On periodicity of solutions for thermocontrol problems with hysteresis-type switches

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Abstract

Mathematical models of thermocontrol processes occurring in chemical reactors and climate control systems are considered. In the models under consideration, the temperature inside a domain is controlled by a thermostat acting on the boundary. The feedback is based on temperature measurements performed by thermal sensors inside the domain. The solvability of the corresponding nonlinear nonlocal problems and the periodicity of solutions are studied.

Contents

1 Introduction

We consider mathematical models of thermocontrol processes in which the temperature inside a domain is controlled by a thermostat acting on the boundary. The feedback is based on temperature measurements performed by thermal sensors inside the domain. The processes under consideration occur in chemical reactors and climate control systems.

The temperature distribution in the domain obeys the heat equation, while the boundary condition involves a control function (the Dirichlet, the Neumann, and the Robin boundary conditions are considered). The control function satisfies an ordinary differential equation whose right-hand side is a nonlinear operator H depending on the "mean" temperature over the domain and taking the values 0 or 1. There are two temperature thresholds w_1 and w_2 ($w_1 < w_2$). If the "mean" temperature is less than or equal to w_1, then $H = 1$ (the heating is switched on); if the "mean" temperature is greater than or equal to w_2, then $H = 0$ (the heating is switched off); if the "mean" temperature is greater than w_1 and less than w_2, then H takes the same value as "just before." Thus, the presence of the operator H provides the so-called hysteresis phenomenon, while the thermal sensors inside the domain cause nonlocal effects.

Thermocontrol models similar to ours were originally proposed in [7, 8]. By transforming the problem into an equivalent set-valued integro-differential equation, the existence of a solution was proved. The existence and uniqueness of solutions for two-phase Stefan problems with the Robin boundary condition involving a hysteresis control were studied in [3, 5, 11].

The question whether *periodic* solutions exist turns out to be much more difficult. In [6], a one-dimensional thermocontrol problem is considered under the assumption that the temperature of the thermostat changes by jump. Thus, there is no coupling with an ordinary differential equation in this case. The existence of a periodic solution is proved. Its uniqueness in a class of the so-called "two-phase" periodic solutions is established. Periodicity of solutions of a one-dimensional problem in the case where the thermostat changes its temperature continuously, was considered in [15]. The existence of a periodic solution was proved. The periodicity of solutions for a one-dimensional Stefan problem with hysteresis-type boundary conditions was investigated in [9]. The existence of periodic solutions in the multidimensional case is an unsolved problem.

In the present paper, we prove the existence and uniqueness result for the heat equation and investigate the *periodicity* of solutions in the multidimensional case. A pair consisting of the temperature $w(x,t)$ and the control function $u(t)$ both periodic in time with the same period is called a *strong periodic solution* (see Definitions 4.1 and 4.2). Along with a strong periodic solution, we introduce the notion of a *mean-periodic solution*, which is a pair $(w(x,t),u(t))$ such that the "mean" temperature and the control function are both periodic in time with the same period (see Definition 4.3). The main result is the so-called *conditional existence*. We show that the existence of a mean-periodic solution implies the existence of a strong periodic solution with the same period.

Moreover, we prove that, for any initial temperature distribution (with the "mean" value from the interval $[w_1,w_2)$), there exists a unique mean-periodic solution, provided that the Neumann boundary condition is considered and the "uniform" distribution of thermal sensors inside the domain is assumed. It follows from the result about conditional existence that a strong periodic solution also exists in this case.

The paper is organized as follows. The setting of the problem is given in Sec. 2. In Sec. 3, we prove the existence and uniqueness of a strong solution of the thermocontrol problem for any initial temperature distribution and control value. We use the theory of initial boundary-value problems for linear parabolic equations to prove the existence of solutions between the times at which the operator H "switches" and to estimate the intervals between the switches from below. The main result about the conditional existence is proved in Sec. 4. We introduce a nonlinear operator G of the shift of the temperature function by a period T. The operator G is defined for all initial temperature distributions providing a given T-periodic "mean" temperature and a given T-periodic control function. We prove that the domain of definition of the operator G is a nonempty closed set, while the operator G is a contraction map. Application of the Banach fixed-point theorem yields the desired result. Finally, in Sec. 5, we consider an example in which the boundary condition is of the Neumann type and the "uniform" distribution of thermal sensors inside the domain is assumed. We show that the "mean" temperature satisfies an ordinary differential equation, which simplifies the situation (there is vast literature devoted to ordinary differential equations with hysteresis operators, e.g., [2, 4, 14, 17] and others). In this case, we prove the existence of a mean-periodic solution (hence, a strong periodic solution) for the problem in question.

The results of the paper were earlier announced in [10] without proofs.

2 Setting of the Problem

1. Let $Q \subset \mathbb{R}^n$ $(n \ge 1)$ be a bounded domain with boundary Γ of class C^∞. Let $w(x,t)$ be the temperature at the point $x \in Q$ at the moment $t \ge 0$ satisfying the heat equation

$$w_t(x,t) = \Delta w(x,t) - p(x)w(x,t) \quad ((x,t) \in Q_T) \tag{2.1}$$

with the initial condition

$$w(x,0) = \varphi(x) \quad (x \in Q), \tag{2.2}$$

where $Q_T = Q \times (0,T)$, $T > 0$, $p \in C^\infty(\mathbb{R}^n)$, and $p(x) \ge 0$.

The boundary condition contains a real-valued control function $u(t)$ (to be defined below) which regulates the temperature on the boundary, the heat flux through the boundary, or the embient temperature:

$$-\gamma \frac{\partial w}{\partial \nu} = \sigma(x)(w(x,t) - w_e(x)) - K(x)(u(t) - u_c) \quad ((x,t) \in \Gamma_T), \tag{2.3}$$

where $\Gamma_T = \Gamma \times (0,T)$, ν is the outward normal to Γ_T at the point (x,t), $\gamma \ge 0$, $\sigma, w_e, K \in C^\infty(\mathbb{R}^n)$ are real-valued functions, $\sigma(x) \ge 0$, $\sigma(x) \ge \sigma_0 > 0$ if $\gamma = 0$, and $u_c > 0$.

For any function $v(x,t)$, we denote

$$v_m(t) = \int_Q m(x)v(x,t)\,dx,$$

where the weight function $m \in L_\infty(Q)$, $m(x) \not\equiv 0$, is determined by characteristics of the thermal sensors.

We assume that the control function $u(t)$ satisfies the following Cauchy problem:

$$u'(t) + au(t) = H(w_m)(t) \quad (t \in (0,T)), \tag{2.4}$$

$$u(0) = u_0, \tag{2.5}$$

where $a > 0$, $u_0 \in \mathbb{R}$, w is the function satisfying relations (2.1)–(2.3), and H is the hysteresis operator to be defined below (cf. [12, Chap. 5, Sec. 28]).

We denote by $BV(0,T)$ the Banach space of real-valued functions having finite total variation on the segment $[0,T]$ and by $C_r[0,T)$ the linear space of functions which are continuous on the right in $[0,T)$. Fix two numbers $w_1 < w_2$. We introduce the *hysteresis operator*

$$H : C[0,T] \to BV(0,T) \cap C_r[0,T)$$

by the following rule. For any $g \in C[0,T]$, the function $z = H(g) : [0,T] \to \{0,1\}$ is defined as follows. Let $X_t = \{t' \in (0,t] : g(t') = w_1 \text{ or } w_2\}$; then

$$z(0) = \begin{cases} 1 & \text{if } g(0) < w_2, \\ 0 & \text{if } g(0) \geq w_2 \end{cases}$$

and for $t \in (0,T]$

$$z(t) = \begin{cases} z(0) & \text{if } X_t = \varnothing, \\ 1 & \text{if } X_t \neq \varnothing \text{ and } g(\max X_t) = w_1, \\ 0 & \text{if } X_t \neq \varnothing \text{ and } g(\max X_t) = w_2. \end{cases}$$

To be definite, we assume that

$$w_1 \leq \int_Q m(x)\varphi(x)\,dx < w_2. \tag{2.6}$$

2. Denote by $W_2^k(Q)$ $(k \in \mathbb{N})$ the Sobolev space with the norm

$$\|v\|_{W_2^k(Q)} = \left(\sum_{|\alpha| \leq k} \int_Q |D^\alpha v(x)|^2\,dx \right)^{1/2}.$$

By $\mathring{W}_2^k(Q)$ we denote the closure in $W_2^k(Q)$ of the set $C_0^\infty(Q)$ consisting of infinitely differentiable functions supported in Q.

