RATTLING IN SPATIALLY DISCRETE DIFFUSION EQUATIONS WITH HYSTERESIS*

PAVEL GUREVICH† AND SERGEY TIKHOMIROV‡

Abstract. The paper treats a reaction-diffusion equation with hysteretic nonlinearity on a one-dimensional lattice. It arises as a result of the spatial discretization of the corresponding continuous model with so-called nontransverse initial data and exhibits a propagating microstructure—which we call *rattling*—in the hysteretic component of the solution. We analyze this microstructure and determine the speed of its propagation depending on the parameters of hysteresis and the nontransversality coefficient in the initial data.

Key words. spatially discrete parabolic equations, reaction-diffusion equations, lattice, hysteresis, pattern, rattling

AMS subject classifications. 34K31, 47J40, 35B36, 37L60

DOI. 10.1137/16M106039X

1. Introduction. In [13, 14], Hoppensteadt and Jäger suggested a model for the growth of a colony of bacteria in a Petri plate. Bacteria fixed to an agar gel in a Petri plate grow according to a hysteresis law. The growth rate depends on the relative levels of the two diffusing substances: nutrient and by-product. A distinctive pattern of concentric rings forms after a fixed amount of nutrient is added to the center of the plate; see Figure 1.

Mathematically, the Hoppensteadt–Jäger model is a system of reaction-diffusion equations coupled with an ordinary differential equation, and hysteresis enters the reaction terms. Numerical simulations carried out in [13, 14] were in accordance with the experiment. In a more general context, such systems may describe various hysteretic interactions between different diffusive and nondiffusive substances; see, e.g., [17] for an application in developmental biology.

Rigorous analysis of such systems appears to be very nontrivial because the hysteresis may switch at different spatial points at different time moments. Since the 1980s, a series of results have been obtained about the existence of solutions; see [18, 2, 19, 16, 1, 15, 20]. However, questions about the uniqueness of solutions and their continuous dependence on initial data as well as a thorough analysis of pattern formation remained open. For the purposes of this paper, we will deal with a scalar reaction-diffusion equation with hysteresis on a one-dimensional domain, which can serve a prototype for a more general system in [13, 14]. Thus, we consider the following problem:

---

†Free University of Berlin, Institute of Mathematics I, Arnimalle 3, 14195 Berlin, Germany, and Peoples' Friendship University of Russia, Miklukho-Maklaya str. 6, Moscow 117198, Russia (gurevich@math.fu-berlin.de).

‡St. Petersburg State University, 7/9 Universitetskaya nab., St. Petersburg 199034, Russia, and Max Planck Institute for Mathematics in the Sciences, Inselstraße 22, 04103 Leipzig, Germany (s.tikhomirov@spbu.ru).
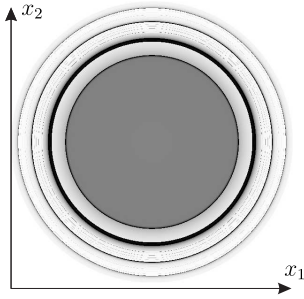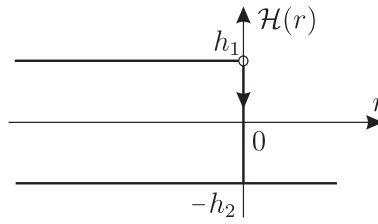
FIG. 1. *Bacteria density at end of experiment.*

FIG. 2. *Hysteresis operator of relay type with threshold 0.*

$$(1.1) \qquad v_\tau = v_{xx} + \mathcal{H}(v), \quad \tau > 0, \ x \in (-1,1),$$

$$(1.2) \qquad v|_{\tau=0} = \varphi(x), \quad x \in (-1,1),$$

where $v = v(x,\tau)$ and $\mathcal{H}(\cdot)$ is the hysteresis operator of relay type; see Figure 2. To define it, we fix $h_1 > 0$ and $h_2 \geq 0$ and set for any function $r \in C[0,\infty)$

$$(1.3) \qquad \mathcal{H}(r)(\tau) = \begin{cases} h_1 & \text{if } r(s) < 0 \text{ for all } s \in [0,\tau], \\ -h_2 & \text{if } r(s) \geq 0 \text{ for some } s \in [0,\tau]. \end{cases}$$

Note that the hysteresis operator is *rate independent*, i.e., for any continuous increasing $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$, we have

$$(1.4) \qquad \mathcal{H}(r(g(\cdot)))(\tau) = \mathcal{H}(r)(g(\tau)), \quad \tau \geq 0.$$

We also note that the standard relay-type hysteresis possesses two thresholds, while the operator given by (1.3) possesses only one threshold, namely, zero, and the second one can be thought to be set to $-\infty$. This technical modification is motivated by the fact that, in this paper, we will be interested in a behavior of solutions near one of the thresholds only.

For a function $v(x,\tau)$ that is continuous with respect to τ for each fixed x, we set $\mathcal{H}(v)(x,\tau) := \mathcal{H}(v(x,\cdot))(\tau)$ for each fixed x. For a function $\varphi(x)$, we also set $\mathcal{H}(\varphi)(x) := h_1$ if $\varphi(x) < 0$ and $\mathcal{H}(\varphi)(x) := -h_2$ if $\varphi(x) \geq 0$.

It turns out that the dynamics of v and $\mathcal{H}(v)$ essentially depends on the so-called (spatial) transversality condition. We say that a function $\varphi(x)$ is *transverse with respect to $\mathcal{H}(\varphi)$* provided the following holds: if $\varphi(x_0) = \varphi'(x_0) = 0$ for some $x_0 \in [-1,1]$, then $\mathcal{H}(\varphi)(x_0) = -h_2$ in a neighborhood of x_0. We say that $v(x,\tau)$ is *transverse with respect to $\mathcal{H}(v)$ on a time interval I* if $v(\cdot,\tau)$ is transverse with respect to $\mathcal{H}(v)(\cdot,\tau)$ for each $\tau \in I$. It was proved in [9, 8] that, for any transverse $\varphi(x)$, the solution $v(x,\tau)$ of problem (1.1), (1.2) exists, is transverse, is unique, and continuously depends on φ on a time interval $(0,T)$. Moreover, the solution can be uniquely extended to its maximal interval of transverse existence $(0, T_{\max})$, where either $T_{\max} = \infty$ or $T_{\max}$ is finite and the solution becomes nontransverse for $t = T_{\max}$. In [10], these results were generalized to systems of reaction-diffusion equations (including the one suggested by Hoppensteadt and Jäger). The notion of spatial transversality was generalized to the multidimensional case in [5]. Regularity of solutions in multi-dimensional domains was considered in [3, 4], where no transversality assumptions were imposed, but only the existence of solutions was required. For a more detailed survey, we refer the reader to [6].
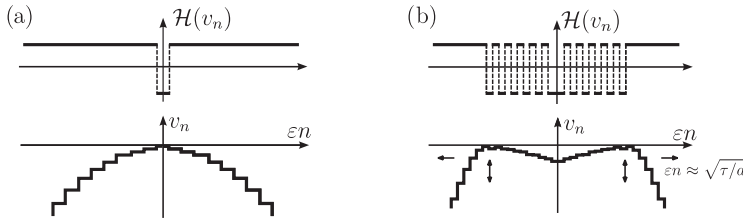
FIG. 3. *The upper graphs represent spatial profiles of the hysteresis $\mathcal{H}(v_n)$ and the lower graphs the spatial profiles of the solution v_n. (a) Nontransverse initial data. (b) Spatial profiles at a moment $\tau > 0$ for $h_1 = h_2 > 0$.*

Although generically the initial data are transverse, it may become nontransverse in a finite time. In particular, this is the case for the Hoppensteadt–Jäger model, where numerics indicates that all the concentric rings in Figure 1 get formed via a nontransversality. To understand problem (1.1), (1.2) in the nontransverse case, we fix some $c > 0$ and take $\varphi(x) = -cx^2 + o(x^2)$ in a neighborhood $\mathcal{B}(0)$ of $x = 0$. By choosing a smaller neighborhood if needed, we have $\varphi(x) < 0$ for $x \in \mathcal{B}(0) \setminus \{0\}$. Now for any $\varepsilon > 0$, setting[1] $v_n(\tau; \varepsilon) := v(\varepsilon n, \tau)$, we replace the continuous model (1.1), (1.2) in $\mathcal{B}(0)$ by the discrete one

$$\begin{cases} \dfrac{dv_n}{d\tau} = \dfrac{\Delta v_n}{\varepsilon^2} + \mathcal{H}(v_n), & \tau > 0, \ n = -N_\varepsilon, \ldots, N_\varepsilon, \\ v_n(0; \varepsilon) = -c(\varepsilon n)^2 + o(\varepsilon^2 n^2), & n = -N_\varepsilon, \ldots, N_\varepsilon, \end{cases}$$

where $\Delta v_n := v_{n-1} - 2v_n + v_{n+1}$ and $N_\varepsilon \to \infty$ as $\varepsilon \to 0$. Since we are interested in small ε and in the behavior in a small neighborhood $\mathcal{B}(0)$, we consider the next approximation by omitting $o(\varepsilon^2 n^2)$ in the initial data and replacing N_ε by ∞. This yields

(1.5)
$$\begin{cases} \dfrac{dv_n}{d\tau} = \dfrac{\Delta v_n}{\varepsilon^2} + \mathcal{H}(v_n), & \tau > 0, \ n \in \mathbb{Z}, \\ v_n(0; \varepsilon) = -c(\varepsilon n)^2, & n \in \mathbb{Z}. \end{cases}$$

By our definition of the relay $\mathcal{H}$, we have $\mathcal{H}(v_0)(\tau) = -h_2$ for all $\tau \geq 0$, while, for $n \neq 0$, $\mathcal{H}(v_n)(\tau) = h_1$ unless the input $v_n(\tau)$ achieves the zero threshold; if this happens at a finite time moment, the relay *switches* at this moment and since then the relay output $\mathcal{H}(v_n)(\tau)$ equals $-h_2$.

