are the spatial topology of hysteresis and transversality. We illustrate them by the following situation. Suppose that, for a function $u(x,t)$, there exists a continuous function $b(t)$, $t \in [0,T]$, taking values in $(0,1)$ such that

$$\mathcal{H}(u(x,\cdot))(t) = \begin{cases} H_1(u(x,t)), & 0 \le x \le b(t), \\ H_2(u(x,t)), & b(t) < x \le 1. \end{cases} \tag{1.3}$$

Then we say that u *preserves spatial topology* of hysteresis in the sense that, for all t, there are exactly two subintervals $(0,b(t))$ and $(b(t),1)$; hysteresis is given by $H_1(u)$ on one of them and by $H_2(u)$ on the other. In this situation, u is said to be *transverse* on $[0,T]$ if

1. $u(x,t) < \beta$ for $x \in [0,b(t)]$,
2. $u(x,t) > \alpha$ for $x \in (b(t),1]$,
3. $u_x(b(t),t) > 0$ whenever $u(b(t),t) = \alpha$.

In the end of section 2, we formulate the main results of the paper: Theorem 2.16 (local existence of transverse solutions preserving topology), Theorem 2.18 (global existence of transverse solutions), and Theorem 2.19 and Corollary 2.20 (continuous dependence on initial data). To make the exposure complete, we also formulate Theorem 2.22 (uniqueness of transverse solutions), which is proved in [5].

In section 3, we collect auxiliary results. First, we recall a theorem on the well-posedness for linear parabolic problems and then for semilinear problems. We show that hysteresis $\mathcal{H}$ can be treated as a continuous operator in suitable functional spaces, provided that its domain consists of spatially transverse functions (although the function $v(x,t) = \mathcal{H}(u(x,\cdot))(t)$ still may have jumps at different spatial points at different time moments). This allows us to prove Theorem 3.9, which is the main tool in the study of problem (1.2). This theorem deals with the auxiliary problem

$$\begin{cases} u_t = u_{xx} + f(u, v_0), \quad x \in (0,1),\ t > 0, \\ u_x|_{x=0} = u_x|_{x=1} = 0, \\ u|_{t=0} = \varphi(x). \end{cases} \tag{1.4}$$

The meaning of v_0 is as follows. Suppose we are given functions $u_0(x,t)$ and $b_0(t)$. Then

$$v_0(x,t) = \begin{cases} H_1(u_0(x,t)), & 0 \le x \le b_0(t), \\ H_2(u_0(x,t)), & b_0(t) < x \le 1. \end{cases} \tag{1.5}$$

One can think of $b_0(t)$ as of a free boundary defining the spatial topology of $v_0(x,t)$ in the sense similar to the above. However, v_0 need not coincide with hysteresis $\mathcal{H}(u_0)$. (It does only if u_0 preserves spatial topology and the corresponding subintervals are divided by the point $b_0(t)$.) Theorem 3.9 states, in particular, that a solution u of problem (1.4) exists and is unique on some time interval $[0,T]$, where $T > 0$ does not depend on u_0 and b_0.

In section 4, we prove the main results from section 2. The local existence is proved by means of Theorem 3.9 and the Schauder fixed-point theorem for a map defined on pairs $(u_0, b_0) \in C([0,1] \times [0,T]) \times C[0,T]$. To prove global existence, we show that any local transverse solution can be continued as long as it remains transverse. The continuous dependence of solutions is also based on Theorem 3.9.

In section 5, we generalize the dissipativity condition for the nonlinearities $f(u,v)$ and $H_1(u), H_2(u)$ (cf. Example 2.12).

2. Setting of the problem.

2.1. Functional spaces. Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a domain with piecewise smooth boundary. We denote by $L_q(\Omega)$ the standard Lebesgue space with $q > 1$.

For an integer $l \ge 0$, we denote by $C^l(\overline{\Omega})$ the space of functions which are continuous and have continuous derivatives up to order l in $\overline{\Omega}$. For $l = 0$, we write $C(\overline{\Omega})$. Denote by $C^\gamma(\overline{\Omega})$, $0 < \gamma < 1$, the Hölder space with the norm

$$\|v\|_{C^\gamma(\overline{\Omega})} = \|v\|_{C(\overline{\Omega})} + \sup_{x,y \in \Omega} \frac{|v(x) - v(y)|}{|x-y|^\gamma}.$$

If $\Omega = (a,b)$, we will write $L_q(a,b)$, $C[a,b]$, $C^l[a,b]$, and $C^\gamma[a,b]$. If $\Omega = (0,1)$, we will write L_q, C, C^l, and C^γ, respectively. We also write $C^\infty = \bigcap_{l=1}^\infty C^l$.

For a natural l, denote by $W_q^l = W_q^l(0,1)$ the space with the norm

$$\|v\|_{W_q^l} = \sum_{j=0}^{l} \|v^{(j)}\|_{L_q},$$

where $v^{(j)}$ is the generalized derivative of order j.

For any noninteger $l > 0$, denote by $W_q^l = W_q^l(0,1)$ the space with the norm

$$\|v\|_{W_q^l} = \|v\|_{W_q^{[l]}} + \left(\int_0^1 dx \int_0^1 \frac{|v^{([l])}(x) - v^{([l])}(y)|^q}{|x-y|^{1+q(l-[l])}} dy\right)^{1/q},$$

where $[l]$ is the integer part of l.

Let $Q_T = (0,1) \times (0,T)$. Denote by $C^{1,0}(\overline{Q}_T)$ the space of function $u(x,t)$ such that $u, u_x \in C(\overline{Q}_T)$. We also introduce the anisotropic Sobolev space $W_q^{2,1}(Q_T)$ with the norm

$$\|u\|_{W_q^{2,1}(Q_T)} = \left(\int_0^T \|u(\cdot,t)\|_{W_q^2}^q \, dt + \int_0^T \|u_t(\cdot,t)\|_{L_q}^q \, dt\right)^{1/q}.$$

The following embedding results hold (see, e.g., [12, Chap. 2]).

LEMMA 2.1. *If $u \in W_q^{2,1}(Q_T)$ with $q > 3$, then $u, u_x \in C^\gamma(\overline{Q}_T)$ and*

$$\|u\|_{C^\gamma(\overline{Q}_T)} + \|u_x\|_{C^\gamma(\overline{Q}_T)} \le c\|u\|_{W_q^{2,1}(Q_T)}$$

for any $0 \le \gamma < 1 - 3/q$, where $c > 0$ depends on q, γ, and T but not on u.

LEMMA 2.2. *If $\varphi \in W_q^{2-2/q}$ with $q > 3$, then $\varphi, \varphi' \in C^\gamma$ and*

$$\|\varphi\|_{C^\gamma} + \|\varphi'\|_{C^\gamma} \le c\|\varphi\|_{W_q^{2-2/q}}$$

for any $0 \le \gamma < 1 - 3/q$, where $c > 0$ depends on q and γ but not on φ.

Throughout the paper, we fix q and γ such that

$$q > 3, \quad 0 < \gamma < 1 - 3/q. \tag{2.1}$$

In what follows, we will consider solutions of parabolic problems in the space $W_q^{2,1}(Q_T)$. To define a space of initial data, we will use the fact that if $u \in W_q^{2,1}(Q_T)$, then the trace $u|_{t=t_0}$ is well defined and belongs to $W_q^{2-2/q}$ for all $t_0 \in [0,T]$ (see Lemma 2.4 in [12, Chap. 2]).

Moreover, due to [16, sections 4.3.3 and 4.4.1], a set of functions from $W_q^{2-2/q}$ satisfying the zero Neumann boundary conditions is well defined and is a subspace of $W_q^{2-2/q}$. We denote it by $\mathcal{W}$ and equip with the norm of $W_q^{2-2/q}$. We will further assume that the initial function φ belongs to $\mathcal{W}$.

2.2. Hysteresis. In this section, we introduce a hysteresis operator defined for functions of time variable t. Then we extend the definition to a spatially distributed hysteresis acting on a space of functions of time variable t and space variable x.

We fix two numbers α and β such that $\alpha < \beta$. The numbers α and β will play a role of thresholds for the hysteresis operator. Next, we introduce continuous functions

$$H_1 : (-\infty, \beta] \mapsto \mathbb{R}, \qquad H_2 : [\alpha, \infty) \mapsto \mathbb{R}.$$

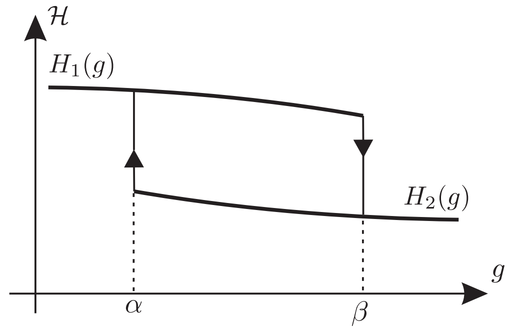

Fig. 2.1. *The hysteresis operator* $\mathcal{H}$.

It is convenient to extend them to $\mathbb{R}$ as follows:

$$\begin{aligned} H_1(u) &= H_1(\beta) \text{ for } u > \beta, \\ H_2(u) &= H_2(\alpha) \text{ for } u < \alpha. \end{aligned} \tag{2.2}$$

We assume throughout that the following condition holds.

Condition 2.3. *There is* $\sigma \in (0,1]$ *such that, for any* $U > 0$, *there exists* $M = M(U) > 0$ *with the following property*:

$$|H_j(u) - H_j(\hat{u})| \le M|u - \hat{u}|^\sigma, \quad j = 1, 2, \quad |u|, |\hat{u}| \le U.$$

We fix $T > 0$ and denote by $C_r[0,T)$ the linear space of functions which are continuous on the right in $[0,T)$. For any $\zeta_0 \in \{1,2\}$ (*initial configuration*) and $g \in C[0,T]$ (*input*), we introduce the *configuration* function

$$\zeta : \{1,2\} \times C[0,T] \to C_r[0,T), \quad \zeta(t) = \zeta(\zeta_0, g)(t)$$

as follows. Let $X_t = \{t' \in (0,t] : g(t') = \alpha \text{ or } \beta\}$. Then

$$\zeta(0) = \begin{cases} 1 & \text{if } g(0) \le \alpha, \\ 2 & \text{if } g(0) \ge \beta, \\ \zeta_0 & \text{if } g(0) \in (\alpha, \beta) \end{cases}$$

and for $t \in (0,T]$

$$\zeta(t) = \begin{cases} \zeta(0) & \text{if } X_t = \varnothing, \\ 1 & \text{if } X_t \ne \varnothing \text{ and } g(\max X_t) = \alpha, \\ 2 & \text{if } X_t \ne \varnothing \text{ and } g(\max X_t) = \beta. \end{cases}$$

Now we introduce the *hysteresis operator* (cf. [11, 17])

$$\mathcal{H} : \{1,2\} \times C[0,T] \to C_r[0,T)$$

by the following rule. For any initial configuration $\zeta_0 \in \{1,2\}$ and input $g \in C[0,T]$, the function $\mathcal{H}(\zeta_0, g) : [0,T] \to \mathbb{R}$ (*output*) is given by

$$\mathcal{H}(\zeta_0, g)(t) = H_{\zeta(t)}(g(t)),$$

where $\zeta(t)$ is the configuration function defined above (see Figure 2.1).

Remark 2.4. One usually assumes that $H_1(u) - H_2(u)$ is sign constant for $u \in [\alpha, \beta]$. However, we will never use this assumption in our paper.

Now we introduce a *spatially distributed hysteresis*. Assume that the initial configuration and the input function depend on spatial variable $x \in [0,1]$. Denote them by $\xi_0(x)$ and $u(x,t)$, where

$$\xi_0 : [0,1] \mapsto \{1,2\}, \qquad u : [0,1] \times [0,T] \mapsto \mathbb{R}.$$

Let $u(x,\cdot) \in C[0,T]$. Set $\varphi(x) = u(x,0)$.