Let $W_\infty^1(a,b)$ $(a < b)$ denote the space of absolutely continuous functions having the first derivative from $L_\infty(a,b)$ with the norm

$$\|u\|_{W_\infty^1(a,b)} = \max_{t \in [a,b]} |u(t)| + \operatorname*{vrai\,sup}_{t \in (a,b)} |u'(t)|. \tag{2.7}$$

We denote by $W_2^{2,1}(Q \times (a,b))$ $(a < b)$ the anisotropic Sobolev space with the norm

$$\|w\|_{W_2^{2,1}(Q \times (a,b))} = \left(\int_a^b \|w(\cdot,t)\|_{W_2^2(Q)}^2\,dt + \int_a^b \|w_t(\cdot,t)\|_{L_2(Q)}^2\,dt \right)^{1/2}.$$

Set

$$\mathcal{W}(Q_T) = W_2^{2,1}(Q_T) \times W_\infty^1(0,T).$$

Definition 2.1. A pair of functions $(w,u) \in \mathcal{W}(Q_T)$ is called a *strong solution* of problem (2.1)–(2.5) in Q_T if w satisfies Eq. (2.1) a.e. in Q_T and conditions (2.2), (2.3) in the sense of traces and u satisfies Eq. (2.4) a.e. in $(0,T)$ and condition (2.5).

Definition 2.2. Let a pair $(w,u) \in \mathcal{W}(Q_T)$ be a strong solution of problem (2.1)–(2.5) in Q_T. A moment $t_1 \in (0,T)$ is called a *switching time* if either

$$\exists \delta = \delta(t_1) : H(w_m)(\tau) = 1 \text{ for } t_1 - \delta < \tau < t_1 \text{ and } w_m(t_1) = w_2,$$

or

$$\exists \delta = \delta(t_1) : H(w_m)(\tau) = 0 \text{ for } t_1 - \delta < \tau < t_1 \text{ and } w_m(t_1) = w_1.$$

3 Existence and Uniqueness of Strong Solutions

1. Now we prove the existence and uniqueness of strong solutions in the sense of Definition 2.1.

Let $\mathcal{V}$ be the closed affine manifold in $W_2^1(Q)\times\mathbb{R}$ given by

$$\mathcal{V}=\begin{cases} W_2^1(Q)\times\mathbb{R} & \text{if } \gamma>0,\\ \{(\varphi,u_0)\in W_2^1(Q)\times\mathbb{R} : \sigma(x)(\varphi(x)-w_e(x))-K(x)(u_0-u_c)=0\ (x\in\Gamma)\} & \text{if } \gamma=0.\end{cases}$$

Theorem 3.1. *Let $(\varphi,u_0)\in\mathcal{V}$ and condition (2.6) hold. Then there exists a unique strong solution $(w,u)\in\mathcal{W}(Q_T)$ of problem (2.1)–(2.5) in Q_T. Moreover, the set of switching times on the interval $(0,T)$ is finite or empty.*

First, we shall prove some auxiliary results. Consider the following initial boundary-value problem:

$$w_{0t}(x,t)=\Delta w_0(x,t)-p(x)w_0(w,t)\quad((x,t)\in Q_T), \tag{3.1}$$

$$w_0(x,0)=\varphi(x)\quad(x\in Q), \tag{3.2}$$

$$-\gamma\frac{\partial w_0}{\partial\nu}=\sigma(x)w_0(x,t)+k_0(x)\psi(t)+k_1(x)\quad((x,t)\in\Gamma_T), \tag{3.3}$$

where $\varphi\in W_2^1(Q)$, $k_0,k_1\in C^\infty(\mathbb{R}^n)$, and $\psi\in W_\infty^1(0,T)$ are real-valued functions.

If $\gamma=0$, we consider the closed affine subspace of $W_2^1(Q)$ depending on the function $\psi\in W_\infty^1(0,T)$ and given by

$$W_{2,\psi}^1(Q)=\{\varphi\in W_2^1(Q):\sigma(x)\varphi(x)+k_0(x)\psi(0)+k_1(x)=0\ (x\in\Gamma)\}.$$

Definition 3.1. A function $w\in W_2^{2,1}(Q_T)$ is called a *strong solution* of problem (3.1)–(3.3) in Q_T if w satisfies Eq. (3.1) a.e. in Q_T and conditions (3.2), (3.3) in the sense of traces.

Lemma 3.1. *For any $\psi\in W_\infty^1(0,T)$ and*

$$\varphi\in\begin{cases} W_2^1(Q) & \text{if } \gamma>0,\\ W_{2,\psi}^1(Q) & \text{if } \gamma=0,\end{cases}$$

there exists a unique strong solution $w_0\in W_2^{2,1}(Q_T)$ of problem (3.1)–(3.3) in Q_T. Moreover,

$$\|w_0\|_{W_2^{2,1}(Q_T)}\le c_1(\|\varphi\|_{W_2^1(Q)}+\|k_0\|_{W_2^2(Q)}\|\psi\|_{W_\infty^1(0,T)}+\|k_1\|_{W_2^2(Q)}), \tag{3.4}$$

where $c_1>0$ does not depend on φ,ψ,k_0,k_1.

Proof. Consider the auxiliary boundary-value problem

$$\Delta U(x)-p(x)U(x)=\varkappa|\Gamma|/|Q|\quad(x\in Q), \tag{3.5}$$

$$-\gamma\frac{\partial U}{\partial\nu}=\sigma(x)U(x)+1\quad(x\in\Gamma), \tag{3.6}$$

where $\varkappa=-1/\gamma$ if $p(x)\equiv0$ and $\sigma(x)\equiv0$ and $\varkappa=0$ otherwise; $|Q|$ is the n-dimensional Lebesgue measure of Q and $|\Gamma|$ is the $(n-1)$-dimensional Lebesgue measure of Γ. In this case problem (3.5), (3.6) admits a solution $U\in W_2^2(Q)$.

Since the boundary condition (2.3) involves only the traces of the functions $k_0(x)$, and $k_1(x)$ on Γ, we may assume without loss of generality that

$$\left.\frac{\partial\sigma}{\partial\nu}\right|_\Gamma=\left.\frac{\partial k_0}{\partial\nu}\right|_\Gamma=\left.\frac{\partial k_1}{\partial\nu}\right|_\Gamma=0.$$

Set $v_0(x,t)=[k_0(x)\psi(t)+k_1(x)]U(x)$. Since $\psi\in W_\infty^1(0,T)$, it follows that $v_0\in W_2^{2,1}(Q_T)$.

The function $v=w-v_0$ satisfies the relations

$$v_t(x,t)=\Delta v(x,t)-p(x)v(w,t)+f_0(x,t)\quad((x,t)\in Q_T), \tag{3.7}$$

$$v(x,0)=\varphi(x)+\varphi_0(x)\quad(x\in Q), \tag{3.8}$$

$$-\gamma\frac{\partial v}{\partial\nu}=\sigma(x)v(x,t)\quad((x,t)\in\Gamma_T), \tag{3.9}$$

where

$$f_0(x,t)=\Delta[k_0(x)\psi(t)+k_1(x)]U(x)+[k_0(x)\psi(t)+k_1(x)]\varkappa|\Gamma|/|Q|-k_0(x)\psi'(t)U(x),$$
$$\varphi_0(x)=[-k_0(x)\psi(0)-k_1(x)]U(x).$$

Applying Theorem 5.3 in [13] to problem (3.7)–(3.9), we complete the proof. □

The following corollary results from Lemma 3.1 and from the embedding theorem (see Theorem 2.1 in [13]).

Corollary 3.1. *For any* $\psi \in W^1_\infty(0,T)$ *and*

$$\varphi \in \begin{cases} W^1_2(Q) & \text{if } \gamma > 0, \\ W^1_{2,\psi}(Q) & \text{if } \gamma = 0, \end{cases}$$

the trace of the strong solution $w_0 \in W^{2,1}_2(Q_T)$ *of problem* (3.1)–(3.3) *for* $t = \tau$, $0 \le \tau \le T$, *satisfies the inequality*

$$\|w_0(\cdot,\tau)\|_{W^1_2(Q)} \le c_2(\|\varphi\|_{W^1_2(Q)} + \|k_0\|_{W^2_2(Q)}\|\psi\|_{W^1_\infty(0,T)} + \|k_1\|_{W^2_2(Q)}), \tag{3.10}$$

where $c_2 > 0$ *does not depend on* $\varphi, \psi, k_0, k_1, \tau$.

Lemma 3.2. *Let* $\psi \in W^1_\infty(0,T)$ *and*

$$\varphi \in \begin{cases} W^1_2(Q) & \text{if } \gamma > 0, \\ W^1_{2,\psi}(Q) & \text{if } \gamma = 0. \end{cases}$$

Let $w_0 \in W^{2,1}_2(Q_T)$ *be a strong solution of problem* (3.1)–(3.3) *in* Q_T. *Then*

$$\frac{(w_{0m}(t'') - w_{0m}(t'))^2}{c_3\|m\|_{L_\infty(Q)}(\|\varphi\|_{W^1_2(Q)} + \|\psi\|_{W^1_\infty(0,T)} + 1)^2} \le t'' - t' \qquad \forall t', t'',\ 0 \le t' < t'' \le T, \tag{3.11}$$

where $c_3 > 0$ *does not depend on* φ, ψ, t', t''.