A nontrivial dynamics occurs in the case $h_1 > 2c > 0 \geq -h_2$. To indicate the difficulty, note that

$$\frac{dv_0}{d\tau}(0; \varepsilon) = -h_2 - 2c < 0, \qquad \frac{dv_n}{d\tau}(0; \varepsilon) = h_1 - 2c > 0 \text{ for } n \neq 0.$$

Thus, for small $\tau > 0$, the *node* $v_0(\tau; \varepsilon)$ decreases, while all the other *nodes* $v_n(\tau; \varepsilon)$, $n \neq 0$, increase. Now it is not clear which node achieves the threshold 0 and switches first and hence what a further dynamics is. On the other hand, numerics provides quite a specific behavior of the solution, illustrated by Figure 3. The profile given by v_n forms two humps symmetric to each other with respect to the origin. The humps propagate away from the origin, while the whole profile moves downwards

---
[1]Note that, in this notation, ε is a parameter and n plays the role of a spatial variable

and upwards (never going beyond the threshold 0). Whenever the profile touches the threshold 0, the corresponding node switches. If we denote by $p(n)$ and $q(n)$ the number of nodes in the set $\{v_1, \ldots, v_n\}$ that switch and do not switch, respectively, on the time interval $[0, \infty)$, then numerics suggests that

$$(1.6) \qquad \lim_{n \to \infty} \frac{q(n)}{p(n)} = \frac{h_2}{h_1}.$$

Moreover,

(1.7)
  if $h_2/h_1 = p_2/p_1$, where p_1 and p_2 are coprime integers, then
  for any j large enough, the set $\{v_{j+1}, \ldots, v_{j+p_1+p_2}\}$ contains exactly p_1 nodes
  that switch and p_2 nodes that do not switch on the time interval $[0, \infty)$.

The switching pattern satisfying (1.6) or, more specifically, (1.7) is called a *rattling pattern* or simply *rattling*.

We emphasize that the properties (1.6) and (1.7) do not depend on ε. In fact, in section 2, we will show that ε in (1.5) can be scaled out; see (2.2). In particular, all the numerical observations concerning the dynamics of v_n have been done for $\varepsilon = 1$ and then transferred to an arbitrary ε according to the scaling in (2.2). Therefore, the rattling describes a microstructure visible on the spatial scale of order ε; see the upper graph in Figure 3(b). A similar effect is observable on different d-dimensional lattices, $d \geq 2$, and in slow-fast systems. We comment on these issues and discuss a continuous limit as $\varepsilon \to 0$ in [11, section 1] and [6]. In connection with passing to the limit, let us also mention [12], where a hysteretic behavior of forward-backward diffusion equations is analyzed via the spatial discretization.

The main goal of this paper is to analyze the property (1.6) and to determine the asymptotic speed of propagation of the abovementioned humps formed by the solution profile; see Figure 3(b). Let $\tau_n = \tau_n(\varepsilon)$ be the switching moment of the node $v_n(\tau; \varepsilon)$ if this node switches on the time interval $[0, \infty)$. Then numerics shows that, for any fixed $h_2 \geq 0$, the τ_n's that correspond to the switching nodes satisfy, as $n \to \infty$,

$$(1.8) \qquad \tau_n = a(\varepsilon n)^2 + \begin{cases} \varepsilon^2 O(\sqrt{n}) & \text{if } h_2 = 0, \\ \varepsilon^2 O(n) & \text{if } h_2 > 0, \end{cases}$$

where $a > 0$ and $O(\cdot)$ do not depend on ε.

The case $h_2 = 0$ was studied in detail in [11], where it was proved that all the nodes switch (which is certainly consistent with (1.6)) and the constant a from (1.8) was found. The case $h_2 > 0$ appears much more complicated, especially if h_2/h_1 is irrational. In this paper, using asymptotic methods developed in [11], we find sufficient conditions that guarantee (1.6). More specifically, for any $h_2 > 0$, we will prove that if the switching nodes are *quasi-uniform* (Condition 3.1) and the switching moments grow asymptotically quadratically as in (1.8) (Condition 3.2), then the property (1.6) takes place and the constant a coincides with that found in [11] for $h_2 = 0$. In particular, we will show that a depends on h_1/c but *does not depend* on h_2 or ε. This is the content of Theorem 3.3. We believe that its proof contains essential ingredients that will allow us to prove the existence of the proper switching pattern in the spirit of [11], i.e., without a priori assumptions about infinitely many switchings. This is a subject of future work.

From the viewpoint of applications, Theorem 3.3 may serve as a parameter identification tool. For instance, if one knows (or prepares ad hoc) the tangency constant

c and, by observing the emerging pattern, finds the constant a in (1.8) and the frequency with which the switching nodes enter the pattern, then one can determine the parameters h_1 and h_2 of the hysteresis, using Theorem 3.3.

The paper is organized as follows. In section 2, we reduce problem (1.5) to the problem with $\varepsilon = 1$ by rescaling time $t := \varepsilon^{-2}\tau$ and the solution itself $u_n(t) := \varepsilon^{-2}v_n(\tau;\varepsilon)$. Then the switching moments τ_n (see (1.8)) of $v_n(\tau;\varepsilon)$ get transformed to the switching moments

$$(1.9) \qquad t_n = an^2 + \begin{cases} O(\sqrt{n}) & \text{if } h_2 = 0, \\ O(n) & \text{if } h_2 > 0 \end{cases}$$

of $u_n(t)$. Next, we prove the existence and uniqueness of the solution $u_n(t)$ for the rescaled problem (which does not depend on ε anymore). In section 3, we specify what *quasi-uniform switchings* mean (Condition 3.1) and formulate the main theorem (Theorem 3.3) concerning the property (1.6) and the constant a in (1.8) and (1.9). In section 4, we discuss the notion of quasi-uniform switchings in more detail and give examples of periodic patterns that are observed numerically for rational h_2/h_1, and their generalizations, which we call quasi-periodic. In both cases, we show that the switching nodes are quasi-uniform (i.e., Condition 3.1 holds and our main theorem applies). The rest of the paper is devoted to the proof of our main theorem. In section 5, we prove preliminary estimates for u_n, $\dot{u}_n$, $\Delta u_n := u_{n-1} - 2u_n + u_{n+1}$, and $\nabla u_n := u_{n+1} - u_n$. In particular, we show the boundedness of $|\nabla u_n(t_n)|$ uniformly with respect to $n \in \mathbb{Z}$. Sections 6 and 7 are central in the proof of our main result. In section 6, we prove that a limit $p_* := \lim_{n\to\infty} p(n)/n$ exists and obtain an asymptotic expansion of $u_n(t_n)$ as $n \to \infty$. This expansion is based on the series representation of the solution $u_n(t)$ via the discrete Green's function $\mathcal{Y}_n(t)$, the asymptotic properties of the latter, and a proximity of emerging series to certain integrals. In section 7, we obtain an asymptotic expansion of $\nabla u_n(t_n)$ in a similar way. Since $u_n(t_n) = 0$ and $|\nabla u_n(t_n)|$ is bounded, the leading order terms in the asymptotic expansions of $u_n(t_n)$ and $\nabla u_n(t_n)$ must vanish. This yields two equalities for a and p_*. Finally, in section 8, we show that these equalities imply that $p_* = h_1/(h_1 + h_2)$ (which is equivalent to (1.6)) and provide the unique possibility for choosing a (which appears to be independent of h_2). This concludes the proof of the main theorem. The paper also contains Appendix A, where discrete Green's function $\mathcal{Y}_n(t)$ is defined and some facts concerning its asymptotics are collected.

2. Existence and uniqueness of solutions. For a sequence $\{w_n\}_{n\in\mathbb{Z}}$ of real numbers, we will use throughout the notation

$$(2.1) \qquad \nabla w_n := w_{n+1} - w_n, \qquad \Delta w_n := \nabla w_n - \nabla w_{n-1} = w_{n-1} - 2w_n + w_{n+1}.$$

We begin with scaling out ε in problem (1.5). Setting

$$(2.2) \qquad t := \varepsilon^{-2}\tau, \quad u_n(t) := \varepsilon^{-2}v_n(\tau;\varepsilon),$$

and using the rateindependence of the hysteresis (see (1.4)), we obtain

$$\mathcal{H}(v_n)(\tau) = \mathcal{H}(\varepsilon^2 u_n(\varepsilon^{-2}\cdot))(\tau) = \mathcal{H}(u_n(\varepsilon^{-2}\cdot))(\tau) = \mathcal{H}(u_n)(\varepsilon^{-2}\tau) = \mathcal{H}(u_n)(t).$$

Hence, problem (1.5) can be rewritten as follows:

$$(2.3) \qquad \dot{u}_n = \Delta u_n + \mathcal{H}(u_n), \quad t > 0, \ n \in \mathbb{Z},$$

$$(2.4) \qquad u_n(0) = -cn^2, \quad n \in \mathbb{Z},$$

where $\dot{} = d/dt$. Problem (2.3), (2.4) does not involve ε, which justifies the fact that $u_n(t)$ in (2.2) does not depend on ε.

We assume that the following condition holds.

CONDITION 2.1. $-h_2 \leq 0 < 2c < h_1$.

We note that the function $\mathcal{H}(r)(t)$ may have discontinuities (actually, at most one) even if $r \in C^\infty[0, \infty)$. Therefore, one cannot expect that a solution of problem (2.3), (2.4) is continuously differentiable on $[0, \infty)$. Thus, we define a solution as follows.

DEFINITION 2.2. *We say that a sequence* $\{u_n(t)\}_{n \in \mathbb{Z}}$ *is a* solution *of problem* (2.3), (2.4) *on the time interval* $(0, T)$, $T > 0$, *if*
1. $u_n \in C[0, T]$ *for all* $n \in \mathbb{Z}$,
2. *for each* $t \in [0, T]$, *there are constants* $A, \alpha \geq 0$ *such that* $\sup_{s \in [0,t]} |u_n(s)| \leq Ae^{\alpha|n|}$ *for all* $n \in \mathbb{Z}$,
3. *there is a finite sequence* $0 = \theta_0 < \theta_1 < \cdots < \theta_J = T$, $J \geq 1$, *such that* $u_n \in C^1[\theta_j, \theta_{j+1}]$ *for all* $n \in \mathbb{Z}$ *and* $j = 0, \ldots, J-1$,
4. *the equations in* (2.3) *hold in* (θ_j, θ_{j+1}) *for all* $n \in \mathbb{Z}$ *and* $j = 0, \ldots, J-1$,
5. $u_n(0) = -cn^2$ *for all* $n \in \mathbb{Z}$.