DEFINITION 2.5. *We say that the function $\varphi(x)$ and the initial configuration $\xi_0(x)$ are* consistent *if, for any $x \in [0,1]$,*

$$\xi_0(x) \in \begin{cases} \{1\} & \text{if } \varphi(x) \le \alpha, \\ \{2\} & \text{if } \varphi(x) \ge \beta, \\ \{1,2\} & \text{if } \varphi(x) \in (\alpha,\beta). \end{cases}$$

Assume that $\xi_0(x)$ and $\varphi(x)$ are consistent. Then we can define the function

$$v(x,t) = \mathcal{H}(\xi_0(x), u(x,\cdot))(t), \tag{2.3}$$

which is called *spatially distributed hysteresis*. The *spatial configuration* is given by

$$\xi(x,t) = \zeta(\xi_0(x), u(x,\cdot))(t). \tag{2.4}$$

Note that the consistency of $\xi_0(x)$ and $\varphi(x)$ and the fact that $\xi(x,\cdot)$ is right continuous guarantee that

$$\lim_{t\to 0} \xi(x,t) = \xi_0(x).$$

In this paper, we shall deal will spatially distributed hysteresis whose spatial configuration has finitely many discontinuity points in x for each t. In particular, we assume throughout that the following holds.

CONDITION 2.6. *The initial configuration $\xi_0(x)$ has finitely many discontinuity points in $(0,1)$, i.e., there are points $0 = \overline{b}_0 < \overline{b}_1 < \cdots < \overline{b}_M < \overline{b}_{M+1} = 1$ $(M \ge 1)$ such that*

1. $\xi_0(x) = \text{const}$ *for $x \in (\overline{b}_i, \overline{b}_{i+1})$, $i = 0, \dots, M$,*
2. $\xi_0(\overline{b}_i + 0) \ne \xi_0(\overline{b}_i - 0)$*, $i = 1, \dots, M$.*

Let us introduce notions of transversality and hysteresis spatial topology, which will play a central role further on.

DEFINITION 2.7. *We say that a function $\varphi \in C^1$ is* transverse *(with respect to a spatial configuration $\xi_0(x)$) if it is consistent with $\xi_0(x)$ and the following hold:*

1. *if $\varphi(\overline{x}) = \alpha$ and $\varphi'(\overline{x}) = 0$ for some $\overline{x} \in [0,1]$, then $\xi_0(\overline{x}) = 1$ in a neighborhood of $\overline{x}$;*
2. *if $\varphi(\overline{x}) = \beta$ and $\varphi'(\overline{x}) = 0$ for some $\overline{x} \in [0,1]$, then $\xi_0(\overline{x}) = 2$ in a neighborhood of $\overline{x}$.*

DEFINITION 2.8. *We say that a function $u \in C^{1,0}(\overline{Q}_T)$ is* transverse on $[0,T]$ *(with respect to a spatial configuration $\xi(x,t)$) if, for every fixed $t \in [0,T]$, the function $u(\cdot,t)$ is transverse with respect to the spatial configuration $\xi(\cdot,t)$.*

DEFINITION 2.9. *We say that a function $u \in C^{1,0}(\overline{Q}_T)$* preserves spatial topology (of a spatial configuration $\xi(x,t)$) on $[0,T]$ *if there is $M > 0$ such that, for $t \in [0,T]$, there are continuous functions*

$$0 \equiv b_0(t) < b_1(t) < \cdots < b_M(t) < b_{M+1}(t) \equiv 1 \tag{2.5}$$

with the properties

1. $\xi(x,t) = \text{const}$ *for* $x \in (b_i(t), b_{i+1}(t))$, $i = 0, \dots, M$,
2. $\xi(b_i(t)+0,t) \neq \xi(b_i(t)-0,t)$, $i = 1, \dots, M$.

We will also say in this case that u is topology preserving.

2.3. Reaction-diffusion equations with hysteresis. The main object of this paper is the reaction-diffusion equation

$$u_t = u_{xx} + f(u,v), \quad (x,t) \in Q_T, \tag{2.6}$$

where $v = v(x,t)$ represents the spatially distributed hysteresis given by (2.3),

$$v(x,t) = \mathcal{H}(\xi_0(x), u(x,\cdot))(t).$$

We also impose the Neumann boundary conditions and the initial conditions

$$u_x|_{x=0} = u_x|_{x=1} = 0, \tag{2.7}$$

$$u|_{t=0} = \varphi(x), \quad x \in (0,1). \tag{2.8}$$

Along with Conditions 2.3 and 2.6, we assume that the right-hand side f, the hysteresis branches $H_j(u)$, and the initial data satisfy the following conditions.

CONDITION 2.10 (Lipschitz continuity). *For any bounded set $B \subset \mathbb{R}^2$, there is a constant $L = L(B) > 0$ such that*

$$|f(u_1,v_1) - f(u_2,v_2)| \le L(|u_1 - u_2| + |v_1 - v_2|) \quad \forall (u_j, v_j) \in B, \ j = 1, 2.$$

CONDITION 2.11 (dissipativity). *For any sufficiently large $U > 0$ and for all u such that $|u| \le U$, we have*

$$f(U, H_j(u)) < 0, \quad f(-U, H_j(u)) > 0, \quad j = 1, 2.$$

Example 2.12. The simplest example of $f(u,v)$ satisfying Conditions 2.10 and 2.11 is $f(u,v) = -du + v$, where $d > 0$ and

$$\lim_{U \to +\infty} \left(-dU + \max_{|u| \le U} H_j(u) \right) < 0, \qquad \lim_{U \to -\infty} \left(-dU + \max_{|u| \le U} H_j(u) \right) > 0.$$

The simplest example of the hysteresis in this case is given by

$$H_1(u) \equiv h_1, \quad H_2(u) \equiv h_2,$$

where h_1 and h_2 are arbitrary real numbers (without any sign restrictions).

Moreover, results in section 5 imply that if $h_1 \le 0 \le h_2$, then one can set $d = 0$.

The natural assumption on the initial data is as follows.

CONDITION 2.13 (consistency). *The initial spatial configuration $\xi_0(x)$ and the initial data $\varphi(x)$ are consistent in the sense of Definition* 2.5.

To prove the existence of a solution for problem (2.6)–(2.8), we will restrict a class of admissible initial data, namely, we assume the following throughout.

CONDITION 2.14. *The initial function $\varphi(x)$ is transverse in the sense of Definition* 2.7.

We give a definition of a solution of problem (2.6)–(2.8), assuming that $\varphi \in \mathcal{W}$.

DEFINITION 2.15. *A function $u(x,t)$ is called a* solution of problem (2.6)–(2.8) (in Q_T) *if $u \in W_q^{2,1}(Q_T)$, $v(x,t)$ is measurable, u and v satisfy* (2.6) *for a.e.* $(x,t) \in Q_T$, *and conditions* (2.7) *and* (2.8) *are satisfied in the sense of traces.*

It follows from this definition and from Lemma 2.1 that any solution $u(x,t)$ belongs to $C(\overline{Q}_T)$. Therefore, the function $v(x,t)$ is well defined by (2.3) and belongs to $L_\infty(Q_T)$.

2.4. Main results. Our main results are the following three theorems. In all of them, we assume that Conditions 2.3–2.14 hold and that q and γ satisfy (2.1).

THEOREM 2.16 (local existence). *There is $T > 0$ such that the following hold.*

1. *Any solution $u \in W_q^{2,1}(Q_T)$ of problem (2.6)–(2.8) in Q_T is transverse and preserves spatial topology. For each $t \in [0, T]$, the function $v(\cdot, t)$ has exactly M discontinuity points $b_1(t) < \cdots < b_M(t)$ in $(0, 1)$. Moreover,*
   (a) $b_i(0) = \overline{b}_i$,
   (b) $b_i \in C^\gamma[0, T]$.
2. *There is at least one transverse topology preserving solution $u \in W_q^{2,1}(Q_T)$ of problem (2.6)–(2.8) in Q_T.*

The next theorem deals with continuation of solutions to their maximal intervals of transverse existence.

DEFINITION 2.17. *We say that $[0, T_{\max})$, $T_{\max} \le \infty$, is a* maximal interval of transverse existence *of a solution u of problem (2.6)–(2.8) if*

1. *for any $T < T_{\max}$, the function u is a transverse solution of problem (2.6)–(2.8) in Q_T,*
2. *either $T_{\max} = \infty$, or $T_{\max} < \infty$ and u is a solution in $Q_{T_{\max}}$, but $u(\cdot, T_{\max})$ is not transverse with respect to $\xi(\cdot, T_{\max})$.*

Note that a solution need not be topology preserving inside its maximal interval of transverse existence.

THEOREM 2.18 (continuation). *Let $u \in W_q^{2,1}(Q_{t_0})$ be a transverse topology preserving solution of problem (2.6)–(2.8) in Q_{t_0}. Then it can be continued to a maximal interval of transverse existence $[0, T_{\max})$, where $T_{\max} > t_0$ may depend on continuation.*

The following theorem shows that if a transverse topology preserving solution is unique, then it continuously depends on initial function φ and initial configuration ξ_0.

THEOREM 2.19 (continuous dependence on initial data). *We assume that the following hold.*

1. *There is $T > 0$ such that problem (2.6)–(2.8) with initial function $\varphi \in \mathcal{W}$ and initial configuration $\xi_0(x)$ defined by its discontinuity points $\overline{b}_1 < \cdots < \overline{b}_M$ admits a unique transverse topology preserving solution $u \in W_q^{2,1}(Q_s)$ in Q_s for any $s \le T$.*
2. *Let $\varphi_n \in \mathcal{W}$, $n = 1, 2, \dots$, be a sequence of other initial functions such that $\|\varphi - \varphi_n\|_{\mathcal{W}} \to 0$ as $n \to \infty$.*
3. *Let $\xi_{0n}(x)$, $n = 1, 2, \dots$, be a sequence of other initial configurations defined by their discontinuity points $\overline{b}_{1n} < \cdots < \overline{b}_{Mn}$ such that $\xi_{0n}(x) = \xi_0(x)$ for $x \in (0, \min(b_1, b_{1n}))$ and $\overline{b}_{jn} - \overline{b}_j \to 0$ as $n \to \infty$, $j = 1, \dots, M$.*

Then, for all sufficiently large n, problem (2.6)–(2.8) with initial function φ_n and initial configuration ξ_{0n} has at least one transverse topology preserving solution $u_n \in W_q^{2,1}(Q_T)$. Each sequence of such solutions satisfies

$$\|u_n - u\|_{W_q^{2,1}(Q_T)} \to 0, \quad \|b_{jn} - b_j\|_{C[0,T]} \to 0, \quad j = 1, \dots, M, \quad n \to \infty,$$

where $b_j(t)$ and $b_{jn}(t)$ are the, respective, discontinuity points of the configuration functions $\xi(t)$ and $\xi_j(t)$, $j = 1, \dots, M$.

As it was mentioned above, a solution u need not be topology preserving on its maximal interval of transverse existence $[0, T_{\max})$. But Theorem 2.19 deals only with topology preserving solutions. The reason is the following. Let $t_{\max}$ ($t_{\max} < T_{\max}$) be a moment at which the solution u changes topology. Although Theorem 2.19 guarantees that u_n remain transverse and topology preserving on any interval $[0, T] \subset$

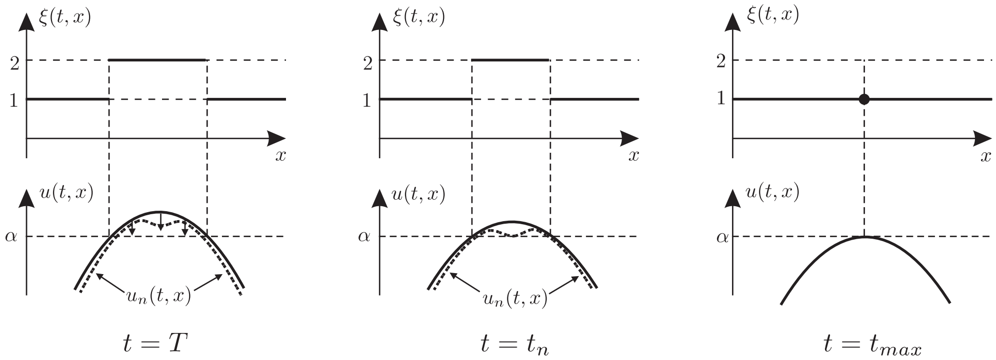

FIG. 2.2. *Approximation u_n becomes nontransverse at a moment $t_n \in (T, t_{\max})$.*

$[0, t_{\max})$ and approximate u, it may happen that each u_n becomes nontransverse at some moment $t_n \in (T, t_{\max}]$ for all $n \geq N = N(T)$. This situation is illustrated by Figure 2.2.