Proof. Using the Cauchy–Bunyakovskii inequality and Lemma 3.1, we obtain

$$\begin{aligned} |w_{0m}(t'') - w_{0m}(t')| &= \left| \int_Q m(x)\,dx \int_{t'}^{t''} w_{0t}(x,t)\,dt \right| \le \|m\|_{L_\infty(Q)}(t''-t')^{1/2}|Q|^{1/2}\|w_{0t}\|_{L_2(Q_T)} \\ &\le \|m\|_{L_\infty(Q)}(t''-t')^{1/2}|Q|^{1/2}\|w_0\|_{W^{2,1}_2(Q_T)} \\ &\le c_3^{1/2}\|m\|_{L_\infty(Q)}(t''-t')^{1/2}(\|\varphi\|_{W^1_2(Q)} + \|\psi\|_{W^1_\infty(0,T)} + 1), \end{aligned}$$

where $|Q|$ is the n-dimensional Lebesgue measure of Q and $c_3 > 0$ does not depend on φ, ψ, t', t''. This inequality implies (3.11). □

2. Now we can prove Theorem 3.1. We will prove it by induction.

I. Consider the following Cauchy problem for the control function $u(t)$:

$$u'(t) + au(t) = H(w_m)(t) \quad (t > t_*), \tag{3.12}$$

$$u(t_*) = u_*. \tag{3.13}$$

As long as $H = \text{const}$, the solution of problem (3.12), (3.13) has the form

$$u(t) = \left(u_* - \frac{H}{a}\right)e^{-a(t-t_*)} + \frac{H}{a}, \qquad t > t_*. \tag{3.14}$$

We note that

$$|u(t)| \le \max(1/a, |u_*|), \quad |u'(t)| \le 1 + a|u(t)| \le \max(2, 1 + a|u_*|) \quad \forall t \ge t_*. \tag{3.15}$$

II. Denote $u_1(t) = u(t)$ ($t \in [0,T]$), where $u(t)$ is given by (3.14) with $H = 1$, $t_* = 0$, and $u_* = u_0$. By virtue of (3.15), we have

$$|u_1(t)| \le \max(1/a, |u_0|) = a_1, \quad |u_1'(t)| \le \max(2, 1 + a|u_0|) = a_2 \quad \forall t \ge 0, \tag{3.16}$$

where $a_1, a_2 > 0$ may depend on u_0, but do not depend on t.

Consider problem (3.1)–(3.3) with $\psi(t) = u_1(t) - u_c$ ($t \in [0,T]$), $k_0(x) = -K(x)$, and $k_1(x) = -\sigma(x)w_e(x)$. By virtue of Lemma 3.1, there exists a unique strong solution w_0 of problem (3.1)–(3.3) in Q_T.

We define the set

$$S_1 = \{t \in (0,T) : w_{0m}(t) = w_2\}.$$

If $S_1 = \varnothing$, then, by virtue of condition (2.6), (w_0, u_1) is a unique strong solution of problem (2.1)–(2.5) in Q_T.

III. Let $S_1 \neq \varnothing$. Denote $t_1 = \inf\limits_{t\in(0,T)} S_1$ (i.e., t_1 is the first switching moment). Clearly, $t_1 < T$. Lemma 3.2 and relations (3.16) imply that $w_{0m} \in C^{1/2}[0,T]$ and

$$t_1 \geq \frac{\left(w_2 - \int_Q m(x)\varphi(x)\,dx\right)^2}{c_3\|m\|_{L_\infty(Q)}(\|\varphi\|_{W_2^1(Q)} + a_1 + a_2 + u_c + 1)^2} = \delta.$$

Thus, (w_0, u_1) is a unique strong solution of problem (2.1)–(2.5) in $Q \times (0, t_1)$, where $\delta \leq t_1 < T$.

Denote

$$u_2(t) = \begin{cases} u_1(t), & t \in [0, t_1], \\ u(t), & t \in [t_1, T], \end{cases}$$

where $u(t)$ is given by (3.14) with $H = 0$, $t_* = t_1$, and $u_* = u_1(t_1)$. Using (3.15), we have

$$\begin{aligned} |u_2(t)| &\leq \max(1/a, |u_1(t_1)|) \leq \max(1/a, |u_0|) = a_1 && \forall t \geq t_1, \\ |u_2'(t)| &\leq \max(2, 1 + a|u_1(t_1)|) \\ &\leq \max(2, 1 + a\max(1/a, |u_0|)) = \max(2, 1 + a|u_0|) = a_2 && \forall t \geq t_1. \end{aligned} \tag{3.17}$$

Consider problem (3.1)–(3.3) with $\psi(t) = u_2(t) - u_c$ ($t \in [0,T]$), $k_0(x) = -K(x)$, and $k_1(x) = -\sigma(x)w_e(x)$. By virtue of Lemma 3.1, there exists a unique strong solution w_0 of problem (3.1)–(3.3) in Q_T.

We define the set

$$S_2 = \{t \in (t_1, T) : w_{0m}(t) = w_1\},$$

If $S_2 = \varnothing$, then (w_0, u_2) is a unique strong solution of problem (2.1)–(2.5) in Q_T.

IV. Let $S_2 \neq \varnothing$. Denote $t_2 = \inf\limits_{t\in(t_1,T)} S_2$ (i.e., t_2 is the second switching moment). Clearly, $t_2 < T$. Lemma 3.2 and relations (3.16), (3.17), and (2.6) imply that $w_{0m} \in C^{1/2}[0,T]$ and

$$\begin{aligned} t_2 - t_1 &\geq \frac{(w_2 - w_1)^2}{c_3\|m\|_{L_\infty(Q)}(\|\varphi\|_{W_2^1(Q)} + a_1 + a_2 + u_c + 1)^2} \\ &\geq \frac{\left(w_2 - \int_Q m(x)\varphi(x)\,dx\right)^2}{c_3\|m\|_{L_\infty(Q)}(\|\varphi\|_{W_2^1(Q)} + a_1 + a_2 + u_c + 1)^2} = \delta. \end{aligned}$$

Thus, (w_0, u_2) is a unique strong solution of problem (2.1)–(2.5) in $Q \times (0, t_2)$, where $2\delta \leq t_2 < T$. Repeating the above procedure finitely many times and taking into account that $\delta > 0$, we prove the existence and uniqueness of a strong solution of problem (2.1)–(2.5) in Q_T. Clearly, the set of switching times is finite. □

4 Conditional Existence of Strong Periodic Solutions

1. In this section, we prove the existence of a strong T-periodic solution (w, u) of problem (2.1), (2.3), (2.4), provided that for some initial values $(\varphi, u_0) \in \mathcal{V}$ there is a strong solution $(\tilde{w}, \tilde{u}) \in \mathcal{W}(Q_T)$ of problem (2.1)–(2.5) in Q_T such that $\tilde{w}_m(0) = \tilde{w}_m(T)$, $\tilde{u}(0) = \tilde{u}(T)$, and $H(\tilde{w}_m)(T) = 1$.

Definition 4.1. A pair (w, u) is called a *strong T-periodic solution of problem* (2.1), (2.3), (2.4) if, for any $T_0 \geq T$, the following holds:

1. $(w, u) \in \mathcal{W}(Q_{T_0})$,
2. the function w satisfies the equation in (2.1) a.e. in Q_{T_0} and the equality in (2.3) on Γ_{T_0},
3. the function u satisfies the equation in (2.4) a.e. in $(0, T_0)$,
4. $w(\cdot, t) = w(\cdot, t + T)$, $u(t) = u(t + T)$, and $H(w_m)(t) = H(w_m)(t + T)$ for $t \in [0, T_0 - T]$.

In a sense, Definition 4.1 is equivalent to the following one.

Definition 4.2. A pair (w, u) is called a *strong T-periodic solution of problem* (2.1), (2.3), (2.4) if

1. $(w, u) \in \mathcal{W}(Q_T)$,
2. the function w satisfies Eq. (2.1) a.e. in Q_T and condition (2.3) in the sense of traces,

3. the function u satisfies Eq. (2.4) a.e. in $(0,T)$,

4. $w(\cdot,0)=w(\cdot,T)$, $u(0)=u(T)$, and $H(w_m)(T)=1$.