We say that a sequence $\{u_n(t)\}_{n \in \mathbb{Z}}$ *is a* solution *of problem* (2.3), (2.4) *on the time interval* $(0, \infty)$ *if it is a solution on* $(0, T)$ *for all* $T > 0$.

Remark 2.3. If $\{u_n(t)\}_{n \in \mathbb{Z}}$ is a solution, then, as we have mentioned above, the function $\mathcal{H}(u_n)(t)$ has at most one discontinuity point for each fixed $n \in \mathbb{Z}$. Hence, the equations in (2.3) imply that each function $\dot{u}_n(t)$ has at most one discontinuity point on $[0, \infty)$.

For the solution, we use the phrasing a *node n switches* (*at a moment t*) if $\mathcal{H}(u_n)$ is discontinuous (at this moment t). If a node switches on the time interval $[0, \infty)$, we call it a *switching node*; otherwise we call it a *nonswitching node*. For a switching node n, we denote the time moment at which it switches by t_n. By $\mathcal{S}(t)$ we denote the set of nodes that switch on the time interval $[0, t]$, i.e.,

$$(2.5) \qquad \mathcal{S}(t) := \{k \in \mathbb{Z} : \mathcal{H}(u_k)(t) = -h_2\},$$

and let $\#(\mathcal{S}(t))$ be the number of elements in $\mathcal{S}(t)$.

The next result provides the existence and uniqueness of solutions as well as a two-sided bound for the switching moments t_n. Its proof uses the discrete Green's function $\mathcal{Y}_n(t)$. We refer the reader to Appendix A for its definition and asymptotic properties, which play a central role in this paper.

THEOREM 2.4. *Let Condition* 2.1 *hold. Then the following is true.*
1. *Problem* (2.3), (2.4) *has a unique solution* $\{u_n(t)\}_{n \in \mathbb{Z}}$ *on the time interval* $(0, \infty)$. *It is given by*

$$(2.6)$$
$$u_n(t) = -cn^2 + (h_1 - 2c)t - (h_1 + h_2) \sum_{k \in \mathcal{S}(t)} \mathcal{Y}_{n-k}(t - t_k), \quad t \in [0, \infty),$$

   where $\mathcal{Y}_n(t)$ *is the discrete Green's function given by* (A.1). *The solution is symmetric:* $u_n(t) = u_{-n}(t)$ *for all* $n \in \mathbb{Z}$.
2. *For each switching node* n, *we have* $t_n \geq \dfrac{cn^2}{h_1 - 2c}$.
3. *The set* $\mathcal{S}(t)$ *is finite for each* $t \geq 0$, *symmetric with respect to the origin,* $\#(\mathcal{S}(t)) \to \infty$ *as* $t \to \infty$, *and, for each switching node* n, *we have* $t_{-n} = t_n$.

Proof. Items 1 and 2 are proved in [11, Theorem 2.5] in the case $h_2 = 0$. Their proof for $h_2 > 0$ is basically the same. For the reader's convenience, we provide a scheme of the proof. We prove existence by steps that are separated by consecutive switching moments $\tau_1 < \tau_2 < \cdots$. On each step, we define $u_n(t)$ by formula (2.6), which obviously provides a solution of problem (2.3), (2.4) on the time interval $(0, \tau_k)$ for each $k \in \mathbb{N}$ if the sequence τ_k is infinite and on the time interval $(0, \infty)$ if this sequence is finite. Furthermore, on each step, we compare the solution $u_n(t)$ with the solution $z_n(t) = -cn^2 + (h_1 - 2c)t$ of problem (2.3), (2.4) with $\mathcal{H}(u_n)$ replaced by h_1 for all $n \in \mathbb{Z}$. It is not difficult to see that $u_n(t) \leq z_n(t)$ on the interval of existence of $u_n(t)$. Therefore, the time moment t_n at which $u_n(t)$ vanishes for the first time is not less than the moment $cn^2/(h_1 - 2c)$ at which $z_n(t)$ vanishes. In particular, this implies that $S(t)$ is finite for any $t > 0$. Let us show that $\#(\mathcal{S}(t)) \to \infty$ as $t \to \infty$. Assume, to the contrary, that there are only finitely many, say $n_0 \in \mathbb{N}$, switchings on the time interval $[0, \infty)$. Denote the last switching moment by $t_* \geq 0$. Then the solution formula (2.6) contains the same n_0 terms in the sum for all $t > t_*$. Hence, since $h_1 - 2c > 0$, Proposition A.1 implies that $u_n(t) \to \infty$ as $t \to \infty$ for all $n \in \mathbb{Z}$, which shows that the $(n_0 + 1)$th switching must happen. This proves that $\#(\mathcal{S}(t)) \to \infty$ as $t \to \infty$ and $\tau_j \to \infty$ as $j \to \infty$.

We have proved that formula (2.6) provides a solution on the time interval $(0, \infty)$ (the growth condition from item 2 of Definition 2.2 holds due to the fact that $\sup_{n \in \mathbb{Z}} |\mathcal{Y}_n(t)| < \infty$ on any finite time interval; see Proposition A.1). The uniqueness is proved by induction for each step. As an example, we consider the first step. Assume there are two different solutions. Let $(0, T)$ be a time interval on which neither of them switches (i.e., $T < \tau_1$). Denote the difference of these two solutions by $w_n(t)$. Then we have

$$\dot{w}_n(t) = \Delta w_n(t), \quad t \in (0, T), \ n \in \mathbb{Z},$$
$$w_n(0) = 0, \quad n \in \mathbb{Z}.$$

If we looked for solutions that are square summable with respect to $n \in \mathbb{Z}$, then the application of the discrete Fourier transform would immediately imply that all $w_n(t) \equiv 0$. However, we are interested in solutions that may have exponential growth with respect to $n \in \mathbb{N}$ (see item 2 in Definition 2.2). We will argue as follows. For each $N \in \mathbb{N}$, we consider the functions

$$\zeta_n^N(t) := w_n(t) \text{ for } |n| \leq N, \qquad \zeta_n^N(t) := 0 \text{ for } |n| \geq N + 1.$$

They satisfy the relations

(2.7)
$$\dot{\zeta}_n^N(t) = \Delta \zeta_n^N(t) + G_n^N(t), \quad t \in (0, T), \ n \in \mathbb{Z},$$
$$\zeta_n^N(0) = 0, \quad n \in \mathbb{Z},$$

where $G_n^N(t) = 0$ for $|n| \leq N - 1$ and $|n| \geq N + 2$, $G_{\pm N}^N(t) = w_{\pm(N+1)}(t)$, and $G_{\pm(N+1)}^N(t) = -w_{\pm N}(t)$. Since no more than finitely many elements in the sequences $\{\zeta_n^N(t)\}_{n \in \mathbb{Z}}$ and $\{G_n^N(t)\}_{n \in \mathbb{Z}}$ are nonzero, we can apply the discrete Fourier transform to (2.7) and obtain

(2.8)
$$\zeta_n^N(t) = \sum_{|k|=N}^{N+1} \int_0^t \dot{\mathcal{Y}}_{n-k}(t - s) \, G_k^N(s) \, ds, \quad t \in (0, T), \ n \in \mathbb{Z}.$$

Now, using (2.8), asymptotic properties of $\mathcal{Y}_n(t)$, and the growth condition for $G_k^N(t)$, which follows from item 2 in Definition 2.2, one can show that, for each fixed $n \geq 0$ and $t > 0$ and all $N \geq n$,

$$|w_n(t)| = |\zeta_n^N(t)| \leq t \sum_{\substack{|k|=N}}^{N+1} \sup_{s \in [0,t]} \left(|\dot{\mathcal{Y}}_{n-k}(s)| \cdot |G_k^N(s)| \right) \to 0 \quad \text{as } N \to \infty$$

(for details, see the proof of [11, Theorem 2.5]). Therefore, $w_n(t) \equiv 0$ for $t \in (0, T)$, which proves uniqueness on the first step, i.e., until the first switching.

The symmetry of $\mathcal{S}(t)$ and the relation $t_{-n} = t_n$ follow from the symmetry of the solution. □

3. Main result. First, we formulate assumptions meaning that the nodes switch according to a certain general pattern. Taking these assumptions for granted, we will rigorously determine specific parameters of this pattern. In particular, we will justify the conjecture (1.6) and explicitly find the leading order term for the pattern propagation speed.

Denote by

$$\mathcal{K} = \{k_0, \pm k_1, \pm k_2, \pm k_3, \dots\}$$

the set of nodes that switch on the time interval $[0, \infty)$. Let $0 = k_0 < k_1 < k_2 < \cdots$ be the corresponding nonnegative nodes. For $n \in \mathbb{N}$, set

$$\mathcal{K}_n := \mathcal{K} \cap [-n, n], \quad \mathcal{K}_n^+ := \mathcal{K} \cap [1, n]$$

and

(3.1) $$p(n) := \# \left(\mathcal{K}_{n-1}^+ \right), \quad q(n) := n - p(n).$$

Thus, $p(n)$ and $q(n)$ are the number of switching and nonswitching nodes, respectively, from the spatial interval $[1, n-1]$. Note that the set $\mathcal{K}$ is infinite and, thus, $p(n) \to \infty$ as $n \to \infty$ by Theorem 2.4. We assume that the switching nodes are quasi-uniform in the following sense (see section 4 below for a further discussion and examples of patterns with this property).

CONDITION 3.1 (quasi-uniformity).

$$\sup_{i=-p(n),\dots,p(n)} \left| \frac{k_i}{n} - \frac{i}{p(n)} \right| \to 0 \text{ as } n \in \mathcal{K}, n \to \infty.$$

In what follows, we will use the notation

(3.2) $$x_{n,k_i} := \frac{k_i}{n}, \quad y_{n,i} := \frac{i}{p(n)}$$

for $n \in \mathcal{K}$ and $i = -p(n), \dots, p(n)$. Obviously, $y_{n,i}$ are uniformly distributed on the interval $[-1, 1]$, while x_{n,k_i} are generally not. Condition 3.1 is equivalent to the following:

(3.3) $$\sup_{i=-p(n),\dots,p(n)} |x_{n,k_i} - y_{n,i}| = 0 \quad \text{as } n \in \mathcal{K}, \ n \to \infty,$$

which motivates the notion of quasi-uniformity; see also Figure 4.