However, if we a priori know that all u_n are transverse on some interval $[0, T] \subset [0, T_{\max})$ (possibly with $T \geq t_{\max}$), then we can prove that u_n approximate u. Now the topology of u can change. This happens if some neighboring discontinuity points $b_j(t)$ and $b_{j+1}(t)$ of $\xi(x, t)$ converge to the same point $\hat{b}_j$ as $t \to t_{\max}$. Then, for $t > t_{\max}$, the point $b_j(t) = b_{j+1}(t) = \hat{b}_j$ is not a discontinuity point of $\xi(x, t)$ any more. However, if we consider the functions $b_j(t)$ and $b_{j+1}(t)$ on the whole interval $[0, T]$ assuming them constant for $t > t_{\max}$, then we can prove that $b_{jn}(t)$ approximate $b_j(t)$ in $C[0, T]$. Here we use the same convention about $b_{jn}(t)$.

Let us formulate this assertion as a corollary from Theorem 2.19.

COROLLARY 2.20.

1. *We fix some time interval $[0, T]$, initial function $\varphi \in \mathcal{W}$, and initial configuration $\xi_0(x)$ defined by its discontinuity points $\overline{b}_1 < \cdots < \overline{b}_M$. Assume that problem* (2.6)–(2.8) *with initial data φ and $\xi_0(x)$ admits a unique transverse solution $u \in W_q^{2,1}(Q_s)$ in Q_s for any $s \leq T$.*
2. *Let assumptions* 2 *and* 3 *of Theorem* 2.19 *hold. Moreover, suppose that, for each n, there exists a transverse solution $u_n \in W_q^{2,1}(Q_T)$ of problem* (2.6)–(2.8) *with initial data φ_n and $\xi_{0n}(x)$.*

Then

$$\|u_n - u\|_{W_q^{2,1}(Q_T)} \to 0, \quad \|b_{jn} - b_j\|_{C[0,T]} \to 0, \quad j = 1, \ldots, M, \quad n \to \infty.$$

For the completeness of exposition, we also formulate the uniqueness theorem, (see [5] for the proof). In the uniqueness theorem, along with Conditions 2.3–2.14 we assume that the following holds.

CONDITION 2.21. *There is $\sigma \in [0, 1)$ such that, for any $U > 0$, there exists $M = M(U) > 0$ with the properties*

$$|H_1(u) - H_1(\hat{u})| \leq \frac{M}{(\beta - u)^\sigma + (\beta - \hat{u})^\sigma} |u - \hat{u}| \quad \forall u, \hat{u} \in [-U, \beta), \tag{2.9}$$

$$|H_2(u) - H_2(\hat{u})| \leq \frac{M}{(u - \alpha)^\sigma + (\hat{u} - \alpha)^\sigma} |u - \hat{u}| \quad \forall u, \hat{u} \in (\alpha, U]. \tag{2.10}$$

THEOREM 2.22 (uniqueness). *Let the functions $H_j(u)$, $j = 1, 2$, additionally satisfy Condition* 2.21. *Assume that $u, \hat{u} \in W_q^{2,1}(Q_{T_0})$ are two transverse solutions of problem* (2.6)–(2.8) *in Q_{T_0} for some T_0. Then $u = \hat{u}$.*

Remark 2.23.

1. Any locally Lipschitz continuous functions $H_1(u)$ and $H_2(u)$ satisfy Condition 2.21. Moreover, this condition covers the important case where non-Lipschitz hysteresis branches $H_1(u)$ and $H_2(u)$ appear as a singular-perturbation limit of a slow-fast system (see [5]).
2. Any $H_1(u)$ and $H_2(u)$ satisfying Condition 2.3 are locally Hölder continuous with exponent $1 - \sigma$ on $(-\infty, \beta]$ and $[\alpha, \infty)$, respectively. Therefore, they satisfy Condition 2.3.

2.5. Technical simplification. For clarity of exposition, we give detailed proofs of the main results for the functions $\varphi(x)$ and $\xi_0(x)$ satisfying the following condition.

CONDITION 2.24.

1. *For some $\overline{b} \in (0, 1)$, one has*

$$\xi_0(x) = \begin{cases} 1, & x \le \overline{b}, \\ 2, & x > \overline{b}. \end{cases} \tag{2.11}$$

2. $\varphi(x) < \beta$ *for* $x \in [0, \overline{b}]$.
3. $\varphi(x) > \alpha$ *for* $x \in (\overline{b}, 1]$.
4. *If* $\varphi(\overline{b}) = \alpha$, *then* $\varphi'(\overline{b}) > 0$.

It follows from this condition that the hysteresis (2.3) at the initial moment is given by

$$v|_{t=0} = \begin{cases} H_1(\varphi(x)), & x \le \overline{b}, \\ H_2(\varphi(x)), & x > \overline{b}. \end{cases}$$

Clearly, Condition 2.6 (with $M = 1$ and $\overline{b}_1 = \overline{b}$) and Conditions 2.13 and 2.14 are satisfied in this case.

We denote by E_m, $m \in \mathbb{N}$, the set of pairs (φ, ξ_0) satisfying Condition 2.13 such that $\varphi \in \mathcal{W}$, $\xi_0(x)$ is of the form (2.11) and the following hold:

1. $\overline{b} \in [1/m, 1 - 1/m]$,
2. $\varphi(x) \le \beta - 1/m^2$ for $x \in [0, \overline{b}]$,
3. $\varphi(x) \ge \alpha + 1/m^2$ for $x \in [\overline{b} + 1/m, 1]$,
4. if $x \in [\overline{b}, \overline{b} + 1/m]$ and $\varphi(x) \in [\alpha, \alpha + 1/m^2]$, then $\varphi'(x) \ge 1/m$,
5. $\|\varphi\|_{\mathcal{W}} \le m$.

Note that item 4 in the definition of E_m yields

$$\varphi(x) \ge \alpha + \frac{1}{m}(x - \bar{b}), \quad x \in [\bar{b}, \bar{b} + 1/m]. \tag{2.12}$$

This observation easily implies $E_m \subset E_{m+1}$. Moreover, Lemma 2.25 below shows that the union of all sets E_m coincides with the set of all transverse data satisfying Condition 2.24. These sets allow us to measure the "level of transversality" of the data. The precise statement is as follows.

LEMMA 2.25.

1. *Functions $\varphi(x)$, $\xi_0(x)$ satisfy Condition* 2.24 *if and only if $(\varphi, \xi_0) \in \bigcup_{m=1}^{\infty} E_m$.*
2. *Let $\varphi_m \in \mathcal{W}$, and let $\xi_m(x)$ be defined analogously to* (2.11) *with some $\overline{b}_m$ instead of $\overline{b}$. If*

(a) $(\varphi_m, \xi_m) \in E_m \setminus E_{m-1}$, $m = 2, 3, \dots$,
(b) $\|\varphi_m - \varphi\|_{\mathcal{W}} \to 0$ *as* $m \to \infty$ *for some* $\varphi \in \mathcal{W}$,
(c) $\overline{b}_m - \overline{b} \to 0$ *as* $m \to \infty$ *for some* $\overline{b} \in [0, 1]$,

then $\overline{b} \in \{0, 1\}$ *or* $\varphi(x)$ *is not transverse with respect to* $\xi_0(x)$*, where* $\xi_0(x)$ *is given by* (2.11).

3. *Let* $(\varphi, \xi_0) \in E_m$ *for some* $m \in \mathbb{N}$. *Then there is* $\varepsilon = \varepsilon(m) > 0$ *such that* $(\tilde\varphi, \tilde\xi_0) \in E_{m+1}$ *whenever* $\|\tilde\varphi - \varphi\|_{\mathcal{W}} \le \varepsilon$, $|\tilde b - \overline{b}| \le \varepsilon$, *and* $\tilde\xi_0$ *is given analogously to* (2.11) *with* $\tilde b$ *instead of* $\overline{b}$.

Proof. 1. Let $\varphi(x)$ and $\xi_0(x)$ satisfy Condition 2.24. Items 1, 2, 5 in the definition of E_m directly follow from Condition 2.24 for a sufficiently large m.

Further, if $\varphi(\overline{b}) \ne \alpha$, then one can choose m such that $\varphi(x) \notin [\alpha, \alpha + 1/m^2]$ for all points $x \in [\overline{b}, \overline{b} + 1/m]$. In this case item 4 in the definition of E_m becomes void and item 3 in the definition of E_m is a trivial consequence of item 3 in Condition 2.24.

Finally, if $\varphi(\overline{b}) = \alpha$, then $\varphi'(\overline{b}) > 0$ and hence there exists $m > 0$ such that $\varphi'(x) \ge 1/m$ for $x \in [\overline{b}, \overline{b} + 1/m]$. This inequality implies item 4 of the definition of E_m and inequalities (2.12). Applying item 3 of Condition 2.13, we can further increase m so that item 3 holds.

If $(\varphi, \xi_0) \in E_m$ for some m, then it is obvious that Condition 2.24 holds.

2. Assertion 2 will follow from assertions 1 and 3. Indeed, if $\overline{b} \in (0, 1)$ and $\varphi(x)$ is transverse with respect to $\xi_0(x)$, then assertion 1 implies that $(\varphi, \xi_0) \in E_{m_0}$ for a sufficiently large m_0. Therefore, by assertion 3, $(\varphi_m, \xi_m) \in E_{m_0+1}$ for all sufficiently large m, which contradicts assumption 2(a) on the sequence (φ_m, ξ_m).

3. Let us prove assertion 3. Lemma 2.2 implies that

$$\|\tilde\varphi - \varphi\|_C \le c\varepsilon, \quad \|\tilde\varphi' - \varphi'\|_C \le c\varepsilon, \quad \|\tilde\varphi'\|_{C^\gamma} \le c(m+1), \tag{2.13}$$

where $c > 0$ does not depend on ε and m. Items 1, 2, 5 of the definition of E_{m+1} are a direct consequence of (2.13), provided that $\varepsilon > 0$ is small enough.

Let us prove item 4 in the definition of E_{m+1}. Consider $x \in [\tilde b, \tilde b + 1/(m+1)]$ such that $\tilde\varphi(x) \in [\alpha, \alpha + 1/(m+1)^2]$. If $x \ge \overline{b}$, then item 4 is a direct consequence of (2.13). Assume $x < \overline{b}$. Then $|x - \overline{b}| \le \varepsilon$, and inequalities (2.13) imply

$$\tilde\varphi(\overline{b}) \le \alpha + \frac{1}{(m+1)^2} + |x - \overline{b}|c(m+1) \le \alpha + \frac{1}{(m+1)^2} + \varepsilon c(m+1).$$

For small enough ε, this implies that $\varphi(\overline{b}) \le \alpha + 1/m^2$ and hence $\varphi'(\overline{b}) \ge 1/m$. This implies $\tilde\varphi'(\overline{b}) \ge 1/m - \varepsilon$ and (by inequalities (2.13))

$$\tilde\varphi'(x) \ge \frac{1}{m} - \varepsilon - c(m+1)\varepsilon^\gamma.$$

The latter inequality implies $\tilde\varphi'(x) \ge 1/(m+1)$ for small enough ε, which completes the proof of item 4.

Let us now prove item 3 in the definition of E_{m+1}. Consider $x \in [\tilde b + 1/(m+1), 1]$. If $x \ge \overline{b} + 1/m$, then the inequality $\varphi(x) \ge \alpha + 1/m^2$ and (2.13) imply $\tilde\varphi(x) \ge 1/(m+1)^2$ for small enough ε. Let $x < \overline{b} + 1/m$. The inequality $|\tilde b - \overline{b}| \le \varepsilon$ implies $x - \overline{b} \ge 1/(m+1) - \varepsilon$. Therefore,

$$\varphi(x) \ge \alpha + \frac{1}{m}\left(\frac{1}{m+1} - \varepsilon\right).$$

Now, using (2.13), we conclude that $\tilde\varphi(x) \ge \alpha + 1/(m+1)^2$, provided that $\varepsilon > 0$ is small enough, which completes the proof of assertion 3. $\square$

We will also need the following auxiliary statement.