Indeed, if a pair (w,u) is a strong T-periodic solution of problem (2.1), (2.3), (2.4) in the sense of Definition 4.1, then its restriction to Q_T is a strong T-periodic solution of problem (2.1), (2.3), (2.4) in the sense of Definition 4.2. If (w,u) is a strong T-periodic solution of problem (2.1), (2.3), (2.4) in the sense of Definition 4.2, then one can extend (w,u) to Q_{T_0} for any $T_0>T$ in such a way that $w(\cdot,t)=w(\cdot,t+T)$ and $u(t)=u(t+T)$ for $t\in[0,T_0-T]$. Due to Theorem 3.1, this extension is a strong T-periodic solution of problem (2.1), (2.3), (2.4) in the sense of Definition 4.1.

Definition 4.3. We say that problem (2.1), (2.3), (2.4) possesses the *mean-periodicity property* if there is a pair $(\varphi,u_0)\in\mathcal{V}$ and a number $T>0$ such that, for any $T_0\ge T$, the strong solution $(\tilde w,\tilde u)\in\mathcal{W}(Q_{T_0})$ of problem (2.1)–(2.5) in Q_{T_0} with the initial values $(\varphi,u_0)\in\mathcal{V}$ satisfies the equalities

$$\tilde w_m(t)=\tilde w_m(t+T),\quad \tilde u(t)=\tilde u(t+T),\quad H(\tilde w_m)(t)=H(\tilde w_m)(t+T),\quad t\in[0,T_0-T].$$

The strong solution $(\tilde w,\tilde u)$ is said to be a *mean-periodic solution* (*with period* T).

Remark 4.1. Unlike the definition of a strong T-periodic solution, in the case of mean-periodic solution, the condition

$$\tilde w_m(0)=\tilde w_m(T),\quad \tilde u(0)=\tilde u(T),\quad H(w_m)(T)=1$$

does not generally imply that

$$\tilde w_m(t)=\tilde w_m(t+T),\quad \tilde u(t)=\tilde u(t+T),\quad H(w_m)(t)=H(w_m)(t+T),\quad t>0.$$

Remark 4.2. In Definition 4.3, one could omit the requirement that $H(\tilde w_m)(t)=H(\tilde w_m)(t+T)$ for $t\in[0,T_0-T]$. In this case, the function $H(\tilde w_m)$ would be periodic for $t\ge T$. However, it is more convenient for our purposes to require that $H(\tilde w_m)$ be periodic for $t\ge 0$ (see the proof of Theorem 4.1).

Theorem 4.1. *Let problem* (2.1), (2.3), (2.4) *possess the mean-periodicity property and* $(\tilde w,\tilde u)$ *be a mean-periodic solution* (*with period* T) *such that*

$$w_1\le \tilde w_m(0)<w_2. \tag{4.1}$$

If $p(x)\equiv 0$ *and* $\sigma(x)\equiv 0$, *we assume that* $m(x)\equiv m_0$, *where* $m_0\ne 0$ *is a constant.*

Then there is a unique strong T*-periodic solution* $(w,\tilde u)$ *of problem* (2.1), (2.3), (2.4) *such that*

$$w_m(t)=\tilde w_m(t)\quad (t\ge 0). \tag{4.2}$$

Proof. I. By assumption, the pair $(\tilde w_m,\tilde u)\in C^{1/2}[0,\infty)\times W^1_\infty(0,\infty)$ is T-periodic.

By Theorem 3.1, for any pair $(\varphi,u_0)\in\mathcal{V}$, where φ satisfies condition (2.6), and for any $T_0>0$ there is a unique strong solution $(w,u)\in\mathcal{W}(Q_{T_0})$ of problem (2.1)–(2.5) in Q_{T_0}. By Lemma 3.2, $w_m\in C^{1/2}[0,T_0]$. Denote

$$\tilde{\mathcal{V}}=\{(\varphi,u_0)\in\mathcal{V}: w_m(t)=\tilde w_m(t),\ u(t)=\tilde u(t),\ H(w_m)(t)=H(\tilde w_m)(t),\ t\ge 0\}. \tag{4.3}$$

We note that if $(\varphi,u_0)\in\tilde{\mathcal{V}}$ and $(\tilde\varphi,\tilde u_0)\in\tilde{\mathcal{V}}$, then $u_0=\tilde u_0=\tilde u(0)$. It is also clear that the set $\tilde{\mathcal{V}}$ is not empty because it contains the pair $(\tilde w(x,0),\tilde u(0))$.

II. Let us prove that the set $\tilde{\mathcal{V}}$ is closed in $\mathcal{V}$. Let $(\varphi^k,u_0^k)\in\tilde{\mathcal{V}}$ $(k=1,2,\dots)$ and $(\varphi^k,u_0^k)=(\varphi^k,\tilde u(0))\to(\varphi^0,\tilde u(0))$ in $\mathcal{V}$ as $k\to\infty$. We denote by $(w^k,\tilde u)\in\mathcal{W}(Q_{T_0})$ and $(w^0,u^0)\in\mathcal{W}(Q_{T_0})$ the corresponding strong solutions of problem (2.1)–(2.5) in Q_{T_0} for any $T_0>0$.

We have to prove that

$$w^0_m(t)=\tilde w_m(t),\quad u^0(t)=\tilde u(t),\ H(w^0_m)(t)=H(\tilde w_m)(t)\quad (t\ge 0). \tag{4.4}$$

Since $w^k_m(t)=\tilde w_m(t)$ $(t\in[0,T])$ for all k, it follows that the first switching time is one and the same for all strong solutions $(w^k,\tilde u)$. We denote this switching time by τ. Let τ_0 be the first switching time corresponding to the strong solution (w^0,u^0).

Denote $t_1=\min(\tau,\tau_0)$. Clearly, neither of the solutions $(w^k,\tilde u)$, (w^0,u^0) "switches" on the interval $(0,t_1)$ and $u^0(0)=\tilde u(0)$. Therefore, $u^0(t)=\tilde u(t)$ $(t\in[0,t_1])$. Hence, the function $v^k=w^k-w^0$ is a strong solution of the following problem in $Q\times(0,t_1)$:

$$v^k_t(x,t)=\Delta v^k(x,t)-p(x)v^k(w,t)\quad ((x,t)\in Q\times(0,t_1)),$$

$$v^k(x,0) = \varphi^k(x) - \varphi^0(x) \quad (x \in Q),$$

$$-\gamma \frac{\partial v^k}{\partial \nu} = \sigma(x) v^k(x,t) \quad ((x,t) \in \Gamma \times (0,t_1)).$$

Using the Cauchy–Bunyakovskii inequality and Corollary 3.1, we obtain

$$|\tilde{w}_m(t) - w_m^0(t)| \le c_4 \|v^k(\cdot,t)\|_{W_2^1(Q)} \le c_4 c_2 \|\varphi^k - \varphi^0\|_{W_2^1(Q)} \to 0, \quad k \to \infty, \quad \forall t \in [0,t_1],$$

i.e., $\tilde{w}_m(t) = w_m^0(t)$ ($t \in [0,t_1]$). Therefore, the functions $\tilde{w}_m(t)$ and $w_m^0(t)$ simultaneously achieve the upper threshold w_2. Hence,

$$\tau = \tau_0 = t_1, \quad w_m^0(t) = \tilde{w}_m(t), \quad u^0(t) = \tilde{u}(t), \quad H(w_m^0)(t) = H(\tilde{w}_m)(t) \quad (t \in [0,t_1]).$$

For any $T_0 > 0$, repeating the above arguments finitely many times, we see that the equalities in (4.4) hold on the interval $[0,T_0]$. This means that the set $\tilde{\mathcal{V}}$ is closed in $\mathcal{V}$.

III. Consider the operator $G : \mathcal{V} \to \mathcal{V}$ given by

$$G(\varphi, u_0) = (w(x,T), u(T)) \quad ((\varphi,u_0) \in \mathcal{V}).$$

It follows from the periodicity of the pair $(\tilde{w}_m, \tilde{u})$, from the definition of the set $\tilde{\mathcal{V}}$, and from the uniqueness part in Theorem 3.1 that $G(\varphi, \tilde{u}(0)) \in \tilde{\mathcal{V}}$ for $(\varphi, \tilde{u}(0)) \in \tilde{\mathcal{V}}$. Let us prove that the operator

$$G : \tilde{\mathcal{V}} \to \tilde{\mathcal{V}}$$

is a contraction map.