Our second assumption concerns the asymptotics of the switching moments t_n and is also motivated by numerics; cf. (1.8).
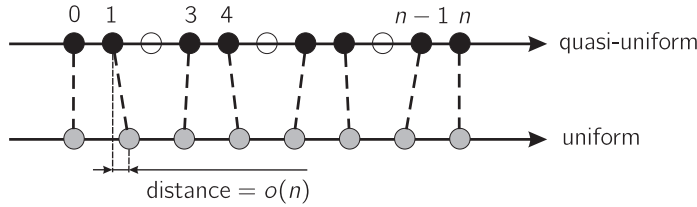
FIG. 4. *Upper axis indicates the nodes* $0, \ldots, n$; *black (white) circles correspond to the nodes that switch (do not switch). Lower axis indicates uniformly distributed gray circles. The number of the gray circles on the lower axis equals the number of black ones on the upper axis. Quasi-uniformity means that the horizontal distance between the respective black and gray circles is $o(n)$ as $n \to \infty$.*

CONDITION 3.2.
1. *There exist $a > 0$ and $A > 0$ such that the switching moment t_k of each node $k \in \mathcal{K}$ satisfies*

$$(3.4) \qquad t_k = ak^2 + \omega_k, \quad |\omega_k| \le A|k|.$$

2. *There exists $N_0 \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that, for any $n \in \mathcal{K} \setminus [-N_0, N_0]$ and $k \in \mathcal{K}_{n-1}$, we have*

$$(3.5) \qquad t_n - t_k \ge \varepsilon_0(n^2 - k^2).$$

Our main result guarantees that if the switching nodes are quasi-uniform and the switching moments grow asymptotically quadratic, then, actually, the proportion between the number of switching and nonswitching nodes satisfies conjecture (1.6). Furthermore, the coefficient a in the leading order term of the asymptotics (3.4) for t_k can be found explicitly.

THEOREM 3.3. *Let Conditions* 2.1, 3.1, *and* 3.2 *hold. Then the following hold.*
1. *There exist limits*

$$(3.6) \qquad p_* := \lim_{n \to \infty} \frac{p(n)}{n}, \quad q_* := \lim_{n \to \infty} \frac{q(n)}{n}.$$

Moreover,

$$(3.7) \qquad p_* = \frac{h_1}{h_1 + h_2}, \quad q_* = \frac{h_2}{h_1 + h_2}, \quad \frac{q_*}{p_*} = \frac{h_2}{h_1}.$$

2. *There exists an infinitely differentiable function $a_* : (2, \infty) \to (0, \infty)$ such that the coefficient a in asymptotics (3.4) is given by $a = a_*(h_1/c)$. In particular, a does not depend on h_2. Furthermore, $a_*(\lambda)$ is monotone decreasing and satisfies the following relations:*

$$(3.8) \qquad a_*(\lambda) \to +\infty \ as \ \lambda \to 2 \quad and \quad a_*(\lambda) \to 0 \ as \ \lambda \to +\infty.$$

Remark 3.4. We emphasize that item 2 in Theorem 3.3 implies that the speed with which the pattern propagates does not depend on h_2, since $a = a_*(h_1/c)$ does not. The function $a_*(\cdot)$ is found in section 8 as a unique root of each of (8.3), (8.4), (8.10). Figure 5 illustrates the dependence of a_* on h_1/c.

The proof of Theorem 3.3 will be given in section 8, after some preparatory estimates in section 5 and asymptotic analysis in sections 6 and 7.
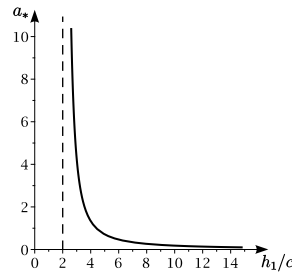
FIG. 5. *Dependence of a_* on h_1/c.*

4. Discussion.

4.1. Discussion of condition 3.1. First, we observe the following. Suppose we are given an arbitrary set $\mathcal{K} = \{k_0, \pm k_1, \pm k_2, \pm k_3, \dots\}$ such that $0 = k_0 < k_1 < k_2 < \cdots$, not necessarily corresponding to the switching pattern for problem (2.3), (2.4). Let us define the function $p(n)$ as in (3.1). Then the following is true.

LEMMA 4.1. *If there exists a limit $\lim_{n \in \mathcal{K}, n \to \infty} p(n)/n > 0$, then*

$$\sup_{i = -p(n), \dots, p(n)} \left| \frac{k_i}{n} - \frac{i}{p(n)} \right| \to 0 \quad as \ n \in \mathcal{K}, \ n \to \infty.$$

Proof. Obviously, it suffices to prove that $\sup_{j \in \mathcal{K}_{n-1}^+} |\frac{j}{n} - \frac{p(j)}{p(n)}| \to 0$ as $n \in \mathcal{K}$, $n \to \infty$. The latter follows by observing that

$$\sup_{j \in \mathcal{K}_{n-1}^+} \left| \frac{j}{n} - \frac{p(j)}{p(n)} \right| = \sup_{\alpha = j/n, \, j \in \mathcal{K}_{n-1}^+} \left| \alpha \left(1 - \frac{p(\alpha n)}{\alpha n} \frac{n}{p(n)} \right) \right| \to 0 \quad as \ n \in \mathcal{K}, n \to \infty. \ \square$$

Now we provide examples of patterns to which Theorem 3.3 applies. We consider *periodic* patterns that are observed numerically for rational h_2/h_1 and their generalizations, which we call *quasi-periodic*. In both cases, using Lemma 4.1, we show that Condition 3.1 holds.

We fix $\alpha \in (0, 1]$ and $\beta \in [0, 1)$. Denote by $\lfloor \cdot \rfloor$ the floor, or integer part, of a real number. The set

$$(4.1) \qquad \mathcal{P} := \{0\} \cup \{n \in \mathbb{Z} \setminus \{0\} : \lfloor |n|\alpha + \beta \rfloor > \lfloor (|n| - 1)\alpha + \beta \rfloor\}$$

is said to be a *quasi-periodic pattern*. If α is rational, then we say that $\mathcal{P}$ is a *periodic pattern*.

It is convenient to have the geometric description of the pattern $\mathcal{P}$ in mind; see Figure 6. On the plane (x, y), consider the horizontal strips

$$(4.2) \qquad \{0 \le y < 1\}, \quad \{1 \le y < 2\}, \quad \{2 \le y < 3\}, \dots$$

and the points $Q_n = (n, n\alpha + \beta)$ lying on the line $y = \alpha x + \beta$. Then $0 \in \mathcal{P}$, and nodes $n > 0$ and $-n < 0$ belong to $\mathcal{P}$ if and only if the points Q_n and Q_{n-1} do not belong to the same interval from (4.2).

Any switching pattern $\mathcal{K}$ satisfying $\mathcal{K} \setminus [-N_0, N_0] = \mathcal{P} \setminus [-N_0, N_0]$ for some $N_0 \in \mathbb{N}$ necessarily consists of quasi-uniform nodes. Indeed, $|p(n) - \lfloor \alpha n \rfloor|$ is bounded uniformly with respect to n. Hence, $\lim_{n \to \infty} p(n)/n = \alpha$. Therefore, due to Lemma 4.1, the pattern $\mathcal{K}$ satisfies Condition 3.1. Note that if α is rational, then the coprime integers p_1 and p_2 from the property (1.7) are defined by the equality $\alpha = p_1/(p_1 + p_2)$.
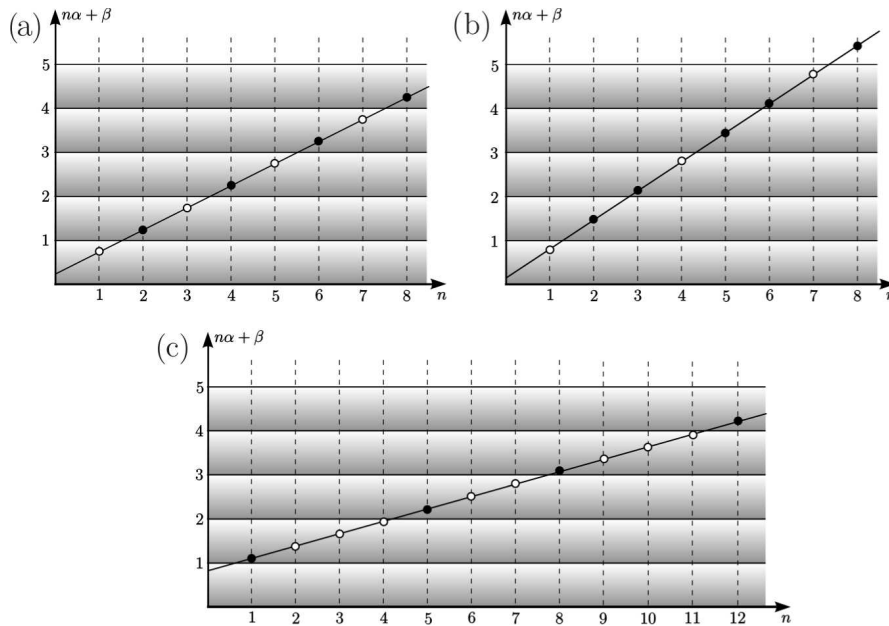
FIG. 6. *Periodic and quasi-periodic patterns*: (a) $\alpha = 1/2$, $\beta = 1/4$; (b) $\alpha = 2/3$, $\beta = 1/5$; (c) $\alpha = \sqrt{2}/5$, $\beta = 4/5$. *The filled points of intersection correspond to the points included in* $\mathcal{P}$.

Remark 4.2. It is easy to show that item 2 in Condition 3.2 is a consequence of the following (simpler to formulate) condition: *there exists* $N_0 \in \mathbb{N}$ *and* $\varepsilon > 0$ *such that, for any* $k_i \in \mathcal{K}$, $k_i > N_0$, *we have* $t_{k_i} - t_{k_{i-1}} \geq \varepsilon k_i$.

4.2. Counterexample. Now we show that the converse of Lemma 4.1 is not true: quiasi-uniformity of the elements of a set $\mathcal{K}$ (Condition 3.1) alone does not guarantee the existence of a limit $\lim_{n \to \infty} p(n)/n$. We will explicitly construct such sets $\mathcal{K}$. To show that they cannot be switching patterns generated by the solution of problem (2.3), (2.4), an additional condition is necessary. In Theorem 3.3, its role is played by Condition 3.2.