LEMMA 2.26. *Let* $(\varphi, \xi_0) \in E_m$ *and* $\overline{b} \in (0,1)$ *be such that* (2.11) *holds. Then, for any functions* $\psi_1, \psi_2 \in C^1$ *satisfying*

$$\|\psi_i - \varphi\|_C + \|\psi_i' - \varphi'\|_C \le \frac{1}{4m^2}, \quad i = 1, 2, \tag{2.14}$$

the following hold.

1. *The equation* $\psi_i(x) = \alpha$ $(i = 1, 2)$ *has no more than one root in* $[\overline{b}, 1]$.
2. *If such a root exists, let us denote it by* a_i; *otherwise, we set* $a_i = \overline{b}$. *Then*

$$a_1, a_2 \in [\overline{b}, \overline{b} + 1/m], \tag{2.15}$$

$$|a_1 - a_2| \le 2m\|\psi_1 - \psi_2\|_C. \tag{2.16}$$

Proof. Item 1 and inclusions (2.15) are trivial consequences of items 3 and 4 in the definition of E_m.

Let us prove item 2. First, let us assume that both a_1, a_2 satisfy $\psi_{1,2}(a_{1,2}) = \alpha$. Without loss of generality, we can assume that $a_2 > a_1$. Let us fix an arbitrary $x \in [a_1, a_2]$. Item 4 in the definition of E_m and (2.14) imply that $\psi_2(x) \le 0$ and $\psi_1(x) \le \alpha + 1/(2m^2)$. Hence, due to (2.14), $\varphi(x) \le \alpha + 1/m^2$, which implies $\varphi'(x) \ge 1/m$ and, thus, $\psi_1'(x) \ge 1/(2m)$. The latter inequality yields

$$|a_2 - a_1| \le 2m|\psi_1(a_2) - \psi_2(a_2)|.$$

Assume that a_2 satisfies $\psi_2(a_2) = \alpha$ and the equation $\psi_1(x) = \alpha$ has no roots on the interval $[\overline{b}, 1]$. Using inequalities (2.12) and (2.14), we conclude that $\psi_1(\overline{b} + 1/(2m)) > 0$ and hence $\psi_1(\overline{b}) > 0$. Arguing similarly to the previous case, we conclude that $\psi_1'(x) \ge 1/(2m)$ for $x \in [\overline{b}, a_2]$ and hence

$$|a_2 - a_1| = |a_2 - \overline{b}| \le 2m|\psi_1(a_2) - \psi_1(\overline{b})| \le 2m|\psi_1(a_2) - \psi_2(a_2)|.$$

If neither of the equations $\psi_{1,2}(x) = \alpha$ has a root on the interval $[\overline{b}, 1]$, then $a_1 = a_2 = \overline{b}$ and (2.16) is trivial. ∎

3. Auxiliary results.

3.1. Linear parabolic problem. In this section, we formulate a well-known result on the solvability of the following linear parabolic equation in $W_q^{2,1}(Q_T)$:

$$\begin{cases} u_t = u_{xx} + F(x,t), \quad (x,t) \in Q_T, \\ u_x|_{x=0} = u_x|_{x=1} = 0, \\ u|_{t=0} = \varphi(x), \quad x \in (0,1). \end{cases} \tag{3.1}$$

Combining the results of [12, Chap. 4] (including Lemma 2.1 formulated above in our paper) with Theorem 3.1 in [2] and with the interpolation theory in [16, sections 1.14.5, 4.3.3, and 4.4.1], we obtain the following result.

THEOREM 3.1. *Assume that* q *and* γ *satisfy* (2.1). *Fix numbers* $T_0 \ge T > 0$. *Let* $F \in L_q(Q_T)$ *and* $\varphi \in \mathcal{W}$. *Then problem* (3.1) *has a unique solution* $u \in W_q^{2,1}(Q_T)$ *and*

$$\|u\|_{W_q^{2,1}(Q_T)} + \max_{t\in[0,T]} \|u(\cdot,t)\|_{\mathcal{W}} \le c_1(\|\varphi\|_{\mathcal{W}} + \|F\|_{L_q(Q_T)}),$$

$$\|u\|_{C^\gamma(\overline{Q}_T)} + \|u_x\|_{C^\gamma(\overline{Q}_T)} \le c_2(\|\varphi\|_{\mathcal{W}} + \|F\|_{L_q(Q_T)}),$$

where $c_1, c_2 > 0$ *depend on* q, γ, *and* T_0, *but do not depend on* T, φ, *and* F.

3.2. Semilinear parabolic problem. In this section, we consider an auxiliary semilinear initial boundary-value problem

$$\begin{cases} u_t = u_{xx} + f_0(u, x, t), \quad (x,t) \in Q_T, \\ u_x|_{x=0} = u_x|_{x=1} = 0, \\ u|_{t=0} = \varphi(x), \quad x \in (0,1). \end{cases} \tag{3.2}$$

3.2.1. $E_{\infty,T}$-mild solutions. First, we assume that $\varphi \in L_\infty$. We also assume that the function f_0 satisfies the following.

1. $f_0(u,x,t)$ is measurable with respect to $(x,t) \in [0,1] \times [0,\infty)$ for all $u \in \mathbb{R}$.
2. For every bounded set $B \subset \mathbb{R} \times [0,1] \times [0,\infty)$, there exists a constant $L = L(B) > 0$ such that

$$|f_0(u,x,t)| \le L \quad \forall (u,x,t) \in B,$$
$$|f_0(u,x,t) - f_0(v,x,t)| \le L|u - v| \quad \forall (u,x,t),(v,x,t) \in B.$$

We give one result from [14] on the so-called $E_{\infty,T}$-mild solutions of problem (3.2). First, we define the operator $A_0 : D(A_0) \subset L_p \to L_p$ for $p > 1$ by

$$D(A_0) = \{\psi \in C^2 : \psi'(0) = \psi'(1) = 0\},$$
$$A_0\psi = -\psi_{xx} + \psi \quad \forall \psi \in D(A_0).$$

The operator A_0 has a closure A_p in L_p. The operators A_p are generators of analytic semigroups $S_p(t)$ in L_p. By Lemma 1 in [14, p. 15], $S_{p_1}(t) \subset S_{p_2}(t)$ for any $p_1, p_2 \in (1,\infty)$, $p_1 \ge p_2$. By Lemma 2 in [14, p. 19],

$$\sup_{t\in[0,T]} \|S_p(t)\psi\|_{L_\infty} \le \|\psi\|_{L_\infty} \quad \forall \psi \in L_\infty.$$

Therefore, one can define the operators $P(t)$ as the restrictions of $S_p(t)$ to the space L_∞. These operators do not depend on p and are continuous from L_∞ to L_∞. Furthermore, since

$$P(t_1 + t_2) = P(t_1)P(t_2) \quad \forall t_1, t_2 \in [0,\infty),$$

they define a semigroup in L_∞.

Note that it is not a strongly continuous semigroup in L_∞ because, due to Lemma 2 in [14, p. 19], $P(t)\psi \to \psi$ in L_∞ as $t \to 0$ if and only if $\psi \in C$.

DEFINITION 3.2. *Let $T \in (0,\infty]$. An* $E_{\infty,T}$-mild solution *of problem* (3.2) *for initial data $\varphi \in L_\infty$ on the time interval $[0,T)$ is a measurable function $u(x,t)$, $(x,t) \in (0,1) \times (0,T)$, satisfying*

$$u(\cdot,t) \in L_\infty, \qquad \sup_{s\in(0,t)} \|u(\cdot,s)\|_{L_\infty} < \infty \quad \forall t \in (0,T),$$
$$u(\cdot,t) = P(t)\varphi + \int_0^t P(t-s)(f_0(u(\cdot,s),\cdot,s) + u(\cdot,s))\,ds \quad \forall t \in (0,T),$$

where the integral is an absolutely converging Bochner integral in L_∞.

DEFINITION 3.3. *We say that $T \in (0,\infty)$ is a* maximal existence time *for given initial data φ if problem* (3.2) *has an $E_{\infty,T}$-mild solution on the interval $[0,T)$, but for any $T' > T$, it has no $E_{\infty,T'}$-mild solution on the interval $[0,T')$.*

The following lemma is formulated as Theorem 1 in [14, p. 111].

LEMMA 3.4. *Assume that $\varphi \in L_\infty$ and the above conditions on f_0 are satisfied. Then there exists a maximal existence time $T \in (0,\infty]$ and problem* (3.2) *has a unique $E_{\infty,T}$-mild solution on the interval* $[0,T)$. *If T is finite, then*

$$\lim_{t\to T} \|u(\cdot,t)\|_{L_\infty} = \infty.$$

3.2.2. Uniformly bounded solutions. Now we formulate a result which states that if problem (3.2) has a solution $u \in W_q^{2,1}(Q_T)$, then it is bounded uniformly with respect to $T \in (0,\infty)$. It can be proved by regularizing the right-hand side and applying the invariant-rectangles method, which can also be exploited for systems of reaction-diffusion equations (see, e.g., [15]).

LEMMA 3.5. *Let $\varphi \in \mathcal{W}$, $u \in W_q^{2,1}(Q_T)$, and $F(x,t) = f_0(u(x,t),x,t)$ belong to $L_q(Q_T)$. We assume that u satisfies* (3.2) *and the following hold for some $U > 0$.*

1. *$f_0(\cdot,x,t)$ is continuous at the points $\pm U$ uniformly with respect to $(x,t) \in \overline{Q}_T$ (perhaps, after modification on a subset of $\overline{Q}_T$ of measure zero),*
2. *$f_0(U,x,t) < 0$, $f_0(-U,x,t) > 0$ for a.e. $(x,t) \in \overline{Q}_T$,*
3. *$\|\varphi\|_C < U$.*

Then $\|u\|_{C(\overline{Q}_T)} < U$.

3.3. Semilinear parabolic problem with a special nonlinearity. In this section, we specify the nonlinearity $f_0(u,x,t)$ in (3.2).

Let us fix initial data φ and ξ_0 satisfying Condition 2.24. By Lemma 2.25, $(\varphi,\xi_0) \in E_m$ for a sufficiently large m. Let us fix such an m. Next, we choose $U > 0$ such that Condition 2.11 holds for U and

$$\|\varphi\|_C < U. \tag{3.3}$$

Further, we fix $T > 0$ and consider functions $u_0 \in C(\overline{Q}_T)$ and $b_0 \in C[0,T]$ with the following properties:

$$\|u_0\|_{C(\overline{Q}_T)} \le U, \tag{3.4}$$

$$b_0(t) \in [\overline{b}, \overline{b} + 1/m], \quad t \in [0,T]. \tag{3.5}$$

Now we define the function $f_0(u,x,t)$ by

$$f_0(u,x,t) = f(u, v_0(x,t)), \tag{3.6}$$

where

$$v_0(x,t) = \begin{cases} H_1(u_0(x,t)), & 0 \le x \le b_0(t), \\ H_2(u_0(x,t)), & b_0(t) < x \le 1; \end{cases} \tag{3.7}$$

here we use convention (2.2).

Note that, in general, $v_0(x,t) \ne \mathcal{H}(\xi_0(x), u_0(x,\cdot))(t)$. However, the following is true and will be essentially used later on.

Remark 3.6. Suppose that the following hold for each $t \in [0,T]$.

1. $u_0(x,0) = \varphi(x)$.
2. The equation $u(x,t) = \alpha$, on the interval $[\overline{b},1]$ has no more than one root. Let us define a function $a_0(t)$ as follows: if the above root exists, then $a_0(t)$ is equal to this root, and $a_0(t) = \overline{b}$ otherwise.

3. The function

$$b_0(t) = \max_{s\in[0,t]} a_0(s) \tag{3.8}$$

satisfies $b_0(t) \in [\overline{b}, \overline{b} + 1/m]$.

4. The equation $u_0(x,t) = \beta$ on the interval $[0, b_0(t)]$ has no roots.

Then the nonlinearity $v_0(x,t)$ given by (3.7) coincides with the spatially distributed hysteresis defined for $u_0(x,t)$:

$$v_0(x,t) \equiv \mathcal{H}(\xi_0(x), u_0(x,\cdot))(t).$$

The next lemma establishes basic properties of the "max" operator in (3.8).