Consider arbitrary pairs $(\varphi_j, \tilde{u}(0)) \in \tilde{\mathcal{V}}$, $j = 1,2$. Let $(w_j, \tilde{u}) \in \mathcal{W}(Q_T)$, $j = 1,2$, be the strong solutions of problem (2.1)–(2.5) in Q_T with the initial values $(\varphi_j, \tilde{u}(0))$. Clearly the function $w = w_1 - w_2 \in W_2^{2,1}(Q_T)$ satisfies the relations

$$w_t(x,t) = \Delta w(x,t) - p(x) w(x,t) \quad ((x,t) \in Q_T), \tag{4.5}$$

$$w(x,0) = \varphi(x) \quad (x \in Q), \tag{4.6}$$

$$-\gamma \frac{\partial w}{\partial \nu} = \sigma(x) w(x,t) \quad ((x,t) \in \Gamma_T), \tag{4.7}$$

where $\varphi = \varphi_1 - \varphi_2 \in \begin{cases} W_2^1(Q) & \text{if } \gamma > 0, \\ \mathring{W}_2^1(Q) & \text{if } \gamma = 0. \end{cases}$

It follows from Theorem 3.7 in [1, Chap. 1], inequality (3.9) in [1, Chap. 1], Theorem 1.14.5 in [16], and Theorem 4.3.3 in [16] that

$$w \in \begin{cases} C([0,T]; W_2^1(Q)) \cap C^1((0,T]; W_2^2(Q)) & \text{if } \gamma > 0, \\ C([0,T]; \mathring{W}_2^1(Q)) \cap C^1((0,T]; \mathring{W}_2^1(Q) \cap W_2^2(Q)) & \text{if } \gamma = 0 \end{cases}$$

(the differentiability with respect to t follows from the analiticity of the semigroup corresponding to parabolic problem (4.5)–(4.7)). Therefore, multiplying (4.5) by $\overline{w}_t$ and integrating over Q for each fixed $t > 0$ (Fubini's theorem should be taken into account), we obtain

$$\int_Q |w_t|^2 \, dx = -\frac{1}{2} \frac{d}{dt} \left(\int_Q \left(|\nabla w|^2 + p(x)|w|^2 \right) dx + \int_\Gamma \omega \sigma(x) |w|^2 d\Gamma \right), \tag{4.8}$$

where $\omega = \gamma^{-1}$ if $\gamma > 0$ and $\omega = 0$ if $\gamma = 0$.

Denote

$$|\!|\!| v |\!|\!|_{W_2^1(Q)} = \left(\int_Q \left(|\nabla v|^2 + p(x)|v|^2 \right) dx + \int_\Gamma \omega \sigma(x) |v|^2 d\Gamma \right)^{1/2}. \tag{4.9}$$

First, we assume that $p(x)$ and $\sigma(x)$ are not simultaneously identically zero. Then relation (4.9) defines an equivalent norm in $W_2^1(Q)$ for $\gamma > 0$ and in $\mathring{W}_2^1(Q)$ for $\gamma = 0$. Moreover, the elliptic problem corresponding to the parabolic problem under consideration is uniquely solvable.

Using (4.8), (4.5), and the a priori estimate of solutions for elliptic problems, we have

$$\frac{d}{dt} |\!|\!| w |\!|\!|_{W_2^1(Q)}^2 = -2 \int_Q |w_t|^2 \, dx = -2 \int_Q |\Delta w - p(x) w|^2 dx \le -c |\!|\!| w |\!|\!|_{W_2^1(Q)}^2,$$

where $c > 0$ does not depend on w. Applying the Gronwall lemma yields

$$\|w(\cdot, T)\|_{W_2^1(Q)} \le e^{-cT/2}\|\varphi\|_{W_2^1(Q)}. \tag{4.10}$$

Now we assume that $p(x) \equiv 0$ and $\sigma(x) \equiv 0$ (hence, $\gamma > 0$). Relations (4.8) and (4.9) take the form

$$\int_Q |w_t|^2\, dx = -\frac{1}{2}\frac{d}{dt}\int_Q |\nabla w|^2 dx, \tag{4.11}$$

$$\|v\|_{W_2^1(Q)} = \left(\int_Q |\nabla v|^2 dx\right)^{1/2}. \tag{4.12}$$

On the other hand, $m(x) \equiv m_0 \ne 0$ by assumption; therefore,

$$\int_Q \varphi(x)\, dx = \int_Q \varphi_1(x)\, dx - \int_Q \varphi_2(x)\, dx = \frac{1}{m_0}(\tilde w_m(0) - \tilde w_m(0)) = 0. \tag{4.13}$$

Using (4.13) and (4.5)–(4.7), we have for any $\tau > 0$

$$\int_Q w(x,\tau)\, dx = \int_0^\tau \int_Q w_t(x,t)\, dx\, dt = \int_0^\tau \int_Q \Delta w(x,t)\, dx\, dt = 0 \tag{4.14}$$

(because $w(x,t)$ satisfies the Neumann boundary condition). Relation (4.13) defines an equivalent norm in $W_2^1(Q)$ for the functions $w(\cdot,\tau)$ satisfying (4.14). Moreover, the elliptic problem (the Neumann problem for the Poisson equation) corresponding to the parabolic problem under consideration is uniquely solvable in the subspace of $W_2^1(Q)$ consisting of the functions $w(\cdot,\tau)$ satisfying (4.14). Thus, similarly to the above, we obtain estimate (4.10).

IV. We equip the space $\mathcal{V}$ with the norm

$$\|(\varphi, u_0)\|_{\mathcal{V}} = \left(\|\varphi\|^2_{W_2^1(Q)} + |u_0|^2\right)^{1/2} \qquad \forall(\varphi, u_0) \in \mathcal{V}.$$

By using (4.10), we have

$$\begin{aligned}\|G(\varphi_1, \tilde u(0)) - G(\varphi_2, \tilde u(0))\|_{\mathcal{V}} &= \|w(\cdot, T)\|_{W_2^1(Q)} \le e^{-cT/2}\|\varphi_1 - \varphi_2\|_{W_2^1(Q)}\\ &= e^{-cT/2}\|(\varphi_1, \tilde u(0)) - (\varphi_2, \tilde u(0))\|_{\mathcal{V}}\end{aligned}$$

for any $(\varphi_j, \tilde u(0)) \in \tilde{\mathcal{V}}$, $j = 1, 2$. Since $e^{-cT/2} < 1$, it follows that G is a contraction map on $\tilde{\mathcal{V}}$.

Since the operator G takes a nonempty closed set $\tilde{\mathcal{V}}$ into itself and is a contraction map, it remains to apply the Banach fixed-point theorem. ☐

Remark 4.3. Estimate (4.10) (and, therefore, the contractivity of the mapping G) could also be proved by using the standard Fourier method for parabolic initial boundary-value problems. The constant c in (4.10) would then appear to be the first positive eigenvalue of the corresponding elliptic problem (slightly different equivalent norms for $W_2^1(Q)$ should be used).

Corollary 4.1. *Let the hypotheses of Theorem* 4.1 *be fulfilled. If* $(w, \tilde u)$ *is a strong* T*-periodic solution of problem* (2.1), (2.3), (2.4) *such that*

$$w_m(t) = \tilde w_m(t) \quad (t \in [0, T]),$$

then

$$\|\tilde w(\cdot, t) - w(\cdot, t)\|_{W_2^1(Q)} \to 0 \quad \textit{as} \quad t \to \infty. \tag{4.15}$$

Proof. Let $\tilde{\mathcal{V}}$ and G be the same as in the proof of Theorem 4.1. We have

$$(\tilde w(\cdot, kT), \tilde u(kT)) = G^k(\tilde w(\cdot, 0), \tilde u(0)) \in \tilde{\mathcal{V}} \quad (k = 0, 1, 2, \dots),$$

whereas

$$(w(\cdot, kT), \tilde u(kT)) = (w(\cdot, 0), \tilde u(0)) \in \tilde{\mathcal{V}}$$

is a fixed point of the operator G. Therefore,

$$\|\tilde w(\cdot, kT) - w(\cdot, kT)\|_{W_2^1(Q)} \to 0 \quad \text{as} \quad k \to \infty. \tag{4.16}$$

Now we can prove (4.15). Due to (4.16), for an arbitrary $\varepsilon > 0$, there is a number $k_\varepsilon \in \mathbb{N}$ such that

$$\|\tilde{w}(\cdot, kT) - w(\cdot, kT)\|_{W_2^1(Q)} \le \varepsilon/c_2 \quad \forall k \ge k_\varepsilon, \tag{4.17}$$

where c_2 is the constant from estimate (3.10). For any fixed $\tau \ge k_\varepsilon T$, we set $k = [\tau/T]$ $(k \ge k_\varepsilon)$, where $[\cdot]$ denotes the integer part of a number. Set

$$w_k(x,t) = w(x, t+kT), \quad \tilde{w}_k(x,t) = \tilde{w}(x, t+kT).$$

Clearly, the function $v_k = \tilde{w}_k - w_k$ is a strong solution of the problem

$$v_{kt}(x,t) = \Delta v_k(x,t) - p(x)v_k(w,t) \quad ((x,t) \in Q_T),$$
$$v_k(x,0) = \tilde{w}_k(x,0) - w_k(x,0) \quad (x \in Q),$$
$$-\gamma \frac{\partial v_k}{\partial \nu} = \sigma(x) v_k(x,t) \quad ((x,t) \in \Gamma_T).$$