The idea of our construction is as follows. We split the half-line $(0, \infty)$ into intervals $(M_j, M_{j+1}]$, $j = 0, 1, 2, \ldots$, whose lengths $L_{j+1} := M_{j+1} - M_j$ *grow fast* as $j \to \infty$. For each j, we include in the set $\mathcal{K} \cap (M_j, M_{j+1}]$ nodes with frequency $p_j \in (0, 1)$. The frequencies p_j are chosen in such a way that they *slowly oscillate* between two different limits: $\liminf_{j \to \infty} p_j$ and $\limsup_{j \to \infty} p_j$. The whole construction is balanced in such a way that the nodes from $\mathcal{K}$ are quasi-uniform (since the oscillation of p_j is slow enough), but $p(n)/n$ has no limit (since p_j has no limit).

Let us make the above argument rigorous. We consider a sequence of frequencies $p_j \in \mathbb{Q}$, $p_j \in (0, 1)$, possessing the *slow oscillation* property

$$(4.3) \qquad \lim_{j \to \infty} \frac{p_j}{p_{j+1}} = 1, \quad \liminf_{j \to \infty} p_j < \limsup_{j \to \infty} p_j$$

and a sequence of interval lengths $L_j \in \mathbb{N}$ such that $L_j p_j \in \mathbb{N}$ satisfying the *fast growth* property

$$(4.4) \qquad p_{j+1} L_{j+1} \geq (L_1 + \cdots + L_j)(j + 1).$$

We define the intervals $(M_j, M_{j+1}]$ by setting $M_0 := 0$ and $M_j := L_1 + \cdots + L_j$. Finally, we define $\mathcal{K}$ as the set consisting of 0 and all the integer numbers $n = M_j + r$ and $-n$ satisfying

$$(4.5) \qquad j \geq 0, \quad r \in [1, L_{j+1}], \quad \lfloor rp_{j+1} \rfloor > \lfloor (r-1)p_{j+1} \rfloor;$$

cf. (4.1). In particular, for $R \in [1, L_{j+1}]$, we have $\#(\mathcal{K} \cap (M_j, M_j + R]) = \lfloor Rp_{j+1} \rfloor$, which motivates the term "frequency" for p_j.

The following lemma asserts that the set $\mathcal{K}$ possesses the desired properties.

LEMMA 4.3. *The set $\mathcal{K}$ satisfies Condition* 3.1, *but*

$$(4.6) \qquad \liminf_{n \to \infty} \frac{p(n)}{n} \leq \liminf_{j \to \infty} p_j, \quad \limsup_{n \to \infty} \frac{p(n)}{n} \geq \limsup_{j \to \infty} p_j.$$

Proof. 1. First, we prove that

$$\sup_{i = -p(n+1), \ldots, p(n+1)} \left| \frac{k_i}{n} - \frac{i}{p(n+1)} \right| \to 0 \text{ as } n \in \mathcal{K}, n \to \infty,$$

which implies Condition 3.1 because $p(n) \to \infty$ as $n \to \infty$. Set

$$l_j := L_j p_j = \#(\mathcal{K} \cap (M_{j-1}, M_j]), \quad m_j := l_1 + \cdots + l_j = \#(\mathcal{K} \cap (0, M_j]),$$

$$b_j := \left| \frac{M_j p_{j+1}}{m_j} - 1 \right|.$$

It is easy to show that (4.3) and (4.4) yield

$$(4.7) \qquad \lim_{j \to \infty} \frac{M_j}{M_{j+1}} = 0, \quad \lim_{j \to \infty} \frac{m_j}{m_{j+1}} = 0, \quad \lim_{j \to \infty} \frac{p_j M_j}{m_j} = 1, \quad \lim_{j \to \infty} b_j = 0.$$

Now, for $i = 1, \ldots, p(n+1)$, we represent

$$n = M_j + r \text{ with } r \in [1, L_{j+1}], \quad i = m_s + \rho \text{ with } \rho \in [1, l_{s+1}],$$

$$(4.8) \qquad p(n+1) = m_j + \lfloor rp_{j+1} \rfloor, \qquad\qquad k_i = M_s + \left\lceil \frac{\rho}{p_{s+1}} \right\rceil,$$

where $\lceil x \rceil := \min_{u \in \mathbb{Z}} \{u \geq x\}$. In particular,

$$(4.9) \qquad s < j \quad \text{or} \quad (s = j \text{ and } \rho \leq rp_{j+1}).$$

If $s \leq j - 2$, then

$$(4.10) \qquad \frac{k_i}{n} \leq \frac{M_{s+1}}{M_j} \leq \frac{M_{j-1}}{M_j} \to 0 \quad \text{as } j \to \infty,$$

$$(4.11) \qquad \frac{i}{p(n+1)} \leq \frac{m_{s+1}}{m_j} \leq \frac{m_{j-1}}{m_j} \to 0 \quad \text{as } j \to \infty.$$

It remains to consider the case $s = j - 1$ or $s = j$. We have

$$(4.12) \qquad \frac{k_i}{n} - \frac{i}{p(n+1)} = \frac{M_s + \left\lceil \dfrac{\rho}{p_{s+1}} \right\rceil}{M_j + r} - \frac{m_s + \rho}{m_j + \lfloor rp_{j+1} \rfloor}.$$

We rewrite the first term in the right-hand side in (4.12) as follows:

$$(4.13) \qquad \frac{M_s + \left\lceil \dfrac{\rho}{p_{s+1}} \right\rceil}{M_j + r} = \frac{M_s + \dfrac{\rho}{p_{s+1}}}{M_j + r}(1 + \delta_1),$$

where

$$(4.14) \qquad |\delta_1| \leq \frac{1}{M_s}.$$

Similarly,

$$(4.15) \qquad \begin{aligned} \frac{M_s + \dfrac{\rho}{p_{s+1}}}{M_j + r} &= \frac{M_s p_{s+1} + \rho}{M_j p_{s+1} + r p_{s+1}} = \frac{m_s \dfrac{M_s p_{s+1}}{m_s} + \rho}{\dfrac{p_{s+1}}{p_{j+1}}\left(m_j \dfrac{M_j p_{j+1}}{m_j} + r p_{j+1}\right)} \\ &= \frac{p_{j+1}}{p_{s+1}} \cdot \frac{m_s + \rho}{m_j + r p_{j+1}} \cdot \frac{1 + \delta_2}{1 + \delta_3}, \end{aligned}$$

where

$$(4.16) \qquad |\delta_2| \leq b_s, \quad |\delta_3| \leq b_j.$$

We also rewrite the second term in the right-hand side in (4.12) as follows:

$$(4.17) \qquad \frac{m_s + \rho}{m_j + \lfloor r p_{j+1} \rfloor} = \frac{m_s + \rho}{m_j + r p_{j+1}} \cdot \frac{1}{1 + \delta_4},$$

where

$$(4.18) \qquad |\delta_4| \leq \frac{1}{m_j}.$$

Now, taking into account (4.9), (4.12), (4.13), (4.15), and (4.17), we obtain

$$\begin{aligned} \left| \frac{k_i}{n} - \frac{i}{p(n+1)} \right| &= \frac{m_s + \rho}{m_j + r p_{j+1}} \left| \frac{(1 + \delta_1)(1 + \delta_2)}{1 + \delta_3} \cdot \frac{p_{j+1}}{p_{s+1}} - \frac{1}{1 + \delta_4} \right| \\ &\leq \left| \frac{(1 + \delta_1)(1 + \delta_2)}{1 + \delta_3} \cdot \frac{p_{j+1}}{p_{s+1}} - \frac{1}{1 + \delta_4} \right|. \end{aligned}$$

Since $s = j - 1$ or $s = j$, relations (4.3), (4.7), (4.14), (4.16), and (4.18) imply that the latter expression tends to 0 as $j \to \infty$.

2. In order to prove relations (4.6), we consider the subsequence $n_j = M_j$. Note that due to (4.8) the equality $p(n_j + 1)/n_j = m_j/M_j$ holds, and (4.3) together with the third relation in (4.7) imply (4.6). □

5. Elementary estimates. We begin with some straightforward estimates for u_n, $\dot{u}_n$, Δu_n, and ∇u_n (see notation (2.1)). In particular, our goal is to prove the following result about the boundedness of $\nabla u_n(t_n)$.

PROPOSITION 5.1. *Let Condition* 2.1 *hold. Then, for each node* $n \in \mathcal{K}$, *the following relations hold at the switching moment* t_n:

$$\nabla u_n(t_n) \in [-(2h_1 + h_2), 0], \quad \nabla u_{n-1}(t_n) \in [0, (2h_1 + h_2)].$$

The proof will be given in the end of this section as a consequence of several lemmas (in which we assume that Condition 2.1 holds).

LEMMA 5.2. $u_n(t) \leq 0$ for all $n \in \mathbb{Z}$, $t \geq 0$.

Proof. Using the solution formula (2.6) and the relations (A.2) for the discrete Green's function $\mathcal{Y}_n(t)$, it is easy to see that, on a sufficiently small time interval, all the nodes satisfy $u_m(t) \leq 0$, $m \in \mathbb{Z}$. Let $\theta > 0$ be a time moment at which at least one of the nodes, say u_n, is on the threshold 0, while all the nodes still satisfy $u_m(\theta) \leq 0$, $m \in \mathbb{Z}$. Then $\Delta u_n(\theta) \leq 0$. If $\Delta u_n(\theta) < 0$ or if $\Delta u_n(\theta) = 0$ and $-h_2 < 0$, then $\dot{u}_n(\theta + 0) < 0$ and $u_n(t)$ must decrease on $(\theta, \theta + \varepsilon)$ for a sufficiently small $\varepsilon > 0$.

It remains to consider the case $h_2 = 0$, $u_n(\theta) = 0$ for some $n \in \mathbb{Z}$, and $\Delta u_n(\theta) = 0$. If $u_n(\theta) = 0$ for all $n \in \mathbb{Z}$, then Theorem 2.4 implies that $u_n(t) \equiv 0$ for $n \in \mathbb{Z}$ and $t \in [\theta, \infty)$.

