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Pavel Gurevich & Sergey Tikhomirov

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Symmetric Periodic Solutions of Parabolic Problems With Discontinuous Hysteresis

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Abstract We develop a framework for treating the long-term behavior of solutions for parabolic equations in multidimensional domains with discontinuous hysteresis. Bearing in mind the thermostat model, we concentrate in this paper on the prototype heat equation with hysteresis in the boundary condition. We provide an algorithm for constructing all periodic solutions with exactly two switchings on the period and study their stability. Coexistence of several periodic solutions with different stability properties is proved to be possible. A mechanism of appearance and disappearance of periodic solutions is investigated.

Keywords Hysteresis · Parabolic equation · Periodic solution · Stability · Bifurcation

1 Introduction

Hysteresis operators arise in mathematical description of various physical processes [4,19,26]. Models with hysteresis for ordinary differential equations were considered by many authors (see e.g., [1,2,6,19,22,24,25]). Partial differential equations with hysteresis have also been actively studied during the last decades (see [4,26] and the references therein). The primary focus has been on the well-posedness of the corresponding problems and related issues (existence of solutions, uniqueness, regularity, etc.). However, many questions remain open, especially those related to the periodicity and long-time behavior of solutions.

In this paper, we deal with parabolic problems containing a discontinuous hysteresis operator in the boundary condition. Such problems describe processes of thermal control arising in chemical reactors and climate control systems. The temperature regulation in a domain is

P. Gurevich (✉) · S. Tikhomirov
Institute for Mathematics I, Free University of Berlin, Arnimallee 3, 14195 Berlin, Germany
e-mail: gurevichp@gmail.com

S. Tikhomirov
Departamento de Matematica, Pontificia Universidade Catolica do Rio de Janeiro, Rua Marques de Sao Vicente, 225, Edificio Cardeal Leme, Gavea, Rio de Janeiro CEP 22453-900, Brazil
e-mail: sergey.tikhomirov@gmail.com

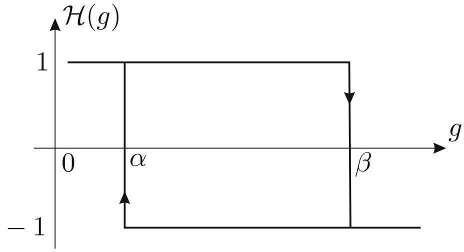


Fig. 1 The hysteresis operator $\mathcal{H}$

performed via heating (or cooling) elements on the boundary of the domain. The regime of the heating elements on the boundary is based on the registration of thermal sensors inside the domain and obeys a hysteresis law.

Let $v(x, t)$ denote the temperature at the point x of a bounded domain $Q \subset \mathbb{R}^n$ at the moment t. We define the *mean temperature* $\hat{v}(t)$ by the formula

$$\hat{v}(t) = \int_Q m(x)v(x, t)\, dx,$$

where m is a given function from the Sobolev space $H^1(Q)$ (see Condition 2.1 for another technical assumption on $m(x)$).

In our prototype model, we assume that the function $v(x, t)$ satisfies the heat equation

$$v_t(x, t) = \Delta v(x, t) \quad (x \in Q,\ t > 0) \tag{1.1}$$

and a boundary condition which involves a hysteresis operator $\mathcal{H}$ depending on the mean temperature $\hat{v}$.

The hysteresis $\mathcal{H}(\hat{v})(t)$ is defined as follows (cf. [19,26] and the accurate definition and Fig. 1 in Sect. 2). One fixes two temperature thresholds α and β ($\alpha < \beta$). If $\hat{v}(t) \le \alpha$, then $\mathcal{H}(\hat{v})(t) = 1$ (the heating is switched on); if $\hat{v}(t) \ge \beta$, then $\mathcal{H}(\hat{v})(t) = -1$ (the cooling is switched on); if the mean temperature $\hat{v}(t)$ is between α and β, then $\mathcal{H}(\hat{v})(t)$ takes the same value as "just before." We say that the hysteresis operator *switches* when it jumps from 1 to -1 or from -1 to 1. The corresponding time moment is called the *switching moment.* Note that the hysteresis phenomenon takes place along with the nonlocal effect caused by averaging of the function $v(x, t)$ over Q.

To be definite, let us assume that one regulates the heat flux through the boundary ∂Q. Then the boundary condition is of the form

$$\frac{\partial v}{\partial \nu} = K(x)\mathcal{H}(\hat{v})(t) \quad (x \in \partial Q,\ t > 0), \tag{1.2}$$

where ν is the outward normal to ∂Q at the point x, K is a given smooth real-valued function (distribution of the heating elements on the boundary).