Using the relation $0 \le \tau - kT < T$, Corollary 3.1, and inequality (4.17), we obtain

$$\begin{aligned}\|\tilde{w}(\cdot,\tau) - w(\cdot,\tau)\|_{W_2^1(Q)} &= \|v_k(\cdot, \tau - kT)\|_{W_2^1(Q)} \le c_2 \|\tilde{w}_k(\cdot,0) - w_k(\cdot,0)\|_{W_2^1(Q)} \\ &= c_2 \|\tilde{w}(\cdot,kT) - w(\cdot,kT)\|_{W_2^1(Q)} \le \varepsilon,\end{aligned}$$

which completes the proof. □

Corollary 4.2. *Let the hypotheses of Theorem* 4.1 *be fulfilled. Assume that* $\tilde{u}(t) \not\equiv$ const. *Then, for any* $\varphi_0 \in [w_1, w_2)$*, there is a strong* T*-periodic solution* (w_0, u) *of problem* (2.1), (2.3), (2.4) *such that*

$$w_{0m}(0) = \varphi_0.$$

Proof. I. Let $(w, \tilde{u})$ be the strong T-periodic solution of problem (2.1), (2.3), (2.4) constructed in Theorem 4.1. Let us show that there exists a moment τ such that $w_m(\tau) = \varphi_0$.

I.a. First, we assume that $\tilde{u}(0) \ne 1/a$. Clearly, there exists a switching time t_1 such that $w_m(t_1) = w_2$ because otherwise the function $\tilde{u}(t)$ is strictly monotone for $t > 0$ (see (3.14) for $t_* = 0$, $H = 1$, and $u_* = \tilde{u}(0) \ne 1/a$) and cannot be periodic.

Similarly, there exists the second switching time t_2 such that $w_m(t_2) = w_1$. Indeed, otherwise the function $\tilde{u}(t)$ is either constant or strictly monotone for $t > t_1$. Taking into account that $u(t)$ is not constant for $t \in (0, t_1)$, we see that in both cases $u(t)$ cannot be periodic for $t > 0$. Analogously, there is the third switching time t_3 such that $w_m(t_3) = w_2$.

I.b. Now we assume that $\tilde{u}(0) = 1/a$. Clearly, there exists a switching time t_1 such that $w_m(t_1) = w_2$ because otherwise $H(w_m)(t) \equiv 1$ and therefore $\tilde{u}(t) \equiv 1/a$ (see (3.14) for $t_* = 0$, $H = 1$, and $u_* = \tilde{u}(0) = 1/a$).

Further, there exists the second switching time t_2 such that $w_m(t_2) = w_1$. Indeed, otherwise the function $\tilde{u}(t)$ is strictly monotonically decreasing for $t > t_1$ (see (3.14) for $t_* = t_1$, $H = 0$, and $u_* = \tilde{u}(t_1) = 1/a$) and cannot be periodic. Similarly, there is the third switching time t_3 such that $w_m(t_3) = w_2$.

II. Since $w_m(t)$ is continuous by Lemma 3.2, it follows that, in both cases I.a and I.b, there exists a moment $\tau \in [t_2, t_3)$ such that $w_m(\tau) = \varphi_0$ and $H(w_m)(\tau) = 1$. Then $(w_0(x,t), u(x,t)) = (w(x, t+\tau), \tilde{u}(t+\tau))$ is the desired solution. □

5 Uniform Temperature Measurements

1. In this section, we consider a thermocontrol problem which possesses the mean-periodicity property. Hence, by Theorem 4.1, it also admits a strong periodic solution.

We consider problem (2.1)–(2.5) with $p(x) \equiv 0$, $\sigma(x) \equiv 0$, $\gamma = 1$, and $m(x) \equiv m_0 \ne 0$:

$$w_t(x,t) = \Delta w(x,t) \quad ((x,t) \in Q_T), \tag{5.1}$$
$$w(x,0) = \varphi(x) \quad (x \in Q), \tag{5.2}$$
$$\frac{\partial w}{\partial \nu} = K(x)(u(t) - u_c) \quad ((x,t) \in \Gamma_T), \tag{5.3}$$

the control function $u(t)$ satisfies the Cauchy problem

$$u'(t) + au(t) = H(w_m)(t) \quad (t > 0), \tag{5.4}$$
$$u(0) = u_0, \tag{5.5}$$

where $u_c, a > 0$, $H(w_m)(t)$ $(t \geq 0)$ is the above hysteresis operator, and the mean temperature is given by

$$w_m(t) = m_0 \int_Q w(x,t)\,dx.$$

To prove the mean-periodicity property, we will show that the mean temperature $w_m(t)$ satisfies an ordinary differential equation. Integrating Eq. (5.1) over Q, we have

$$w_m'(t) = m_0 \int_Q \Delta w\,dx = m_0 \int_\Gamma \frac{\partial w}{\partial \nu}\,d\Gamma.$$

Combining this equality with the boundary-value condition (5.3), we obtain the differential equation

$$w_m'(t) = k(u(t) - u_c) \quad (t > t_*), \tag{5.6}$$

where $k = m_0 \int_\Gamma K(x)\,dx$ and $t_* \geq 0$ is arbitrary. In what follows, we assume that $k > 0$.

The initial condition for the function $w_m(t)$ has the form

$$w_m(t_*) = \varphi_* = m_0 \int_Q w(x, t_*)\,dx. \tag{5.7}$$

For $t_* = 0$, we denote

$$\varphi_0 = \varphi_* = m_0 \int_Q \varphi(x)\,dx$$

and assume that (cf. (2.6))

$$w_1 \leq w_m(0) = \varphi_0 < w_2.$$

Clearly, the function $w_m(t)$ increases if $u(t) > u_c$ and decreases if $u(t) < u_c$. If $u(t) = u_c$ at some point t, then this point is critical for $w_m(t)$. Using (3.14), we can write the solution of problem (5.6), (5.7) in the form

$$w_m(t) = \frac{k}{a}\left(u_* - \frac{H}{a}\right)\left(1 - e^{-a(t-t_*)}\right) + k\left(\frac{H}{a} - u_c\right)(t - t_*) + \varphi_*, \tag{5.8}$$

where $u_* = u(t_*)$.

2. We consider the case where $0 < u_c < 1/a$. Assume that the initial control u_0 is such that (see Fig. 5.1)

$$u_c \leq u_0 \leq 1/a.$$

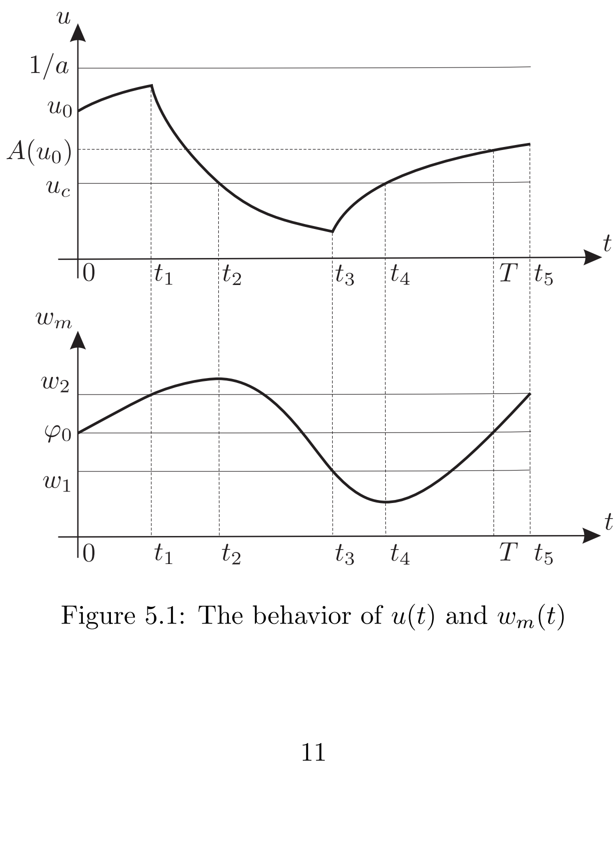

Figure 5.1: The behavior of $u(t)$ and $w_m(t)$

As the process starts, we have

$$u(t) = \left(u_0 - \frac{1}{a}\right) e^{-at} + \frac{1}{a}, \qquad t > 0, \tag{5.9}$$

$$w_m(t) = \frac{k}{a}\left(u_0 - \frac{1}{a}\right)\left(1 - e^{-at}\right) + k\left(\frac{1}{a} - u_c\right) t + \varphi_0, \qquad t > 0$$

(see (3.14) and (5.8)), i.e., $u(t) \equiv 1/a$ (if $u_0 = 1/a$) or $u(t)$ increases (if $u_0 < 1/a$). In both cases, $u(t) - u_c > 0$ for $t > 0$; hence, $w_m(t)$ increases due to (5.6). Since $\lim_{t\to\infty} w(t) = +\infty$, it follows that there is a moment $t_1 > 0$ such that $w_m(t_1) = w_2$. The operator H "switches" at this moment, i.e., $H = 0$ for $t > t_1$.