LEMMA 3.7. *Let $\lambda = 0$ or γ. Then the following hold.*

1. *If $a \in C^\lambda[0,T]$ and $b(t) = \max_{s\in[0,t]} a(s)$, then $b \in C^\lambda[0,T]$ and*

$$\|b\|_{C^\lambda[0,T]} \le \|a\|_{C^\lambda[0,T]}.$$

2. *If $a_j \in C[0,T]$ and $b_j(t) = \max_{s\in[0,t]} a_j(s)$, $j = 1, 2$, then*

$$\|b_1 - b_2\|_{C[0,T]} \le \|a_1 - a_2\|_{C[0,T]}.$$

Proof. We leave details to the reader. ∎

In what follows, we will need the continuous dependence of v_0 defined by (3.7) on u_0 and b_0. Denote by $\mathcal{R}$ the set of pairs $(u_0, b_0) \in C(\overline{Q}_T) \times C[0,T]$ satisfying conditions (3.4) and (3.5).

LEMMA 3.8. *For any $(u_0, b_0), (\hat{u}_0, \hat{b}_0) \in \mathcal{R}$, let v_0 be defined by (3.7) and $\hat{v}_0$ by (3.7), where u_0 and b_0 are replaced by $\hat{u}_0$ and $\hat{b}_0$, respectively. Then, for any $p \in [1, \infty)$,*

$$\|v_0 - \hat{v}_0\|_{L_p(Q_T)} \le c_0 \left(T^{1/p} \|u_0 - \hat{u}_0\|^\sigma_{C(\overline{Q}_T)} + \|b_0 - \hat{b}_0\|^{1/p}_{L_1(0,T)} \right), \tag{3.9}$$

where σ is the constant in Condition 2.3 and $c_0 > 0$ depends on U and p, but does not depend on u_0, b_0, T.

Proof. We fix some $t \in [0,T]$ and assume that $b_0(t) \le \hat{b}_0(t)$ for this t. Then, using (3.7) and omitting the arguments of the integrands, we have

$$\int_0^1 |v_0 - \hat{v}_0|^p \, dx = \int_0^{b_0(t)} |H_1(u_0) - H_1(\hat{u}_0)|^p \, dx + \int_{\hat{b}_0(t)}^1 |H_2(u_0) - H_2(\hat{u}_0)|^p \, dx$$
$$+ \int_{b_0(t)}^{\hat{b}_0(t)} |H_2(u_0) - H_1(\hat{u}_0)|^p \, dx.$$

Using Condition 2.3 in the first two integrals and the boundedness of $H_j(u)$ for $|u| \le U$ in the second integral, we obtain

$$\int_0^1 |v_0 - \hat{v}_0|^p \, dx \le k_1 (\|u_0 - \hat{u}_0\|^{\sigma p}_{C(\overline{Q}_T)} + k_2 |b_0(t) - \hat{b}_0(t)|),$$

where $k_1 > 0$ depends on U and p, but does not depend on u_0, b_0, T.

Integrating the latter inequality with respect to t from 0 to T yields (3.9). ∎

Set

$$V = \max_{|u|\le U} \{H_1(u), H_2(u)\}, \qquad f_U = \max_{|u|\le U, |v|\le V} |f(u,v)|. \tag{3.10}$$

The main step in finding a solution of problem (2.6)–(2.8) is the following theorem.

THEOREM 3.9. *Let f_0 be defined by* (3.6), *and let q and γ satisfy* (2.1). *We fix an arbitrary $T_0 > 0$. Then the following hold.*

1. *Problem* (3.2) *has a unique solution $u \in W_q^{2,1}(Q_T)$ and, for any $T \le T_0$,*

$$\|u\|_{C(\overline{Q}_T)} < U, \tag{3.11}$$

$$\begin{gathered} \|u\|_{W_q^{2,1}(Q_T)} + \max_{t\in[0,T]} \|u(\cdot,t)\|_{\mathcal{W}} \le c_1(\|\varphi\|_{\mathcal{W}} + f_U), \\ \|u\|_{C^\gamma(\overline{Q}_T)} + \|u_x\|_{C^\gamma(\overline{Q}_T)} \le c_2(\|\varphi\|_{\mathcal{W}} + f_U), \end{gathered} \tag{3.12}$$

where f_U is given by (3.10) *and $c_1, c_2 > 0$ depend only on T_0 and do not depend on $m, u_0, b_0, \varphi, u, T$.*

2. *The solution of problem* (3.2) *continuously depends on φ, u_0, and b_0. In other words, if $u_n \in W_q^{2,1}(Q_T)$, $n = 1, 2, \dots$, are solutions of problem* (3.2) *with φ, u_0, b_0, v_0 replaced by $\varphi_n, u_{0n}, b_{0n}, v_{0n}$, then*

$$\|u_n - u\|_{W_q^{2,1}(Q_T)} \to 0, \quad n \to \infty, \tag{3.13}$$

whenever

$$\|\varphi_n - \varphi\|_{\mathcal{W}} + \|u_{0n} - u_0\|_{C(\overline{Q}_T)} + \|b_{0n} - b_0\|_{C[0,T]} \to 0, \quad n \to \infty. \tag{3.14}$$

3. *There is $T = T(m) \in (0, T_0]$ and a natural number $N = N(m, U) \ge m$ which do not depend on u_0, b_0, φ, u, such that the following is true for any $t \in [0, T]$.*
   (a) *The equation $u(x,t) = \alpha$ in $[\overline{b}, 1]$ has no more than one root. If this root exists, we denote it by $a(t)$; otherwise, we set $a(t) = \overline{b}$ (similarly to Lemma* 2.26). *One has $a(t) \in [\overline{b}, \overline{b} + 1/N]$, $a \in C^\gamma[0,T]$, and*

$$\|a\|_{C^\gamma[0,T]} \le a^*, \tag{3.15}$$

   where $a^* > 0$ depends on m, but does not depend on u_0, b_0, φ.
   (b) *The hysteresis $\mathcal{H}(\xi_0, u)$ and its configuration function $\xi(x,t)$ have exactly one discontinuity point $b(t)$; moreover, $b(t) = \max_{s\in[0,t]} a(s)$ and $b \in C^\gamma[0,T]$.*
   (c) *$(u(\cdot,t), \xi(\cdot,t)) \in E_N$.*

Proof. 1. Throughout the proof, we assume that u_0 and b_0 are extended as continuous functions to $[0, T_0]$ in such a way that (3.4) and (3.5) hold on this interval.

It follows from the definition (3.6) of the function f_0 and from the Lipschitz continuity of the function f (Condition 2.10) that the function $f_0(u,x,t)$ satisfies assumptions 1 and 2 in section 3.2. Therefore, by Lemma 3.4, there is $T_1 \in (0, T_0]$ such that problem (3.2) has a unique E_{∞,T_1}-mild solution. Hence, $f_0(u(x,t),x,t)$ is in $L_\infty(Q_{T_1})$, and Theorem 3.1 yields $u \in W_q^{2,1}(Q_{T_1})$.

Now we claim that

$$\|u\|_{C(\overline{Q}_{T_1})} < U. \tag{3.16}$$

Indeed, the function $v_0(x,t)$ in (3.6) is bounded for $(x,t) \in \overline{Q}_{T_1}$; therefore, $f_0(u,x,t) = f(u, v_0(x,t))$ satisfies the assumptions in Lemma 3.5 Thus, Lemma 3.5 implies estimate (3.16).

Combining Lemma 3.4 with estimate (3.16), we see that T_1 can be chosen equal to T_0. Hence, $u \in W_q^{2,1}(Q_{T_0})$ and, by Theorem 3.1 and (3.16) (with any $T \le T_0$ instead of T_1), we obtain estimates (3.12). Part 1 of the theorem is proved.

2. Let us prove (3.13). Assume the contrary: there is an $\varepsilon > 0$ and a subsequence of u_n (which we denote u_n again) such that

$$\|u_n - u\|_{W_q^{2,1}(Q_T)} \geq \varepsilon, \quad n = 1, 2, \ldots. \tag{3.17}$$

2a. Note that (3.14) and Lemma 3.8 imply

$$\|v_{0n} - v_0\|_{L_q(Q_T)} \to 0, \quad n \to \infty. \tag{3.18}$$

Further, by part 1 of the theorem, the u_n are bounded in $W_q^{2,1}(Q_T)$ uniformly with respect to n. Therefore, by the compactness of the embedding $W_q^{2,1}(Q_T) \subset L_q(Q_T)$, there is a subsequence of u_n (which we denote u_n again) fundamental in $L_q(Q_T)$.

For this subsequence, using Theorem 3.1 and the Lipschitz continuity of f (Condition 2.10), we have

$$\begin{aligned}
&\|u_n - u_k\|_{W_q^{2,1}(Q_T)} \\
&\quad \leq k_1 \left(\|\varphi_n - \varphi_k\|_{\mathcal{W}} + \left(\int_{Q_T} |f(u_n, v_{0n}) - f(u_k, v_{0k})|^q \, dxdt \right)^{1/q} \right) \\
&\quad \leq k_2 \left(\|\varphi_n - \varphi_k\|_{\mathcal{W}} + \left(\int_{Q_T} (|u_n - u_k| + |v_{0n} - v_{0k}|)^q \, dxdt \right)^{1/q} \right) \\
&\quad \leq k_2 (\|\varphi_n - \varphi_k\|_{\mathcal{W}} + \|u_n - u_k\|_{L_q(Q_T)} + \|v_{0n} - v_{0k}\|_{L_q(Q_T)}),
\end{aligned} \tag{3.19}$$

where $k_1, k_2 > 0$ do not depend on n and k. The latter inequality, the first convergence in (3.14), relation (3.18), and the fact that u_n is fundamental in $L_q(Q_T)$ imply that u_n is fundamental and, thus, converges to some $\hat{u}$ in $W_q^{2,1}(Q_T)$.

2b. Passing to the limit as $n \to \infty$ and using Lemma 3.8 and Condition 2.10, we conclude that $\hat{u}$ is a solution of the problem

$$\begin{cases} \hat{u}_t = \hat{u}_{xx} + f(\hat{u}, v_0), \quad (x,t) \in Q_T, \\ \hat{u}_x|_{x=0} = \hat{u}_x|_{x=1} = 0, \\ \hat{u}|_{t=0} = \varphi(x), \quad x \in (0,1). \end{cases}$$

But the latter problem has a unique solution due to part 1 of the proof. Hence, $\hat{u} = u$, which contradicts (3.17). Part 2 is proved.

3. Set

$$\Omega = \left\{ (x, \varphi(x)) : x \in [\overline{b}, \overline{b} + 1/m], \ \varphi(x) \in [\alpha, \alpha + 1/m^2] \right\}. \tag{3.20}$$

Consider the two cases: $\Omega \neq \varnothing$ and $\Omega = \varnothing$.

Case I. Let $\Omega \neq \varnothing$. The second inequality in (3.12) implies that

$$\|u(\cdot, t_1) - u(\cdot, t_2)\|_C + \|u_x(\cdot, t_1) - u_x(\cdot, t_2)\|_C \leq c|t_1 - t_2|^\gamma, \tag{3.21}$$

where $c = c_2(m + f_U)$. This inequality implies that, for small enough $\tau(m) > 0$, the functions φ, $\psi_1 = u(\cdot, t_1)$, $\psi_2 = u(\cdot, t_2)$ with $t_1, t_2 \in [0, \tau(m)]$ satisfy the assumptions of Lemma 2.26. Therefore, $a \in C^\gamma$ and

$$\|a\|_{C^\gamma} \leq a^* = 1 + 2mc. \tag{3.22}$$

Set $b(t) = \max_{s\in[0,t]} a(s)$. Note that $b(0) = a(0) = \overline{b}$. It follows from (3.21) and (3.22) (decreasing $\tau(m)$ if necessary) that

$$b(t) \in [\overline{b}, \overline{b} + 1/(2m)], \quad \|b\|_{C^\gamma[0,T]} \le a^*, \tag{3.23}$$

$$u_x(x,t) \ge \frac{1}{m+1} \quad \forall x \in [b(t), b(t) + 1/(2m)]. \tag{3.24}$$

Set $N = N(m,U) = \max\left(2m, [c_1(m + f_U)] + 1\right)$, where c_1 is the constant in (3.12) and $[\cdot]$ stands for the integer part of a number. Then (3.23) and (3.24) imply

$$a(t), b(t) \in [\overline{b}, \overline{b} + 1/N], \tag{3.25}$$

$$u_x(x,t) \ge \frac{1}{N} \quad \forall x \in [b(t), b(t) + 1/N]. \tag{3.26}$$

Now we introduce the function $\xi(x,t)$ as follows: $\xi(x,t) = 1$ for $x \le b(t)$ and $\xi(x,t) = 2$ for $x > b(t)$. We will see below that $\xi(x,t)$ is the configuration function of the hysteresis $H(\xi_0, u)$.