Assume that

$$u_n(\theta) = u_{n\pm 1}(\theta) = u_{n\pm 2}(\theta) = \cdots u_{n\pm(j-1)}(\theta) = 0$$

for some $j \geq 2$, but $u_{n+j}(\theta) < 0$ or $u_{n-j}(\theta) < 0$. Then

$$\underbrace{\Delta \cdots \Delta}_{s \text{ times}} u_n(\theta) = 0, \quad s = 1, \ldots, j-1; \qquad \underbrace{\Delta \cdots \Delta}_{j \text{ times}} u_n(\theta) < 0.$$

Therefore, $d^s u_n(\theta)/dt^s = 0$ for $s = 1, \ldots, j-1$, but $d^j u_n(\theta)/dt^j < 0$ (where d/dt is understood as the right derivative). □

LEMMA 5.3. $\dot{u}_n(t) \in [-(h_1 + h_2), h_1 - 2c]$ for all $n \in \mathbb{Z}$, $t \geq 0$.

Proof. 1. By Theorem 2.4, there is a sequence $0 =: \theta_0 < \theta_1 < \theta_2 < \cdots$ (of switching moments) such that $\lim_{j \to \infty} \theta_j = \infty$ and there are no switchings for $t \in (\theta_j, \theta_{j+1})$, $j = 0, 1, 2, \ldots$.

Set $v_n(t) := \dot{u}_n(t)$ for $t > 0$, $t \neq \theta_j$ $(j = 0, 1, 2, \ldots)$. Note that $v_n \in C[\theta_j, \theta_{j+1}]$, $j = 0, 1, 2, \ldots$. However, if a node n switches at some moment θ_j, then $v_n(t)$ has a jump discontinuity at θ_j.

At the initial moment, we have

$$v_0(0) = -2c - h_2 \in [-(h_1 + h_2), h_1 - 2c],$$
$$v_n(0) = -2c + h_1 \in [-(h_1 + h_2), h_1 - 2c], \quad n \neq 0.$$

Due to (2.3), on the interval $[\theta_0, \theta_1)$, $v_n(t)$ satisfies

$$\dot{v}_n = \Delta v_n, \quad n \in \mathbb{Z}.$$

Therefore, similarly to the proof of Lemma 5.2, we conclude then that $v_n(t) \in [-(h_1 + h_2), h_1 - 2c]$ for all $[\theta_0, \theta_1)$.

2. At the switching moment θ_1, some nodes switch. Let n be one of them. Then $v_n(\theta_1 - 0) \geq 0$ and $\Delta u_n(\theta_1) = \Delta u_n(\theta_1 \pm 0) \leq 0$. Therefore,

$$v_n(\theta_1 + 0) = \Delta u_n(\theta_1 + 0) - h_2 \leq -h_2 < h_1 - 2c,$$

$$\begin{aligned}
v_n(\theta_1 + 0) = \Delta u_n(\theta_1 + 0) - h_2 &= \Delta u_n(\theta_1 - 0) + h_1 - (h_1 + h_2) \\
&= v_n(\theta_1 - 0) - (h_1 + h_2) \geq -(h_1 + h_2).
\end{aligned}$$

If the node n does not switch at θ_1, then $v_n(\theta_1 + 0) = v_n(\theta_1 - 0) \in [-(h_1 + h_2), h_1 - 2c]$.

Now, as in part 1 of the proof, we conclude that $v_n(t) \in [-(h_1 + h_2), h_1 - 2c]$ for all $n \in \mathbb{Z}$ and $t \in [\theta_1, \theta_2)$. Arguing by induction, we complete the proof. □

LEMMA 5.4. $\Delta u_n(t) \in [-(2h_1 + h_2), h_1 + h_2 - 2c]$ *for all* $n \in \mathbb{Z}$, $t \geq 0$.

Proof. The result follows from Lemma 5.3 and (2.3). $\qquad\square$

Proof of Proposition 5.1. By Lemma 5.2, at the switching moment $t = t_n$ ($n \in \mathcal{K}$), we have $\nabla u_n \leq 0$ and $\nabla u_{n-1} \geq 0$. Therefore, using Lemma 5.4, we have at the moment $t = t_n$

$$
\begin{aligned}
\nabla u_n &= \nabla u_{n-1} + \Delta u_n \geq \Delta u_n \geq -(2h_1 + h_2), \\
\nabla u_{n-1} &= \nabla u_n - \Delta u_n \leq -\Delta u_n \leq 2h_1 + h_2.
\end{aligned}
\qquad\square
$$

6. Asymptotics for $u_n(t_n)$. In this section and in the next one, we will use the standard notation $O(\cdot)$ and $o(\cdot)$. For example, given two sequences of real numbers $v_{n,k}$ and $w_{n,k}$ ($n, k \in \mathbb{N}$), we write $v_{n,k} = O(w_{n,k})$ if there exists a constant $C > 0$ such that $|v_{n,k}| \leq C|w_{n,k}|$ for all $n, k \in \mathbb{N}$. Given a sequence of real numbers v_n ($n \in \mathbb{N}$), we write $v_n = o(n)$ if $v_n/n \to 0$ as $n \to \infty$.

The goal of this section is to prove the following result.

PROPOSITION 6.1. *Let Conditions* 2.1, 3.1, *and* 3.2 *hold. Then there exists a limit*

$$
(6.1) \qquad\qquad p_* := \lim_{n \to \infty} \frac{p(n)}{n},
$$

and the numbers a from Condition 3.2 *and* p_* *from* (6.1) *satisfy the equality*

$$
(6.2) \qquad\qquad -c + (h_1 - 2c)a - (h_1 + h_2)p_* I_F(a) = 0,
$$

where

$$
(6.3) \qquad I_F(a) := \int_{-1}^{1} F(a, x)\, dx, \quad F(a, x) := \sqrt{a}\sqrt{1 - x^2}\, f\left(\frac{1}{\sqrt{a}}\sqrt{\frac{1 - x}{1 + x}}\right)
$$

(see Figure 7*), and the function f is defined in* (A.3).

Proof. For $n \in \mathcal{K}$, $n > N_0$ (N_0 is the number from item 2 in Condition 3.2), we will rewrite the equality $u_n(t_n) = 0$. Using (2.6), we have

$$
(6.4) \qquad 0 = u_n(t_n) = -cn^2 + (h_1 - 2c)t_n - (h_1 + h_2) \sum_{k \in \mathcal{K}_{n-1}} \mathcal{Y}_{n-k}(t_n - t_k).
$$

Consider the sum in (6.4). We write $t_n - t_k = a(n^2 - k^2) + \omega_n - \omega_k$ according to Condition 3.2 and expand $\mathcal{Y}_{n-k}$ in the Taylor series about $a(n^2 - k^2)$. Taking into account Proposition A.1 and notation (3.2) for x_{n,k_i}, we obtain
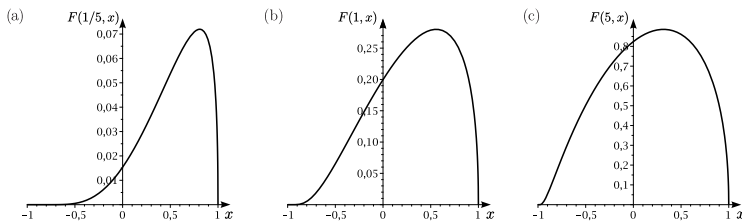

FIG. 7. *Graphs of function $F(a, \cdot)$ for (a) $a = 1/5$, (b) $a = 1$, (c) $a = 5$.*

(6.5)
$$\sum_{k\in\mathcal{K}_{n-1}} \mathcal{Y}_{n-k}(t_n - t_k) = \sum_{k\in\mathcal{K}_{n-1}} \mathcal{Y}_{n-k}(a(n^2 - k^2)) + \Sigma_n$$

$$= n^2 \sum_{i=-p(n)}^{p(n)} \frac{1}{n} F(a, x_{n,k_i}) + \sum_{k\in\mathcal{K}_{n-1}} O\left((n^2 - k^2)^{-1/2}\right) + \Sigma_n,$$

where the function F is defined in (6.3),

(6.6)
$$\Sigma_n := \sum_{k\in\mathcal{K}_{n-1}} \dot{\mathcal{Y}}_{n-k}(\sigma_{n,k})(\omega_n - \omega_k),$$

and $\sigma_{n,k} \geq \min\{\varepsilon_0, a\}(n^2 - k^2)$ due to (3.5). Note that

(6.7)
$$\sum_{k\in\mathcal{K}_{n-1}} O\left((n^2 - k^2)^{-1/2}\right) = O(1).$$

Further, since $\omega_n - \omega_k = O(n)$ by (3.4), we obtain from (6.6) and Proposition A.1

(6.8)
$$\Sigma_n = \sum_{k\in\mathcal{K}_{n-1}} O((n^2 - k^2)^{-1/2})O(n) = O(n).$$

Combining (6.5), (6.7), and (6.8), we obtain

(6.9)
$$\sum_{k\in\mathcal{K}_{n-1}} \mathcal{Y}_{n-k}(t_n - t_k) = n^2 \sum_{i=-p(n)}^{p(n)} \frac{1}{n} F(a, x_{n,k_i}) + O(n).$$

Now we represent the sum in (6.9) as follows (using notation (3.2) for $y_{n,i}$ and recalling that $F(-1) = 0$):

(6.10)
$$\sum_{i=-p(n)}^{p(n)} \frac{1}{n} F(a, x_{n,k_i})$$

$$= \frac{p(n)}{n} \left(\sum_{i=-p(n)+1}^{p(n)} \frac{1}{p(n)} F(a, y_{n,i}) + \sum_{i=-p(n)}^{p(n)} \frac{1}{p(n)} \left(F(a, x_{n,k_i}) - F(a, y_{n,i})\right) \right).$$

The first sum in (6.10) is the Riemann sum for the integral $I_F(a)$ defined in (6.3). Since $p(n) \to \infty$ as $n \to \infty$, we have

(6.11)
$$\sum_{i=-p(n)+1}^{p(n)} \frac{1}{p(n)} F(a, y_{n,i}) = I_F(a) + o(1).$$

Finally, taking into account Condition 3.1 and the uniform continuity of $F(a, x)$ for $x \in [-1, 1]$ (and a fixed a), we estimate the last sum in (6.10) as follows:

(6.12)
$$\left| \sum_{i=-p(n)}^{p(n)} \frac{1}{p(n)} \left(F(a, x_{n,k_i}) - F(a, y_{n,i})\right) \right|$$

$$\leq \frac{2p(n) + 1}{p(n)} \sup_{i=-p(n),\ldots,p(n)} |F(a, x_{n,k_i}) - F(a, y_{n,i})| = o(1).$$

Due to (6.10)–(6.12), relation (6.9) takes the form

$$(6.13) \qquad \sum_{k \in \mathcal{K}_{n-1}} \mathcal{Y}_{n-k}(t_n - t_k) = n^2 \frac{p(n)}{n} I_F(a) + o(n^2).$$

Combining relations (6.4) and (6.13), we conclude that

$$\left(-c + (h_1 - 2c)a - (h_1 + h_2)\frac{p(n)}{n}I_F(a) \right) n^2 + o(n^2) = 0.$$

Since $I_F(a) > 0$, this implies that the limit $p_* := \lim_{n \in \mathcal{K}, \, n \to \infty} \frac{p(n)}{n}$ exists and equality (6.2) holds.