Set

$$u_* = u(t_1), \qquad \varphi_* = w_m(t_1) = w_2.$$

Thus, using (3.14) and (5.8), we obtain

$$u(t) = u_* e^{-a(t-t_1)}, \qquad t > t_1,$$

$$w_m(t) = \frac{k}{a} u_* \left(1 - e^{-a(t-t_1)}\right) - k u_c (t - t_1) + w_2, \qquad t > t_1.$$

Since $u(t)$ decreases, $\lim_{t\to\infty} u(t) = 0$, and $u(t_1) = u_* > u_c$, it follows that there is a moment $t_2 > t_1$ such that $u(t_2) = u_c$. Therefore, using (5.6), we see that $w_m(t)$ continues increasing for $t_1 < t < t_2$ and $w_m(t)$ decreases for $t > t_2$.

Since $\lim_{t\to\infty} w_m(t) = -\infty$, it follows that there is a moment $t_3 > t_2$ such that $w_m(t_3) = w_1$. The operator H "switches" at this moment, i.e., $H = 1$ for $t > t_3$. Moreover, $u(t_3) < u_c$. Set

$$t_* = t_3, \quad u_* = u(t_*) < u_c < 1/a, \quad \varphi_* = w_m(t_*) = w_1.$$

Using (3.14) and (5.8), we obtain

$$u(t) = \left(u_* - \frac{1}{a}\right) e^{-a(t-t_3)} + \frac{1}{a}, \qquad t > t_3,$$

$$w_m(t) = \frac{k}{a}\left(u_* - \frac{1}{a}\right)\left(1 - e^{-a(t-t_3)}\right) + k\left(\frac{1}{a} - u_c\right)(t - t_3) + w_1, \qquad t > t_3.$$

Since $u(t)$ increases, $\lim_{t\to\infty} u(t) = 1/a > u_c$, and $u(t_3) = u_* < u_c$, it follows that there is a moment $t_4 > t_3$ such that $u(t_4) = u_c$. Therefore, using (5.6), we see that $w_m(t)$ continues decreasing for $t_3 < t < t_4$ and $w_m(t)$ increases for $t > t_4$.

Since $\lim_{t\to\infty} w_m(t) = +\infty$, it follows that there is a moment $t_5 > t_4$ such that $w_m(t_5) = w_2$. The operator H "switches" at this moment, i.e., $H = 0$ for $t > t_5$. Moreover, $u_c < u(t_5) < 1/a$. Thus, we have got the situation which occurred at the moment t_1. Therefore, we can continue the process, repeating the above steps, and obtain the points $t_6, \dots, t_{10}, t_{11}, \dots, t_{15}$, and so on.

It follows from Theorem 3.1 that the differences $t_{j+2} - t_j$ $(j = 1, 3, 5)$ are uniformly bounded from below. Using the explicit formulas (3.14) and (5.8) for $w_m(t)$ and $u(t)$, one can show that these differences are also uniformly bounded from above and

$$\frac{1}{ku_c}(w_2 - w_1) \le t_{j+2} - t_j \le \frac{1}{ku_c}\left(w_2 - w_1 + \frac{k}{a^2}\right), \quad j = 1, 5, 9, \dots,$$

$$\frac{1}{\frac{1}{u_c a} - 1}\frac{1}{ku_c}(w_2 - w_1) \le t_{j+2} - t_j \le \frac{1}{\frac{1}{u_c a} - 1}\frac{1}{ku_c}\left(w_2 - w_1 + \frac{k}{a^2}\right), \quad j = 3, 7, 11, \dots.$$

This fact together with (5.6) implies that $w_m \in C^1([0, +\infty))$ and w_m is infinitely differentiable on the intervals $[0, t_1], [t_1, t_3], [t_3, t_5], \dots$.

3. Now we prove that one can choose an initial control $\hat{u}_0$ (depending on the the initial mean temperature φ_0) in such a way that the functions $u(t)$ and $w_m(t)$ are periodic, provided that $w_1 \le \varphi_0 < w_2$.

Theorem 5.1. *Let $m_0 \int_\Gamma K(x)\, d\Gamma > 0$, $0 < u_c < 1/a$, and $w_1 \le \varphi_0 < w_2$. Then there exists a unique initial control $\hat{u}_0$ on the interval $[u_c, 1/a]$ such that the solutions $u(t)$ and $w_m(t)$ of problems (5.4), (5.5) and (5.6), (5.7) with $t_* = 0$, respectively, are both periodic with the same period. This initial control $\hat{u}_0$ satisfies the inequalities $u_c < \hat{u}_0 < 1/a$. The continuously differentiable function $w_m(t)$ is a stable cycle on the phase plane (w_m, w_m').*

Proof. I. Consider an arbitrary initial control $u_0 \in [u_c, 1/a]$. As the process starts ($t = 0$), we have $u_c \le u(0) \le 1/a$, $w_1 \le w_m(0) < w_2$, and $H = 1$ (by assumption).

By construction, there is a moment $T \in (t_4, t_5)$ such that $u_c < u(T) < 1/a$, $w_m(T) = \varphi_0$, and $H(w_m)(T) = 1$ (see Fig. 5.1). If $u(T) = u(0) = u_0$, then the functions $u(t)$ and $w_m(t)$ are periodic with period T.

We introduce the function $A : [u_c, 1/a] \to (u_c, 1/a)$ which maps any initial control $u_0 \in [u_c, 1/a]$ to the point $u(T)$, where $T = T(u_0, \varphi_0)$ is the above moment.

Let us prove that the function A is infinitely differentiable on $[u_c, 1/a]$. Fix an arbitrary initial control $u_0 \in [u_c, 1/a]$ and consider the moments $t_1, \dots, t_4, T$ constructed above.

To find the value $A(u_0) = u(T)$, we make the following three steps.

II.a. We write the equation

$$w_m(t_1, u_0) = w_2$$

and find the moment $t_1 = t_1(u_0)$. Since $w_m(t_1, u_0)$ is infinitely differentiable with respect to t_1 and u_0 and

$$\frac{\partial w_m}{\partial t_1} = k(u(t_1) - u_c) > 0,$$

it follows that the function $t_1 = t_1(u_0)$ is also infinitely differentiable with respect to u_0. Moreover, if $u_0 \ne 1/a$, then, using Eq. (5.6) and relation (5.9), we have

$$\frac{dt_1}{du_0} = -\frac{\partial w_m/\partial u_0}{\partial w_m/\partial t_1} = -\frac{ka^{-1}(1 - e^{-at_1})}{k(u_1 - u_c)} = -\frac{a^{-1}\frac{u_1 - u_0}{a^{-1} - u_0}}{u_1 - u_c}, \tag{5.10}$$

where $u_1 = u(t_1)$.

Substituting the infinitely differentiable function $t_1 = t_1(u_0)$ for the variable t in the function $u(t) = u(t, u_0)$ (see (5.9)), we see that u_1 is a function depending on t_1 and u_0, i.e. $u_1 = u_1(t_1, u_0)$. Since the function u is infinitely differentiable with respect to t and $u_0 \in [u_c, 1/a]$, whereas the function $t_1(u_0)$ is infinitely differentiable with respect to $u_0 \in [u_c, 1/a]$, it follows that u_1 is infinitely differentiable with respect to $u_0 \in [u_c, 1/a]$. Moreover, if $u_0 \ne 1/a$, then, using equality (5.10) and relation (5.9), we have

$$\frac{du_1}{du_0} = \frac{\partial u_1}{\partial t_1}\frac{dt_1}{du_0} + \frac{\partial u_1}{\partial u_0} = -(1 - au_1)\frac{a^{-1}\frac{u_1 - u_0}{a^{-1} - u_0}}{u_1 - u_c} + \frac{u_1 - a^{-1}}{u_0 - a^{-1}} = \frac{a^{-1} - u_1}{a^{-1} - u_0}\frac{u_0 - u_c}{u_1 - u_c}. \tag{5.11}$$