Let us show that $(u(\cdot,t), \xi(\cdot,t)) \in E_N$ on the interval $t \in [0,T]$, provided that $T = T(m)$ is small enough.

i. $b(t) \le 1 - 1/N$. This follows from (3.23).

ii. $u(x,t) \le \beta - 1/N^2$ for $x \in [0, \overline{b}]$. This follows for sufficiently small T from the fact that $u(x,0) = \varphi(x) \le \beta - 1/m^2$ for $x \in [0, \overline{b}]$ and from the Hölder continuity of u (the second inequality in (3.12)).

iii. $u(x,t) \ge \alpha + 1/N^2$ for $x \in [\overline{b} + 1/N, 1]$. This follows for sufficiently small T from the fact that $u(x,0) = \varphi(x) \ge \alpha + 1/(Nm)$ for $x \in [\overline{b} + 1/N, 1]$ and from the Hölder continuity of u (the second inequality in (3.12)).

iv. If $x \in [b(t), b(t) + 1/N]$, then $u(x,t) \ge \alpha$ and $u_x(x,t) \ge 1/N$. The first inequality holds by construction of the function $b(t)$. The second inequality follows from (3.24).

v. $\|u(\cdot,t)\|_{\mathcal{W}} \le N$. This follows for sufficiently small T from the fact that $\|\varphi\|_{\mathcal{W}} \le m$ and from the first inequality in (3.12).

Items i–v together with relations (3.22), (3.23), and (3.25) prove assertions 3(a)–3(c) of the theorem in Case I.

Case II. Let $\Omega = \varnothing$, where Ω is the set in (3.20). Since $u(x,0) = \varphi(x) \ge 1/m^2$ for $x \in [\overline{b}, \overline{b} + 1/m]$, it follows from the Hölder continuity of u (the second inequality in (3.12)) that $u(x,t) \ge 1/(m+1)^2$ for $x \in [\overline{b}, \overline{b} + 1/(m+1)]$, provided that T is sufficiently small.

In this case $a(t) = \overline{b}$ for all $t \in [0,T]$ and it is easy to see that parts 1–5 of the definition of E_N again hold for the pair $(u(\cdot,t), \xi(\cdot,t))$ and for $N = m+1$ on the interval $t \in [0,T]$, provided that T is small enough.

Thus, we have proved assertions 3(a)–3(c) of the theorem in Case II. ∎

Remark 3.10. Let

$$f_0(u,x,t) = f(u, v_0(x,t)), \quad (x,t) \in Q_{T_0},$$

where $v_0(x,t)$ is the spatially distributed hysteresis defined for some $u_0 \in C(\overline{Q}_T)$:

$$v_0(x,t) = \mathcal{H}(\xi_0(x), u_0(x,\cdot))(t).$$

Let $\|u_0\|_{C(\overline{Q}_{T_0})} \le U$ and the function $v_0(x,t)$ be measurable. Then parts 1 and 3 of Theorem 3.9 remain true. The proof is analogous to that for Theorem 3.9.

4. Proof of the main results.

4.1. Existence of solutions. In this section, we prove an analogue of Theorem 2.16 under the assumptions made in section 2.5.

As before, we choose m such that $(\varphi, \xi_0) \in E_m$. It will be convenient to reformulate the theorem.

THEOREM 4.1 (local existence). *Let Condition* 2.24 *be satisfied. Then the following hold for* $T = T(m) > 0$ *from part* 3 *of Theorem* 3.9*.*

1. *Any solution* $u \in W_q^{2,1}(Q_T)$ *of problem* (2.6)–(2.8) *in* Q_T *is transverse and preserves spatial topology. Moreover, it possesses all the properties from part* 3 *of Theorem* 3.9*.*
2. *There is at least one transverse topology preserving solution* $u \in W_q^{2,1}(Q_T)$ *of problem* (2.6)–(2.8) *in* Q_T*.*

Proof. 1. Let $u \in W_q^{2,1}(\overline{Q}_T)$ be an arbitrary solution of problem (2.6)–(2.8). Using Lemma 3.5 (as in the proof of Theorem 3.9), we obtain that $\|u\|_{C(\overline{Q}_T)} < U$. Therefore, by Remark 3.10, the first assertion of the theorem is true.

2. Let us prove the second assertion. Let $\mathcal{R} \subset C(\overline{Q}_T) \times C[0, T]$ be the set defined before Lemma 3.8. Clearly, $\mathcal{R}$ is a closed convex set.

Take any $(u_0, b_0) \in \mathcal{R}$ and define $f_0(u, x, t)$ by formula (3.6). Then Theorem 3.9 implies that problem (3.2) has a unique transverse topology preserving solution $u \in W_q^{2,1}(Q_T) \subset C^\gamma(\overline{Q}_T)$.

Consider the functions $a(t)$ and $b(t)$ (see part 3 of Theorem 3.9). By Theorem 3.9, $a, b \in C^\gamma[0, T]$ and the function $b(t)$ defines the (unique) discontinuity point of the hysteresis $\mathcal{H}(\xi_0, u)$ and of its configuration function $\xi(x, t)$ at each moment $t \in [0, T]$. Moreover $(u, b) \in \mathcal{R}$. Thus, we can define a nonlinear operator $R : \mathcal{R} \to \mathcal{R}$ by the formula $R(u_0, b_0) = (u, b)$.

Let us show that R is continuous. Let a sequence (u_{0n}, b_{0n}) converge to (u_0, b_0). We define $f_{0n}(u, x, t)$ by formula (3.6) with u_{0n}, b_{0n}, v_{0n} instead of u_0, b_0, v_0. Let u_n be a solution of problem (3.2) with the right-hand side f_{0n}. By Theorem 3.9, there is $N = N(m, U) \in \mathbb{N}$ such that

$$(u, \xi), (u_n(\cdot, t), \xi_n(\cdot, t)) \in E_N \quad \forall t \in [0, T],\ n = 1, 2, \dots. \tag{4.1}$$

Let $a_n(t)$ correspond to $u_n(x, t)$ in the same way as $a(t)$ corresponds to $u(x, t)$. Set $b_n(t) = \max_{s \in [0,t]} a_n(s)$ for $t \in [0, T]$. Then

$$R(u_{0n}, b_{0n}) = (u_n, b_n).$$

By construction of T $(\le \tau(m))$ in the proof of assertion 3 of Theorem 3.9, the functions φ, $\psi_1 = u(\cdot, t)$, $\psi_2 = u_n(\cdot, t)$ satisfy the assumptions of Lemma 2.26; hence,

$$|a_n(t) - a(t)| \le 2m\|u_n - u\|_{C(\overline{Q}_T)}.$$

This implies that a_n converges to a in $C[0, T]$. Thus, Lemma 3.7 implies that b_n converges to b in $C[0, T]$. Therefore, the operator R is continuous on $\mathcal{R}$.

The operator R is also compact. Indeed, the map $\mathcal{R} \ni (u_0, b_0) \mapsto u \in C(\overline{Q}_T)$ is compact due to (3.12) and the compactness of the embedding $C^\gamma(\overline{Q}_T) \subset C(\overline{Q}_T)$. The map $\mathcal{R} \ni (u_0, b_0) \mapsto b \in C[0, T]$ is compact due to (3.15), part 1 of Lemma 3.7, and the compactness of the embedding $C^\gamma[0, T] \subset C[0, T]$.

Now the Schauder fixed-point theorem implies that the operator R has a fixed point $(u, b) \in \mathcal{R}$. Taking into account Remark 3.6 and (4.1), we see that u is a transverse topology preserving solution of problem (2.6)–(2.8). ◻

4.2. Continuation of solutions. In this section, we prove an analogue of Theorem 2.18 under the assumptions made in section 2.5.

Proof of Theorem 2.18.

1. We assume that there is $T_0 > 0$ such that $u(x,t)$ cannot be continued to $[0, T_0]$ as a transverse solution of problem (2.6)–(2.8).

Applying Theorem 4.1 and using part 1 of Lemma (2.25), we obtain a sequence $m_k \in \mathbb{N}$ $(k = 1, 2, \dots)$ such that $m_{k+1} > m_k$ and a sequence of time moments t_k $(k = 1, 2, \dots)$ such that $t_{k+1} > t_k$ with the following properties.

1. For each k, the solution $u(x,t)$ of problem (2.6)–(2.8) can be continued as a transverse solution to the time interval $[0, t_k]$.
2. For each k,

$$u(\cdot, t_k) \in E_{m_k} \setminus E_{m_k - 1}. \tag{4.2}$$

Set $T = \lim_{k\to\infty} t_k$. By assumption, $T \le T_0$.

Since u is a solution of problem (2.6)–(2.8) in Q_{t_k} for all k and

$$\|u\|_{W_q^{2,1}(Q_{t_k})} \le c_1(\|\varphi\|_{\mathcal{W}} + f_U)$$

by Remark 3.10 (with the right-hand side not depending on k), it follows that $u \in W_q^{2,1}(Q_T)$ and u is a solution of problem (2.6)–(2.8) in Q_T. Since $u(\cdot, t)$ is a continuous $\mathcal{W}$-valued function, we have

$$\|u(\cdot, t_k) - u(\cdot, T)\|_{\mathcal{W}} \to 0, \quad k \to \infty. \tag{4.3}$$

Denote by $b(t)$ the discontinuity point of the configuration function $\xi(x,t)$. By construction $b(t)$ is continuous and nondecreasing on $[0, t_k]$ for all k. Therefore, $b(t)$ is continuous on $[0, T]$. In particular,

$$b(t_k) - b(T) \to 0, \quad k \to \infty. \tag{4.4}$$

It follows from (4.2)–(4.4) and from part 2 of Lemma 2.25 that $b(T) = 1$ or $u(x,T)$ is not transverse with respect to $\xi(x,T)$.

2. Now we consider several cases.

2.1. First, we assume that $b(T) < 1$. Then $u(x,T)$ is not transverse with respect to $\xi(x,T)$. This happens because the graph of the function $u(x,T)$ touches the line α or β, or because $u(b(T),T) = \alpha$, $u_x(b(T),T) = 0$.

2.2. If $b(T) = 1$, then $u(1,T) = \alpha$. Furthermore, $u_x(1,T) = 0$ due to the Neumann boundary condition (2.7). In this case, either $u(x,T)$ is not transverse (if the transversality fails at some point $x \in [0,1)$ at the same moment $t = T$, in which case $T = T_{\max}$) or $u(x,T)$ is transverse and $\xi(x,T) \equiv 2$, $x \in [0,1]$. In the latter case, we can proceed similarly to the above, but effectively without hysteresis, i.e.,

$$\mathcal{H}(\xi_0(x), u(x,\cdot))(t) \equiv H_2(u(x,t)), \quad t \ge T.$$

Thus, one can find $T_{\max}$ as in part 1 of the proof. ∎

4.3. Continuous dependence of solutions on initial data. In this section we prove an analogue of Theorem 2.19 under the assumptions from section 2.5. The proof will consist of two steps. First, we shall prove the continuous dependence on small time intervals and then on the whole interval $[0, T]$.

4.3.1. Continuous dependence on small time interval. Let the initial data $\varphi(x)$ and $\xi_0(x)$ satisfy Condition 2.24. By part 1 of Lemma 2.25, $(\varphi, \xi_0) \in E_m$ for some $m \in \mathbb{N}$. Let $\varphi_n(x)$ and $\xi_{0n}(x)$ be other initial data satisfying Condition 2.24. Suppose that

$$\|\varphi_n - \varphi\|_{\mathcal{W}} \to 0, \quad \overline{b}_n - \overline{b} \to 0, \quad n \to \infty, \tag{4.5}$$

where $\overline{b}_n$ is the discontinuity point of $\xi_{0n}(x)$.