A similar mathematical model was originally proposed in [9,10]. Generalizations to various phase-transition problems with hysteresis were studied in [4,5,7,15,20]. Some related issues of optimal control were considered in [3]. The most important questions here concern the existence and uniqueness of solutions, the existence of periodic solutions, and long-time behavior of solutions. The latter two questions are especially difficult.

In the case of a *one-dimensional* domain Q (a finite interval, $n = 1$), the periodicity was studied in [8,11,18,23]. Problems with hysteresis on the boundary of a *multidimensional* domain ($n \ge 2$) turn out to be much more complicated. Although one can relatively easily prove the existence (and sometimes uniqueness) of solutions, the issue of finding periodic solutions is still an open question. The main difficulty here is related to the fact that, in general, the solution does not depend on the initial data continuously. The reason is that the solution may intersect the "switching" hyperplane $\{\hat{\varphi} = \alpha\}$ or $\{\hat{\varphi} = \beta\}$ nontransversally (cf. [2,25], where the same phenomenon occurs for ordinary differential equations). This leads to discontinuity of the corresponding Poincaré map. As a result, most methods based on fixed-point theorems do not apply to the Poincaré map.

One possible way to overcome the nontransversality is to consider a continuous model of the hysteresis operator. This was done in [12], where a thermocontrol problem with the Preisach hysteresis operator in the boundary condition was considered and the existence of periodic solutions and global attractors were established. Note that the periodicity and the long-time behavior of solutions were also studied in [17,28] in the situation where a hysteresis operator enters a parabolic equation itself (see also [27] and the references therein).

The first results about periodic solutions of thermocontrol problems in *multidimensional* domains with *discontinuous* hysteresis were obtained in [13]. In [14], a new approach was proposed. It is based on regarding the problem as an infinite-dimensional dynamical system. By using the Fourier method, one can reduce the boundary-value problem for the parabolic equation to infinitely many ordinary differential equations, whose solutions are coupled with each other via the hysteresis operator.

In [14], the existence of a unique periodic solution of the thermocontrol problem is proved for sufficiently large $\beta - \alpha$. This periodic solution possesses certain symmetry, is stable, and is a global attractor. A similar result was established for arbitrary α and β, but $m(x)$ being close to a constant. The idea was to find an invariant region for the corresponding Poincaré map and prove that the Poincaré map is continuous on that region. This turns out to be true for sufficiently large $\beta - \alpha$. However, one can construct examples with small $\beta - \alpha$, where an invariant region exists and even is an attracting set, but the Poincaré map is not continuous on it.

In the present paper, we will show that the requirement for $\beta - \alpha$ to be large enough is essential. We will prove that if $\beta - \alpha$ is small, then unstable periodic solutions may appear. In particular, they may have a saddle structure. To construct those solutions, we will develop a general procedure which yields *all* periodic solutions (with two switchings on the period) in an explicit form. This procedure works even in the presence of discontinuity caused by the above nontransversality. In particular, it allows one to find periodic solutions on which the Poincaré map is discontinuous.

To study stability of periodic solutions (in particular, to find unstable ones), we propose a method which allows one to reduce the original system to an invariant subsystem. The dimension of this subsystem is equal to the number of nonvanishing modes in the Fourier decomposition of the $m(x)$. If $m(x)$ has finitely many nonvanishing modes, then one can explicitly write down the linearization of the reduced system and find all the eigenvalues. They provide complete information about the stability of the periodic solution.

The invariant subsystem corresponding to the nonvanishing modes of $m(x)$ is called *guiding*. The remaining subsystem is called *guided*. We prove that the full system (i.e., the original problem) has a periodic solution whenever the guiding system has one. Moreover, the periodic solution of the full system is a global attractor (is stable, uniformly exponentially stable) whenever the periodic solution of the guiding system possesses those properties. We call these results *conditional existence* of periodic solutions, *conditional attractivity*, and *conditional stability*, respectively. The above "guiding-guided" decomposition is a result of independent interest. It generalizes the results of [13], where $m(x) \equiv$ const (in our terminology, this corresponds to $m(x)$ which has only one nonvanishing mode).

The paper is organized as follows. In Sect. 2, we define the hysteresis operator, formulate the problem, introduce a notion of solution, recall some properties of the solutions, and reduce the problem to an infinite-dimensional dynamical system. In the end of Sect. 2, we define the guiding and the guided subsystems and introduce the corresponding decomposition of the phase space (the Sobolev space $H^1(Q)$). Most results of this section are proved in [14].