To complete the proof, we have to show that the limit in (6.1) exists along any sequence. Fix an arbitrary $\varepsilon > 0$. Then there exists $N \geq 1/\varepsilon$ such that $|\frac{p(k_i)}{k_i} - p_*| \leq \varepsilon$ for all $k_i \in \mathcal{K}$, $k_i \geq N$. Now any $m \geq N$ belongs to some interval $(k_{i-1}, k_i]$, and we have

$$\left| \frac{p(m)}{m} - p_* \right| = \left| \frac{p(k_i)}{m} - p_* \right| \leq \left| \frac{p(k_i)}{k_i} - p_* \right| + \left| \frac{p(k_i)}{k_i} - \frac{p(k_i)}{m} \right|$$

$$\leq \varepsilon + \frac{p(k_i)}{m} \left(1 - \frac{m}{k_i} \right) \leq \varepsilon + \frac{k_i - 1}{k_{i-1}} \left(1 - \frac{k_{i-1}}{k_i} \right).$$

It remains to note that $\dfrac{k_i}{k_{i-1}} \to 1$ as $k_i \in \mathcal{K}$, $k_i \to \infty$ due to Condition 3.1. $\qquad \square$

7. Asymptotics for $\nabla u_n(t_n)$. In this section, we will prove the following result.

PROPOSITION 7.1. *Let Conditions* 2.1, 3.1, *and* 3.2 *hold. Then the numbers a from Condition* 3.2 *and p_* from* (6.1) *satisfy the equality*

$$(7.1) \qquad\qquad -2c - (h_1 + h_2)p_* I_G(a) = 0,$$

where

$$(7.2) \qquad I_G(a) := \int_{-1}^{1} G(a, x)\,dx, \quad G(a, x) := g\left(\frac{1}{\sqrt{a}} \sqrt{\frac{1 - x}{1 + x}} \right)$$

(*see Figure* 8), *and the function g is defined in* (A.3).

Proof. For $n \in \mathcal{K}$, $n > N_0$ (N_0 is the number from item 2 in Condition 3.2), we will find an asymptotic expansion of $\nabla u_n(t_n)$. Using (2.6), we have

$$(7.3) \qquad \nabla u_n(t_n) = -2cn - c - (h_1 + h_2) \sum_{k \in \mathcal{K}_{n-1}} \nabla \mathcal{Y}_{n-k}(t_n - t_k).$$

Consider the last sum in (7.3). We write $t_n - t_k = a(n^2 - k^2) + \omega_n - \omega_k$ according to Condition 3.2 and expand $\nabla \mathcal{Y}_{n-k}$ in the Taylor series about $a(n^2 - k^2)$. Taking into account Proposition A.1 and notation (3.2) for x_{n,k_i}, we obtain

$$(7.4)$$
$$\sum_{k \in \mathcal{K}_{n-1}} \nabla \mathcal{Y}_{n-k}(t_n - t_k) = \sum_{k \in \mathcal{K}_{n-1}} \nabla \mathcal{Y}_{n-k}(a(n^2 - k^2)) + \Theta_n$$

$$= n \sum_{i=-p(n)}^{p(n)} \frac{1}{n}G(a, x_{n,k_i}) + \sum_{k \in \mathcal{K}_{n-1}} O\left((n^2 - k^2)^{-1/2} \right) + \Theta_n,$$
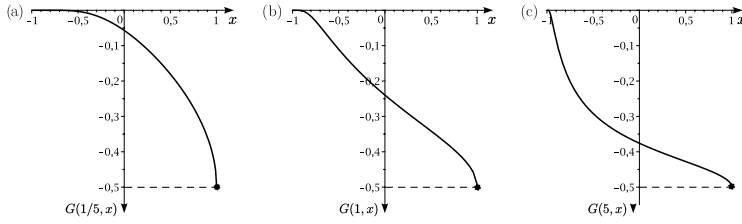
Fig. 8. *Graphs of function $G(a, \cdot)$ for* (a) $a = 1/5$, (b) $a = 1$, (c) $a = 5$.

where the function G is defined in (7.2),

$$(7.5) \qquad \Theta_n := \sum_{k \in \mathcal{K}_{n-1}} \nabla \dot{\mathcal{Y}}_{n-k}(\sigma_{n,k})(\omega_n - \omega_k),$$

and $\sigma_{n,k} \geq \min\{\varepsilon_0, a\}(n^2 - k^2)$ due to (3.5). Since $\omega_n - \omega_k = O(n)$ by (3.4), we obtain from (7.5) and Proposition A.1

$$(7.6) \qquad \Theta_n = \sum_{k \in \mathcal{K}_{n-1}} O((n^2 - k^2)^{-1})O(n) = O(\ln n).$$

Combining (7.4), (6.7), and (7.6), we obtain

$$(7.7) \qquad \sum_{k \in \mathcal{K}_{n-1}} \nabla \mathcal{Y}_{n-k}(t_n - t_k) = n \sum_{i=-p(n)}^{p(n)} \frac{1}{n} G(a, x_{n,k_i}) + O(\ln n).$$

Similarly to the proof of Proposition 6.1 (cf. (6.10)–(6.12)), taking into account the existence of the limit in (6.1), we obtain

$$\sum_{i=-p(n)}^{p(n)} \frac{1}{n} G(a, x_{n,k_i})$$

$$= \frac{p(n)}{n} \left(\sum_{i=-p(n)+1}^{p(n)} \frac{1}{p(n)} G(a, y_{n,i}) + \sum_{i=-p(n)}^{p(n)} \frac{1}{p(n)} \left(G(a, x_{n,k_i}) - G(a, y_{n,i}) \right) \right)$$

$$= p_* I_G(a) + o(1),$$

where p_* is given by (6.1) and the integral $I_G(a)$ is defined in (7.2). Therefore, (7.7) takes the form

$$(7.8) \qquad \sum_{k \in \mathcal{K}_{n-1}} \nabla \mathcal{Y}_{n-k}(t_n - t_k) = n p_* I_G(a) + o(n).$$

Combining relations (7.3) and (7.8), we conclude that

$$\nabla u_n(t_n) = \left(-2c - (h_1 + h_2)p_* I_G(a) \right) n + o(n).$$

On the other hand, $\nabla u_n(t_n) = O(1)$ by Proposition 5.1. Hence, (7.1) follows. $\qquad \square$

8. Proof of the main result: Theorem 3.3. 1. The existence of the limits in (3.6) follows from Proposition 6.1. Let us prove the rest. Denote $b := \frac{h_1+h_2}{h_1} p_*$ and rewrite equalities (6.2) and (7.1) as follows:

$$(8.1) \qquad\qquad -c + (h_1 - 2c)a - bh_1 I_F(a) = 0,$$

$$(8.2) \qquad\qquad -2c - bh_1 I_G(a) = 0.$$

It is proved in [11, Proposition 4.2] that each of the equations

$$(8.3) \qquad\qquad -c + (h_1 - 2c)a_* - h_1 I_F(a_*) = 0,$$

$$(8.4) \qquad\qquad -2c - h_1 I_G(a_*) = 0$$

(with respect to $a_* > 0$) has the same unique root (and a_* as a function of h_1/c possesses all the properties claimed in Theorem 3.3; see also Figure 5). Therefore, equalities (8.1) and (8.2) hold if $a = a_*$ and $b = 1$. Let us prove the opposite: if equalities (8.1) and (8.2) hold for some $a > 0$ and $b \in \mathbb{R}$, then $a = a_*$ and $b = 1$.

Eliminating b from (8.1) and (8.2), we find that

$$(8.5) \qquad\qquad \frac{h_1}{c} = \frac{(2a+1)I_G(a) - 2I_F(a)}{aI_G(a)}.$$

Assume we have proved that the right-hand side in (8.5) satisfies

$$(8.6) \qquad\qquad \frac{(2a+1)I_G(a) - 2I_F(a)}{aI_G(a)} = -\frac{2}{I_G(a)}.$$

Then relations (8.5) and (8.6) imply that $-2c - h_1 I_G(a) = 0$. As we mentioned, by [11, Proposition 4.2], this yields $a = a_*$. Now, comparing (8.2) with $a = a_*$ and (8.4), we conclude that $b = 1$.