By part 3 of Lemma 2.25, there is $n_1 = n_1(m) > 0$ such that

$$(\varphi, \xi_0), (\varphi_n, \xi_{0n}) \in E_{m+1} \quad \forall n \geq n_1(m). \tag{4.6}$$

In what follows, we assume $n \geq n_1(m)$. By Theorem 4.1, there is $T_1 = T_1(m+1) > 0$ for which problem (2.6)–(2.8) has transverse topology preserving solutions $u, u_n \in W_q^{2,1}(Q_{T_1})$ with the initial data φ, ξ_0 and φ_n, ξ_{0n}, respectively. Moreover, any solution of problem (2.6)–(2.8) in Q_{T_1} is transverse and preserves topology.

We introduce the functions $a(t)$ and $a_n(t)$ corresponding to u and u_n as described in part 3 of Theorem 3.9. Then the discontinuity points of the corresponding configuration functions $\xi(x,t)$, $\xi_n(x,t)$ are given by

$$b(t) = \max_{s \in [0,t]} a(s), \quad b_n(t) = \max_{s \in [0,t]} a_n(s), \quad t \in [0, T_1]. \tag{4.7}$$

LEMMA 4.2. *Under the above assumptions and the additional assumption that u is a unique solution of problem* (2.6)–(2.8) *in Q_{T_1} with the initial data φ, ξ_0, we have*

$$\|u_n - u\|_{W_q^{2,1}(Q_{T_1})} \to 0, \quad \|b_n - b\|_{C[0,T_1]} \to 0, \quad n \to \infty.$$

Proof. 1. Assume to the contrary that there is $\varepsilon > 0$ such that

$$\|u_n - u\|_{W_q^{2,1}(Q_{T_1})} \geq \varepsilon, \quad n = 1, 2, \dots, \tag{4.8}$$

for some subsequence of u_n, which we denote u_n again.

Theorem 4.1 implies that, for all sufficiently large n, the functions u_n and a_n are uniformly bounded in $W_q^{2,1}(Q_{T_1})$ and $C^\gamma[0, T_1]$, respectively. Therefore, we can choose subsequences of u_n and a_n (which we denote u_n and a_n again) such that

$$\|u_n - \hat{u}\|_{C^\gamma(\overline{Q}_{T_1})} \to 0, \ \|(u_n)_x - \hat{u}_x\|_{C^\gamma(\overline{Q}_{T_1})} \to 0, \quad n \to \infty, \tag{4.9}$$

$$\|a_n - \hat{a}\|_{C[0,T_1]} \to 0, \quad n \to \infty \tag{4.10}$$

for some function $\hat{u} \in C^\gamma(\overline{Q}_{T_1})$ with $\hat{u}_x \in C^\gamma(\overline{Q}_{T_1})$ and some function $\hat{a} \in C[0, T_1]$.

Set

$$\hat{b}(t) = \max_{s \in [0,t]} \hat{a}(s), \quad t \in [0, T_1].$$

It follows from (4.9), (4.10), and Lemma 2.26 that $\hat{u}$ and $\hat{a}$ are such that

$$\mathcal{H}(\xi(x), \hat{u}(x, \cdot))(t) = \begin{cases} H_1(\hat{u}(x,t)), & 0 \leq x \leq \hat{b}(t), \\ H_2(\hat{u}(x,t)), & \hat{b}(t) < x \leq 1. \end{cases}$$

Combining (4.10) with Lemma 3.7, we obtain

$$\|b_n - \hat{b}\|_{C[0,T_1]} \to 0, \quad n \to \infty. \tag{4.11}$$

2. Now using (4.5), (4.9), (4.11), and part 2 of Theorem 3.9, we see that the sequence of functions u_n satisfying the relations

$$\begin{cases} (u_n)_t = (u_n)_{xx} + f(u_n, \mathcal{H}(\xi_n, u_n)), & (x,t) \in Q_{T_1}, \\ (u_n)_x|_{x=0} = (u_n)_x|_{x=1} = 0, \\ u_n|_{t=0} = \varphi_n(x), \quad x \in (0,1), \end{cases}$$

converges in $W_q^{2,1}(Q_{T_1})$ to $\hat{u}$ satisfying the relations

$$\begin{cases} \hat{u}_t = \hat{u}_{xx} + f(\hat{u}, \mathcal{H}(\xi, \hat{u})), & (x,t) \in Q_{T_1}, \\ \hat{u}_x|_{x=0} = \hat{u}_x|_{x=1} = 0, \\ \hat{u}|_{t=0} = \varphi(x), \quad x \in (0,1). \end{cases}$$

But the latter problem has a unique solution by assumption. Hence, $\hat{u} = u$, which contradicts (4.8).

Thus, the first convergence in the lemma is proved. The second convergence follows from (4.11). $\square$

4.3.2. Continuous dependence on the whole interval without change of topology. Now we shall prove Theorem 2.19 under the assumptions from section 2.5. In particular, we assume that φ and ξ_0 satisfy Condition 2.24. By assumption, problem (2.6)–(2.8) has a unique transverse topology preserving solution u in Q_s for any $s \le T$. We denote by $b(t)$ the (unique) discontinuity point of the corresponding configuration function $\xi(x,t)$.

Further, we assume that φ_n, ξ_{0n}, $n = 1, 2, \dots$, is a sequence of other initial data satisfying Condition 2.24 such that

$$\|\varphi_n - \varphi\|_{\mathcal{W}} \to 0, \quad \overline{b}_n - \overline{b} \to 0, \quad n \to \infty, \tag{4.12}$$

where $\overline{b}_n$ is the discontinuity point of ξ_{0n}.

We have to show that, for all sufficiently large n, problem (2.6)–(2.8) with the initial data φ_n, ξ_{0n} has at least one transverse topology preserving solution $u_n \in W_q^{2,1}(Q_T)$ and each sequence of such solutions satisfies

$$\|u_n - u\|_{W_q^{2,1}(Q_T)} \to 0, \quad \|b_n - b\|_{C[0,T]} \to 0, \quad n \to \infty, \tag{4.13}$$

where $b_n(t)$ is the discontinuity point of $\xi_n(x,t)$.

Proof. 1. By Lemma 2.25, there is $m \in \mathbb{N}$ such that $(u(\cdot,t), \xi(\cdot,t)) \in E_m$ for all $t \in [0,T]$. Let us fix such a number m. Suppose that (4.13) does not hold. Let τ be the infimum of the set of all $s \in [0,T]$ such that at least one of the convergences

$$\|u_n - u\|_{W_q^{2,1}(Q_s)} \to 0, \quad \|b_n - b\|_{C[0,s]} \to 0, \quad n \to \infty, \tag{4.14}$$

does not hold. By assumption $\tau < T$. On the other hand, Lemma 4.2 implies that $\tau \ge T_1$. In particular, this means that

$$\|u_n(\cdot, \tau - T_1/2) - u(\cdot, \tau - T_1/2)\|_{\mathcal{W}} \to 0, \quad b_n(\tau - T_1/2) - b(\tau - T_1/2) \to 0, \quad n \to \infty.$$

Applying Lemma 4.2 again, we obtain the convergence in (4.14) for $s = \min(\tau + T_1/2, T) > \tau$. This contradiction proves (4.13). $\square$

4.3.3. Continuous dependence on the whole interval with change of topology. It remains to prove Corollary 2.20 under the assumptions from section 2.5. In particular, we shall keep notation from section 4.3.2. However, we additionally assume that $t_{\max} > 0$ is the time moment at which the topology of u changes, i.e.,

$$b(t_{\max}) = 1, \qquad u(1, t_{\max}) = 0,$$

and that the solutions u and u_n are transverse on the interval $[0, T]$ with $T \geq t_{\max}$.

LEMMA 4.3. *$\|u - u_n\|_{W_q^{2,1}(Q_{t_{\max}})} \to 0$ and $\|b - b_n\|_{C[0,t_{\max}]} \to 0$ as $n \to \infty$.*

Proof. 1. Let us prove the first convergence in the lemma. Assume to the contrary that there is $\varepsilon > 0$ such that

$$\|u - u_n\|_{W_q^{2,1}(Q_{t_{\max}})} \geq \varepsilon, \quad n = 1, 2, \dots, \tag{4.15}$$

for some subsequence of u_n, which we denote u_n again.

Theorem 2.19 implies that

$$\|u - u_n\|_{W_q^{2,1}(Q_{t_{\max}-\delta})} \to 0, \quad \|b - b_n\|_{C[0,t_{\max}-\delta]} \to 0, \quad n \to \infty, \tag{4.16}$$

for all (small) $\delta > 0$. Therefore,

$$u_n(x,t) \to u(x,t) \quad \text{a.e. } (x,t) \in Q_{t_{\max}}, \tag{4.17}$$

$$b_n(t) \to b(t) \quad \text{a.e. } t \in (0, t_{\max}), \quad n \to \infty. \tag{4.18}$$

Furthermore, Remark 3.10 and Lemma 3.5 imply that the functions u_n are uniformly bounded in $W_q^{2,1}(Q_{t_{\max}})$. Therefore, we can choose a subsequence of u_n (which we denote u_n again) converging in $C(\overline{Q}_{t_{\max}})$. Taking into account (4.17), we see that it converges to u:

$$\|u - u_n\|_{C(\overline{Q}_{t_{\max}})} \to 0, \quad n \to \infty. \tag{4.19}$$

On the other hand, it follows from (4.18), from the uniform boundedness of $b_n(t)$, and from Lebesgue's dominated convergence theorem that

$$\|b - b_n\|_{L_1(0,t_{\max})} \to 0, \quad n \to \infty. \tag{4.20}$$

Combining the convergence of initial data, relations (4.19) and (4.20), and part 2 of Theorem 3.9, we conclude that

$$\|u - u_n\|_{W_q^{2,1}(Q_{t_{\max}})} \to 0, \quad n \to \infty, \tag{4.21}$$

for the chosen subsequence. This contradicts (4.15). Therefore, (4.21) holds for the whole sequence u_n.

2. Now we prove the second convergence in the lemma. Suppose it does not hold. Then, due to the second convergence in (4.16), there is a subsequence of t_n (which we denote t_n again) and $\varepsilon > 0$ such that $t_n \to t_{\max}$ and

$$|b_n(t_n) - b(t_n)| \geq \varepsilon.$$

Combining this with the fact that $b(t_n) \to 1$, we see that there is $b^* < 1$ such that

$$b_n(t_n) \leq b^*. \tag{4.22}$$

On the other hand, since $b(t_{\max}) = 1$, there is a moment $t^* < t_{\max}$ such that

$$b(t^*) > b^*. \tag{4.23}$$

Using the monotonicity of b_n, the convergence in (4.16), and inequality (4.23), we have

$$b_n(t_n) \ge b_n(t^*) > b^*$$

for all sufficiently large n. This contradicts (4.22). $\square$

Proof of Corollary 2.20. Due to Lemma 4.3, it remains to show that

$$\|u_n - u\|_{W_q^{2,1}((0,1)\times(t_{\max},T))} \to 0, \quad n \to \infty, \tag{4.24}$$

$$\|b_n - b\|_{C[t_{\max},T]} \to 0, \quad n \to \infty. \tag{4.25}$$

1. Let us prove (4.24). Assume to the contrary that there is $\varepsilon > 0$ such that

$$\|u_n - u\|_{W_q^{2,1}((0,1)\times(t_{\max},T))} \ge \varepsilon, \quad n = 1, 2, \dots, \tag{4.26}$$

for some subsequence of u_n, which we denote u_n again.