II.b. Now we consider the process on the interval $[t_1, t_3]$ (when $H = 0$). We can assume that it starts anew, i.e., the initial control is u_1, the initial mean temperature is w_2, and $H = 0$. Similarly to step II.a, we can show that the moment t_3 is an infinitely differentiable function of u_1 (i.e., $t_3 = t_3(u_1)$) and the control value u_3 at the moment t_3 is an infinitely differentiable function of t_3 and u_1 (i.e., $u_3 = u_3(t_3, u_1)$). Therefore, u_3 (as a function of u_1) is infinitely differentiable with respect to u_1. Similarly to (5.11), we obtain

$$\frac{du_3}{du_1} = \frac{\partial u_3}{\partial t_3}\frac{dt_3}{du_1} + \frac{\partial u_3}{\partial u_1} = \frac{u_3}{u_1}\frac{u_1 - u_c}{u_3 - u_c}. \tag{5.12}$$

II.c. Finally, we consider the process on the interval $[t_3, T]$ (when $H = 1$ again). We assume that the process starts anew, i.e., the initial control is u_3, the initial mean temperature is w_1, and $H = 1$. As in step II.a, it follows from the equality

$$w_m(T, u_3) = \varphi_0$$

that the function $T = T(u_3)$ is infinitely differentiable with respect to u_3. Since $u_3 < 1/a$, we obtain

$$\frac{dT}{du_3} = -\frac{a^{-1}\frac{u - u_3}{a^{-1} - u_3}}{u - u_c},$$

where $u = u(T, u_3)$ (it suffices to replace u_0 and u_1 by u and u_3, respectively, in (5.10)). Therefore, as in step II.a, we see that the function $u(T(u_3), u_3)$ is infinitely differentiable with respect to u_3 and

$$\frac{du}{du_3} = \frac{\partial u}{\partial T}\frac{dT}{du_3} + \frac{\partial u}{\partial u_3} = \frac{a^{-1} - u}{a^{-1} - u_3}\frac{u_3 - u_c}{u - u_c}. \tag{5.13}$$

III. Steps II.a–II.c show that u is an infinitely differentiable function of variable $u_0 \in [u_c, 1/a]$. It follows from (5.10)–(5.13) that if $u_0 < 1/a$, then

$$\frac{dA(u_0)}{du_0} = \frac{du}{du_0} = \frac{du}{du_3}\frac{du_3}{du_1}\frac{du_1}{du_0} = \frac{a^{-1} - u_1}{a^{-1} - u_0}\frac{u_3}{u_1}\frac{a^{-1} - u}{a^{-1} - u_3}\frac{u_0 - u_c}{u - u_c}. \tag{5.14}$$

Therefore,

$$\begin{aligned} \frac{dA}{du_0} &= 0 \qquad \text{for } u_0 = u_c, \\ \frac{dA}{du_0} &> 0 \qquad \text{for } u_c < u_0 < 1/a. \end{aligned} \tag{5.15}$$

Thus, $A(u_0)$ is an infinitely differentiable function on the interval $[u_c, 1/a]$, $A(u_0)$ increases on $(u_c, 1/a)$, and $A(u_0)$ maps $[u_c, 1/a]$ to $(u_c, 1/a)$. Therefore, the graph of the function $A(u_0)$ intersects the diagonal with the end points (u_c, u_c) and $(1/a, 1/a)$ of the square $[u_c, 1/a] \times [u_c, 1/a]$ at least at one point $(\hat{u}_0, \hat{u}_0) = (\hat{u}_0, A(\hat{u}_0))$.

Clearly, the solution $u(t), w_m(t)$ with the initial values $\hat{u}_0, \varphi_0$, respectively, is periodic with period T. Moreover, $u(T) = u(0) = \hat{u}_0$ in this case, and relation (5.14) implies that

$$\left.\frac{dA}{du_0}\right|_{u_0=\hat{u}_0} = \frac{a^{-1} - u_1}{a^{-1} - u_3}\,\frac{u_3}{u_1} < 1. \tag{5.16}$$

Therefore, the graph of the function $A(u_0)$ intersects the diagonal only at one point, i.e., there exists a unique value $\hat{u}_0$ on the interval $[u_c, 1/a]$ generating a periodic solution $u(t), w_m(t)$; moreover, $u_c < u_0 < 1/a$.

It follows from (5.15) and (5.16) that the sequence

$$u_0, A(u_0), A(A(u_0)), \ldots,$$

where $u_0 < \hat{u}_0$, increases and tends to $\hat{u}_0$, while the sequence

$$u_0, A(u_0), A(A(u_0)), \ldots,$$

where $u_0 > \hat{u}_0$, decreases and also tends to $\hat{u}_0$. This means that any trajectory

$$(w_m(t), w_m'(t)) = (w_m(t), k(u(t) - u_c))$$

tends to the limit-cycle trajectory defined by the initial values $\hat{u}_0, \varphi_0$ (see Fig. 5.2). □

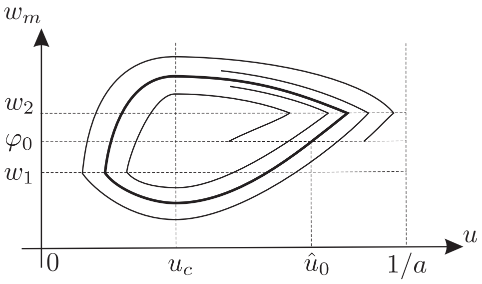

Figure 5.2: A stable-cycle trajectory $(w_m(t), u(t))$

Theorem 5.2. *1. Let $m_0 \int_\Gamma K(x)\,d\Gamma > 0$, $u_c < 1/a$, and $w_1 \le \varphi_0 < w_2$. Then there exists a unique strong periodic solution (w, u) of problem* (5.1), (5.3), (5.4) *such that*

$$u(0) \in [u_c, 1/a], \qquad w_m(0) = \varphi_0.$$

Moreover, the initial control $u(0)$ satisfies the inequalities $u_c < u(0) < 1/a$.

2. If $(\tilde{w}, \tilde{u})$ is a mean-periodic solution of problem (5.1), (5.3), (5.4) *such that*

$$\tilde{u}(0) \in [u_c, 1/a], \qquad \tilde{w}_m(0) = \varphi_0,$$

then

$$\tilde{u}(t) \equiv u(t), \qquad \tilde{w}_m(t) \equiv w_m(t), \qquad \|\tilde{w}(\cdot, t) - w(\cdot, t)\|_{W_2^1(Q)} \to 0 \quad \text{as} \quad t \to \infty,$$

where (w, u) is the strong periodic solution from assertion 1.

Proof. I. The existence of a strong periodic solution (w, u) of problem (5.1), (5.3), (5.4) and the inequalities $u_c < u(0) < 1/a$ result from Theorems 5.1 and 4.1. Before we show the uniqueness of a strong periodic solution (see part III below), let us prove assertion 2.

II. Let $(\tilde{w}, \tilde{u})$ be a mean-periodic solution of problem (5.1), (5.3), (5.4) such that

$$\tilde{u}(0) \in [u_c, 1/a], \qquad \tilde{w}_m(0) = \varphi_0.$$

Then $\tilde{u}(0) = u(0)$ by Theorem 5.1. It follows from (3.14) and (5.8) that the mean temperature $w_m(t)$ and the control function $u(t)$ depend only on the values φ_0 and u_0. Therefore,

$$(\tilde{w}_m(t), \tilde{u}(t)) \equiv (w_m(t), u(t)). \tag{5.17}$$

It follows from relation (5.17) and Corollary 4.1 that

$$\|\tilde{w}(\cdot, t) - w(\cdot, t)\|_{W_2^1(Q)} \to 0 \quad \text{as} \quad t \to \infty.$$

Assertion 2 is proved.

III. To complete the proof of assertion 1, it remains to show that if $(\hat{w}, \hat{u})$ is a strong periodic solution of problem (5.1), (5.3), (5.4) such that

$$\hat{u}(0) \in [u_c, 1/a], \qquad \hat{w}_m(0) = \varphi_0,$$

then $(\hat{w}, \hat{u}) = (w, u)$.

Clearly, $(\hat{w}, \hat{u})$ is a mean-periodic solution of problem (5.1), (5.3), (5.4). Therefore, due to part II of the proof (cf. (5.17)), we have

$$(\hat{w}(\cdot, 0), \hat{u}(0)) \in \tilde{\mathcal{V}}, \qquad (w(\cdot, 0), u(0)) \in \tilde{\mathcal{V}},$$

where $\tilde{\mathcal{V}}$ is the set from the proof of Theorem 4.1. Moreover, both pairs

$$(\hat{w}(\cdot, 0), \hat{u}(0)) \in \tilde{\mathcal{V}}, \quad (w(\cdot, 0), u(0)) \in \tilde{\mathcal{V}}$$

are fixed points of the operator G from the proof of Theorem 4.1. Hence,

$$(\hat{w}(\cdot, 0), \hat{u}(0)) = (w(\cdot, 0), u(0)).$$

Now Theorem 3.1 implies that $(\hat{w}, \hat{u}) = (w, u)$. □

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