2. Let us prove (8.6). We rewrite it as

$$(8.7) \qquad\qquad (2a+1)I_G(a) - 2I_F(a) + 2a = 0.$$

Making the change of variables $y = \frac{1}{\sqrt{a}}\sqrt{\frac{1-x}{1+x}}$ in the integral in (7.2), integrating by parts, and using (A.4) and (A.5), we have

$$(8.8) \qquad I_G(a) = -\int_0^\infty \left(\frac{1-ay^2}{1+ay^2}\right)' g(y)\,dy = \int_0^\infty \frac{1-ay^2}{1+ay^2} h(y)\,dy - \frac{1}{2}.$$

Making the same change of variables in the integral in (6.3) and additionally using (A.4), one can show (see details in [11, section B]) that

$$(8.9) \qquad\qquad I_F(a) = \int_0^\infty \frac{2a+1}{1+ay^2} h(y)\,dy - \frac{1}{2}.$$

Due to (8.8) and (8.9), the left-hand side of relation (8.7) equals

$$(2a+1)I_G(a) - 2I_F(a) + 2a = (2a+1)\left(\frac{1}{2} - \int_0^\infty h(y)\,dy\right) = 0.$$

3. It remains to prove the properties of the function $a_* = a_*(h_1/c)$ stated in item 2 of Theorem 3.3. In order to prove that the function a_* is infinitely differentiable and decreasing, due to (8.4), it suffices to show that $I_G(a)$ is infinitely differentiable
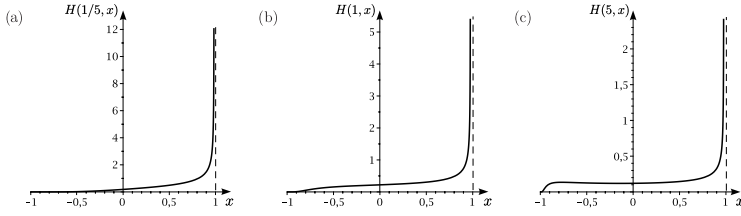
FIG. 9. *Graphs of function $H(a, \cdot)$ for* (a) $a = 1/5$, (b) $a = 1$, (c) $a = 5$.

and $I_G'(a) < 0$ for $a > 0$. The former easily follows from the definition of I_G in (7.2) and the properties of the function g described in Appendix A. To see the latter, we use (7.2) and (A.4) to obtain

$$I_G'(a) = -\frac{1}{2a^{3/2}} \int_{-1}^{1} h\left(\frac{1}{\sqrt{a}}\sqrt{\frac{1-x}{1+x}}\right) \sqrt{\frac{1-x}{1+x}} \, dx < 0.$$

In order to prove (3.8), it is enough to show that

$$I_G(a) \to -1 \text{ as } a \to +\infty \quad \text{and} \quad I_G(a) \to 0 \text{ as } a \to 0,$$

which trivially follows from (7.2), (A.5), and the properties of the function g described in Appendix A.

Theorem 3.3 is proved.

Remark 8.1. Finding a_* from (8.3) or (8.4) requires evaluation of the integral $I_F(a)$ or $I_G(a)$ (see (6.3) and (7.2)) involving the function f or g, respectively. In turn, the latter two are defined via the integral of the function h; see (A.3). From the computational point of view, it would be desirable to avoid evaluating multiple integrals. In our case, it appears to be possible. Similarly to (6.3) and (7.2), set

$$H(a, x) := \frac{1}{\sqrt{a}\sqrt{1-x^2}} h\left(\frac{1}{\sqrt{a}}\sqrt{\frac{1-x}{1+x}}\right), \quad I_H(a) := \int_{-1}^{1} H(a, x) \, dx$$

(see Figure 9), where the function h is defined in (A.3). Then [11, Proposition 4.2] implies that the equation

(8.10) $$(h_1 - 2c) - h_1 I_H(a_*) = 0, \quad a_* > 0,$$

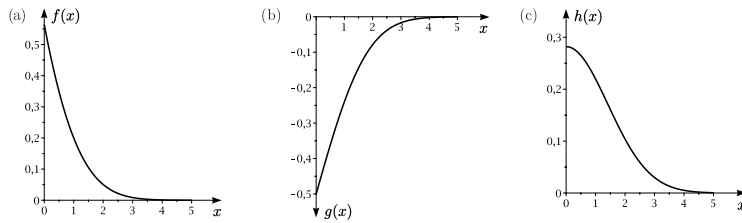has a unique root, which coincides with the root of (8.3) and (8.4).

Appendix A. Discrete Green's function $\mathcal{Y}_n(t)$.

In this appendix, we introduce one of our main tools, namely, the *discrete Green's function*

(A.1) $$\mathcal{Y}_n(t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - e^{-2t(1-\cos\theta)}}{2(1 - \cos\theta)} e^{in\theta} \, d\theta, \quad t \geq 0.$$

For convenience, we set $\mathcal{Y}_n(t) = 0$ for $t < 0$. One can check that $\mathcal{Y}_n \in C^\infty[0, \infty)$ and satisfies

(A.2) $$\begin{cases} \dot{\mathcal{Y}}_0 = \Delta\mathcal{Y}_0 + 1, & t > 0, \\ \dot{\mathcal{Y}}_n = \Delta\mathcal{Y}_n, & t > 0, \ n \neq 0, \\ \mathcal{Y}_n(0) = 0, & n \in \mathbb{Z}. \end{cases}$$

FIG. 10. *Graphs of functions f, g, and h.*

The proof of our main result, namely, Theorem 3.3, is essentially based on asymptotic properties of $\mathcal{Y}_n(t)$ and its derivatives. To describe these properties, we will use the functions $h, f, g : \mathbb{R}^+ \to \mathbb{R}$ given by

$$(A.3) \qquad h(x) := \frac{1}{2\sqrt{\pi}} \, e^{-x^2/4}, \quad f(x) := 2x \int_x^\infty y^{-2} h(y) \, dy, \quad g(x) := f'(x).$$

Note that $f, g, h \in C^\infty[0, \infty)$ and decay to zero as $x \to +\infty$, together with all their derivatives, faster than exponentially; see Figure 10. Moreover, they satisfy the relations

$$(A.4) \qquad\qquad\qquad h(x) = g'(x) = f''(x),$$

$$(A.5) \qquad\qquad\qquad g(0) = -\frac{1}{2},$$

$$(A.6) \qquad\qquad 2h(x) + xg(x) - f(x) = 0.$$

PROPOSITION A.1 (see [7]). *For any $t_* > 0$, there exist constants $C_1, \ldots, C_4 > 0$ such that, for all $t \geq t_*$ and $n = 0, 1, 2, \ldots$, we have*

$$\left| \mathcal{Y}_n(t) - \sqrt{t} f\left(\frac{n}{\sqrt{t}} \right) \right| \leq \frac{C_1}{\sqrt{t}}, \quad \left| \nabla \mathcal{Y}_n(t) - g\left(\frac{n}{\sqrt{t}} \right) \right| \leq \frac{C_2}{\sqrt{t}},$$

$$\left| \dot{\mathcal{Y}}_n(t) \right| \leq \frac{C_3}{\sqrt{t}}, \qquad\qquad\qquad \left| \nabla \dot{\mathcal{Y}}_n(t) \right| \leq \frac{C_4}{t}.$$

If $n \leq -1$, then $\mathcal{Y}_n(t) = \mathcal{Y}_{-n}(t)$ and $\nabla \mathcal{Y}_n(t) = -\nabla \mathcal{Y}_{-(n+1)}(t)$.

REFERENCES

[1] T. AIKI AND J. KOPFOVÁ, *A mathematical model for bacterial growth described by a hysteresis operator*, in Recent Advances in Nonlinear Analysis, World Sci., Hackensack, NJ, 2008, pp. 1–10.

[2] H. W. ALT, *On the thermostat problem*, Control Cybernet., 14 (1985), pp. 171–193.

[3] D. APUSHKINSKAYA AND N. URALTSEVA, *On regularity properties of solutions to the hysteresis-type problem*, Interfaces Free Bound., 17 (2015), pp. 93–115.

[4] D. APUSHKINSKAYA AND N. URALTSEVA, *Free boundaries in problems with hysteresis*, Philos. Trans. A, 373 (2015), p. 20140271.

[5] M. CURRAN, *Local Well-Poseness of a Reaction-Diffusion Equation with Hysteresis*, Masters Thesis, Fachbereich Mathematik und Informatik, Freie Universität Berlin, Berlin, 2014.

[6] M. CURRAN, P. GUREVICH, AND S. TIKHOMIROV, *Recent advances in reaction-diffusion equations with non-ideal relays*, in Control of Self-Organizing Nonlinear Systems, Springer, Cham, Switzerland, 2016, pp. 211–234.

[7] P. GUREVICH, *Asymptotics of parabolic Green's functions on lattices*, Algebra i Analiz., 28 (2016), pp. 21–60 (in Russian); St. Petersburg Math. J., 28 (2017); (in English).

[8] P. GUREVICH, R. SHAMIN, AND S. TIKHOMIROV, *Reaction-diffusion equations with spatially distributed hysteresis*, SIAM J. Math. Anal., 45 (2013), pp. 1328–1355.

[9] P. GUREVICH AND S. TIKHOMIROV, *Uniqueness of transverse solutions for reaction-diffusion equations with spatially distributed hysteresis*, Nonlinear Anal., 75 (2012), pp. 6610–6619.

[10] P. GUREVICH AND S. TIKHOMIROV, *Systems of reaction-diffusion equations with spatially distributed hysteresis*, Math. Bohem., 139 (2014), pp. 239–257.

[11] P. GUREVICH AND S. TIKHOMIROV, *Spatially Discrete Reaction-Diffusion Equations with Discontinuous Hysteresis*, preprint, arXiv:1504.02385 [math.AP], 2015.

[12] M. HELMERS AND M. HERRMANN, *Interface dynamics in discrete forward-backward diffusion equations*, Multiscale Model. Simul., 11 (2013), pp. 1261–1297.

[13] F. C. HOPPENSTEADT AND W. JÄGER, *Pattern formation by bacteria*, in Biological Growth and Spread Lecture Notes in Biomath. 38, Springer, Berlin, 1980, pp. 68–81.

[14] F. C. HOPPENSTEADT, W. JÄGER, AND C. POPPE, *A hysteresis model for bacterial growth patterns*, in Modelling of Patterns in Space and Time, Lecture Notes in Biomath. 55, Springer, Berlin, 1984, pp. 123–134.

[15] A. M. IL'IN AND B. A. MARKOV, *Nonlinear diffusion equation and Liesegang rings*, Dokl. Akad. Nauk, 440 (2011), pp. 164–167 (in Russian), Dokl. Math., 84 (2011), pp. 730–733 (in English).

[16] J. KOPFOVÁ, *Hysteresis in biological models*, J. Phys. Conf. Ser., 55 (2006), pp. 130–134.

[17] A. MARCINIAK-CZOCHRA, *Receptor-based models with hysteresis for pattern formation in hydra*, Math. Biosci., 199 (2006), pp. 97–119.

[18] A. VISINTIN, *Evolution problems with hysteresis in the source term*, SIAM J. Math. Anal.,17 (1986), pp. 1113–1138.

[19] A. VISINTIN, *Differential Models of Hysteresis*, Springer, Berlin, 1994.

[20] A. VISINTIN, *Ten issues about hysteresis*, Acta Appl. Math., 132 (2014), pp. 635–647.