Remark 3.10 and Lemma 3.5 imply that the functions u_n are uniformly bounded in $W_q^{2,1}(Q_{t_{\max}})$. Therefore, there is a subsequence of u_n (which we denote u_n again) and a function $\hat{u}$ such that

$$\|u_n - \hat{u}\|_{C([0,1]\times[t_{\max},T])} \to 0, \quad n \to \infty. \tag{4.27}$$

Further, we have, on the time interval $[t_{\max}, T]$,

$$\mathcal{H}(\xi_{0n}, u_n) = \begin{cases} H_1(u_n), & 0 \le x \le b_n(t), \\ H_2(u_n), & b_n(t) < x \le 1, \end{cases}$$

due to the transversality of u_n (here we set $b_n(t) = 1$ for $t > t_{n,\max}$ if $b_n(t_{n,\max}) = 1$). Therefore, similarly to Lemma 3.8, we obtain

$$\begin{aligned} &\|\mathcal{H}(\xi_{0n}, u_n) - H_1(\hat{u})\|_{L_p((0,1)\times(t_{\max},T))} \\ &\qquad \le c_0(T - t_{\max})^{1/p}(\|\hat{u} - u_n\|^{\sigma}_{C([0,1]\times[t_{\max},T])} + |b_n(t_{\max}) - 1|). \end{aligned}$$

Together with Lemma 4.3 and relation (4.27), this yields

$$\|\mathcal{H}(\xi_{0n}, u_n) - H_1(\hat{u}))\|_{L_p((0,1)\times(t_{\max},T))} \to 0, \quad n \to \infty. \tag{4.28}$$

On the other hand, by Lemma 4.3,

$$\|u_n|_{t=t_{\max}} - u|_{t=t_{\max}}\|_{\mathcal{W}} \to 0, \quad n \to \infty. \tag{4.29}$$

Using (4.27)–(4.29) and Theorem 3.1, we see that (cf. the proof of part 2 of Theorem 3.9)

$$\|u_n - \hat{u}\|_{W_q^{2,1}((0,1)\times(t_{\max},T))} \to 0, \quad n \to \infty, \tag{4.30}$$

and $\hat{u}$ is a solution of the problem

$$\begin{cases} \hat{u}_t = \hat{u}_{xx} + f(\hat{u}, H_1(\hat{u})), & (x,t) \in (0,1) \times (t_{\max}, T)), \\ \hat{u}_x|_{x=0} = \hat{u}_x|_{x=1} = 0, \\ \hat{u}|_{t=t_{\max}} = u|_{t=t_{\max}}, & x \in (0,1). \end{cases}$$

Due to the uniqueness assumption, $\hat{u} = u$ in $(0,1) \times (t_{\max}, T))$. Together with (4.30) this contradicts (4.26).

2. To prove (4.25), we note that, for all $t \in [t_{\max}, T]$,

$$|b_n(t) - b(t)| = b(t_{\max}) - b_n(t) \le b(t_{\max}) - b_n(t_{\max}) \to 0, \quad n \to \infty,$$

due to Lemma 4.3. ◻

5. Some generalizations. In this section, we generalize Condition 2.11 to the following one (cf. Example 2.12).

CONDITION 5.1 (generalized dissipativity).

1. *For all sufficiently large* u*,*

$$f(u, H_2(u)) \le 0, \quad f(-u, H_1(-u)) \ge 0;$$

2. *there is a Lipschitz continuous function* $h(u)$ *such that* $uh(u) > 0$ *for* $u \ne 0$ *and, for any (small)* $\mu > 0$*, there exists* $U_\mu > 0$ *such that the function*

$$f_\mu(u, v) = f(u, v) - \mu h(u)$$

satisfies

$$f_\mu(U_\mu, H_j(u)) < 0, \quad f_\mu(-U_\mu, H_j(u)) > 0 \quad \forall |u| \le U_\mu,\ j = 1, 2.$$

Let us prove Theorem 2.16 under Condition 5.1.

From now on we assume that $(\varphi, \xi_0) \in E_m$ for some m (see Lemma 2.25) and $0 < \mu \le 1$. Due to part 2 of Condition 5.1, f_μ satisfies Condition 2.11. Therefore, by Theorems 2.16 and 2.18, problem (2.6)–(2.8) with the right-hand side f_μ has a solution u_μ, which can be continued to a maximal interval of transverse existence $[0, T_{\mu,\max})$.

Now the important step is to prove the boundedness of the solutions u_μ uniformly with respect to μ.

Let us fix $U > \max(-\alpha, \beta)$ such that part 1 in Condition 5.1 holds for all $u \ge U$ and that $\|\varphi\|_{\mathcal{W}} < U$.

LEMMA 5.2. *The solutions* u_μ *satisfy* $\|u_\mu\|_{C(\overline{Q}_{T_{\mu,\max}})} < U$.

Proof. Part 1 of Condition 5.1 and the assumption $uh(u) > 0$ for $u \ne 0$ imply the strict inequalities

$$f_\mu(U, H_2(U)) < 0, \quad f_\mu(-U, H_1(-U)) > 0.$$

Thus, setting $F(x,t) = f_\mu(u_\mu(x,t), \mathcal{H}(u_\mu(x,\cdot))(t))$, we can proceed analogously to the proof of Lemma 3.5 (with obvious modifications due to another definition of F). ◻

Let $[0, t_{\mu,\max})$ be a maximal interval, on which the solution u_μ both remains transverse and preserves spatial topology. We claim that there is $T > 0$ such that $t_{\mu,\max} \ge T$ for all $\mu > 0$. Indeed, suppose that there is a subsequence of $t_{\mu,\max}$ (which we denote $t_{\mu,\max}$ again) such that $t_{\mu,\max} \to 0$ as $\mu \to 0$. By Theorem 3.1 and Lemma 5.2,

$$\|u_\mu\|_{W_q^{2,1}(Q_{t_{\mu,\max}})} + \max_{t \in [0, t_{\mu,\max}]} \|u_\mu(\cdot, t)\|_{\mathcal{W}} \le c_1(\|\varphi\|_{\mathcal{W}} + f_U + h_U),$$

$$\|u_\mu\|_{C^\gamma(\overline{Q}_{t_{\mu,\max}})} + \|(u_\mu)_x\|_{C^\gamma(\overline{Q}_{t_{\mu,\max}})} \le c_2(\|\varphi\|_{\mathcal{W}} + f_U + h_U),$$

where f_U is defined in (3.10), $h_U = \max_{|u| \le U} |h(u)|$, and $c_2 > 0$ does not depend on $\mu, t_{\mu,\max}, \varphi$.

The latter estimate shows that u_μ remains transverse and preserves spatial topology for all sufficiently small $t_{\mu,\max}$. This contradiction proves that $t_{\mu,\max} \geq T > 0$.

Applying Theorem 3.1 and Lemma 5.2 again, we obtain the estimates

$$\|u_\mu\|_{C(\overline{Q}_T)} < U, \tag{5.1}$$

$$\begin{gathered} \|u_\mu\|_{W_q^{2,1}(Q_T)} + \max_{t\in[0,T]} \|u_\mu(\cdot,t)\|_{\mathcal{W}} \leq c_1(\|\varphi\|_{\mathcal{W}} + f_U + h_U), \\ \|u_\mu\|_{C^\gamma(\overline{Q}_T)} + \|(u_\mu)_x\|_{C^\gamma(\overline{Q}_T)} \leq c_2(\|\varphi\|_{\mathcal{W}} + f_U + h_U), \end{gathered} \tag{5.2}$$

where $c_1, c_2 > 0$ do not depend on μ.

For each u_μ, we have a corresponding function $a_\mu(t)$ (cf. part 3 of Theorem 3.9). It follows from (5.2) and from the fact that $a_\mu \in C^\gamma[0,T]$ that one can choose subsequences, which we denote u_μ and a_μ again, converging to some functions $u(x,t)$ and $a(t)$ in $C(\overline{Q})$ and $C[0,T]$, respectively.

Using (5.2), one can show that the function $a(t)$ corresponds to $u(x,t)$ in the same sense as the functions $a_\mu(t)$ correspond to $u_\mu(x,t)$. In particular, this means that the corresponding hysteresis operators are given by

$$\mathcal{H}(\xi_0,u) = \begin{cases} H_1(u), & 0 \leq x \leq b(t), \\ H_2(u), & b(t) < x \leq 1, \end{cases} \qquad \mathcal{H}(\xi_0,u_\mu) = \begin{cases} H_1(u_\mu), & 0 \leq x \leq b_\mu(t), \\ H_2(u_\mu), & b_\mu(t) < x \leq 1, \end{cases}$$

where $b(t) = \max_{s\in[0,t]} a(s)$ and $b_\mu(t) = \max_{s\in[0,t]} a_\mu(s)$.

By Lemma 3.7, $\|b_\mu - b\|_{C[0,T]} \to 0$ as $\mu \to 0$. Thus, applying Remark 3.6 and Lemma 3.8, we see that $\mathcal{H}(\xi_0,u_\mu)$ converges to $\mathcal{H}(\xi_0,u)$ in $L_q(Q_T)$. Now the Lipschitz continuity of f and the estimate $|h(u_\mu)| \leq h_U$ imply that $f_\mu(u_\mu,\mathcal{H}(\xi_0,u_\mu))$ converges to $f(u,\mathcal{H}(\xi_0,u))$ in $L_q(Q_T)$.

Denote by $\hat{u}$ the solution of the linear parabolic problem

$$\begin{cases} \hat{u}_t = \hat{u}_{xx} + f(u,\mathcal{H}(\xi_0,u)), & (x,t) \in Q_T, \\ \hat{u}_x|_{x=0} = \hat{u}_x|_{x=1} = 0, \\ \hat{u}|_{t=0} = \varphi(x), & x \in (0,1). \end{cases}$$

By Theorem 3.1, $u_\mu \to \hat{u}$ in $W_q^{2,1}(Q_T)$, and hence in $C(\overline{Q}_T)$. Therefore, $u = \hat{u}$, u is a solution of problem (2.6)–(2.8) with the right-hand side f, and estimates (5.1) and (5.2) yield the same estimates for u. Theorem 2.16 is proved.

The number T which we have obtained above depends only on m. Therefore, the continuation theorem (Theorem 2.18) and the theorem on continuous dependence of solutions on initial data (Theorem 2.19) under Condition 5.1 are proved similarly to sections 4.2 and 4.3 (with the help of estimates (5.1) and (5.2), which now hold with u). The proof of Theorem 2.22 does not depend on the dissipativity condition (see [5]).

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REFERENCES

[1] H. W. Alt, *On the thermostat problem*, Control Cybernet., 14 (1985), pp. 171–193.
[2] A. Ashyralyev and P. E. Sobolevskii, *Well-Posedness of Parabolic Difference Equations*, Oper. Theory, Adv. Appl. 69, Birkhäuser Verlag, Basel, 1994.

[3] M. Brokate and J. Sprekels, *Hysteresis and Phase Transitions*, Springer, New York, 1996.

[4] L. C. Evans and M. Portilheiro, *Irreversibility and hysteresis for a forward-backward diffusion equation*, Math. Models Methods Appl. Sci., 14 (2004), pp. 1599–1620.

[5] P. Gurevich and S. Tikhomirov, *Uniqueness of transverse solutions for reaction-diffusion equations with spatially distributed hysteresis*, Nonlinear Anal., 75 (2012), pp. 6610–6619.

[6] P. Gurevich and S. Tikhomirov, *Systems of Reaction-Diffusion Equations with Spatially Distributed Hysteresis*, preprint, arXiv:1210.6252, 2012.

[7] F. C. Hoppensteadt and W. Jäger, *Pattern formation by bacteria*, in Biological Growth and Spread, Lecture Notes in Biomath. 38, 1980, pp. 68–81.

[8] F. C. Hoppensteadt, W. Jäger, and C. Poppe, *A hysteresis model for bacterial growth patterns*, in Modelling of Patterns in Space and Time, Lecture Notes in Biomath. 55, Springer, Berlin, 1984, pp. 123–134.

[9] A. M. Il'in and B. A. Markov, *Nonlinear diffusion equation and Liesegang rings*, Dokl. Akad. Nauk, 440 (2011), pp. 164–167.

[10] J. Kopfova, *Hysteresis in biological models*, J. Phys. Conference Series, 55 (2006), pp. 130–134.

[11] M. A. Krasnosel'skii and A. V. Pokrovskii, *Sistemy s Gisterezisom*, Nauka, Moscow, 1983.

[12] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*. Nauka, Moscow, 1967.

[13] P. I. Plotnikov, *Passing to the limit with respect to the viscosity in an equation with variable parabolicity direction*, Differential Equations, 30 (1994), pp. 614–622.

[14] F. Rothe, *Global Solutions of Reaction-Diffusion Systems*, Lecture Notes in Mathematics 1072, 1984, Springer-Verlag, Berlin, 1984.

[15] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, 2nd ed., Grundlehren der Mathematischen Wissenschaften 258, Springer-Verlag, New York, 1994.

[16] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*. 2nd ed., Johann Ambrosius Barth, Heidelberg, 1995.

[17] A. Visintin, *Differential Models of Hysteresis*, Springer-Verlag. Berlin, Heidelberg, 1994.

[18] A. Visintin, *Evolution problems with hysteresis in the source term*, SIAM J. Math. Anal., 17 (1986), pp. 1113–1138.