In Sect. 3, we give a notion of periodic solution with two switchings on the period. By using the Poincaré maps of the guiding system and the full system, we prove conditional existence of periodic solutions, conditional attractivity, and conditional stability. The latter two results are proved under assumption that the periodic solution of the guiding system intersects the hyperplanes $\{\hat{\varphi} = \alpha\}$ and $\{\hat{\varphi} = \beta\}$ at the switching moments transversally. The transversality implies the continuity (and even the Fréchet differentiability) of the Poincaré maps in a neighborhood of the periodic solution. However, we require neither that this neighborhood be invariant under the Poincaré map, nor that the Poincaré map be continuous in a (bigger) invariant neighborhood (which exists due to [14]).

In Sect. 4, we show that any periodic solution with two switchings on the period possesses a symmetry in the phase space. By using this symmetry, we develop an algorithm which allows us

1. to construct *all* periodic solutions (with two switchings on the period) in an explicit form for any given α and β;
2. to find a sufficient condition under which periodic solutions exist for all sufficiently small $\beta - \alpha$.
3. to define bifurcation points where periodic solutions may appear or disappear; a role of a bifurcation parameter is played either by the period or by the difference $\beta - \alpha$;

Furthermore, using the results about the guiding-guided decomposition from Sect. 3, we construct examples in which periodic solutions are stable or unstable, respectively. In the "unstable" case, we show that they may have a saddle structure.

As a conclusion, we note that the developed method can also be applied to the study of the Dirichlet or Robin boundary conditions. Moreover, one can study the problem where the heat flux through the boundary (in the case of the Neumann boundary condition) changes continuously. Mathematically, this means that the boundary condition (1.2) is replaced by

$$\frac{\partial v}{\partial \nu} = K(x)u(t) \quad (x \in \partial Q,\ t > 0),$$
$$au'(t) + u(t) = \mathcal{H}(\hat{v})(t)$$

with $a > 0$ (cf. [9,10,13,23]).

2 Setting of the Problem: Reduction to Infinite Dynamical System

2.1 Setting of the Problem

Let $Q \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with smooth boundary. Let $L_2 = L_2(Q)$. Denote by $H^1 = H^1(Q)$ the Sobolev space with the norm

$$\|\psi\|_{H^1} = \left(\int_Q (|\psi(x)|^2 + |\nabla \psi(x)|^2)\, dx \right)^{1/2}.$$

Let $H^{1/2} = H^{1/2}(\partial Q)$ be the space of traces on ∂Q of the functions from H^1.

Consider the sets $Q_T = Q \times (0, T)$ and $\Gamma_T = \partial Q \times (0, T)$, $T > 0$. Fix functions $K \in H^{1/2}$ and $m \in H^1$ and real numbers α and β, $\beta > \alpha$.

For any function $\varphi(x)$ or $v(x, t)$ $(x \in Q, t \geq 0)$, the symbol $\hat{}$ will refer to the "average" of the function:

$$\hat{\varphi} = \int_Q m(x)\varphi(x)\,dx, \qquad \hat{v}(t) = \int_Q m(x)v(x,t)\,dx.$$

Let $v(x, t)$ denote the temperature at the point $x \in Q$ at the moment $t \geq 0$ satisfying the heat equation

$$v_t(x, t) = \Delta v(x, t) \quad ((x, t) \in Q_T) \tag{2.1}$$

with the initial condition

$$v|_{t=0} = \varphi(x) \quad (x \in Q) \tag{2.2}$$

and the boundary condition

$$\left.\frac{\partial v}{\partial \nu}\right|_{\Gamma_T} = K(x)\mathcal{H}(\hat{v})(t) \quad ((x, t) \in \Gamma_T). \tag{2.3}$$

Here ν is the outward normal to Γ_T at the point (x, t) and $\mathcal{H}$ is a hysteresis operator, which we now define.

We denote by $BV(0, T)$ the Banach space of real-valued functions having finite total variation on the segment $[0, T]$ and by $C_r[0, T)$ the linear space of functions which are continuous on the right in $[0, T)$. We introduce the *hysteresis operator* (cf. [19,26])

$$\mathcal{H} : C[0, T] \to BV(0, T) \cap C_r[0, T)$$

by the following rule. For any $g \in C[0, T]$, the function $h = \mathcal{H}(g) : [0, T] \to \{-1, 1\}$ is defined as follows. Let $X_t = \{t' \in (0, t] : g(t') = \alpha \text{ or } \beta\}$; then

$$h(0) = \begin{cases} 1 & \text{if } g(0) < \beta, \\ -1 & \text{if } g(0) \geq \beta \end{cases}$$

and for $t \in (0, T]$

$$h(t) = \begin{cases} h(0) & \text{if } X_t = \varnothing, \\ 1 & \text{if } X_t \neq \varnothing \text{ and } g(\max X_t) = \alpha, \\ -1 & \text{if } X_t \neq \varnothing \text{ and } g(\max X_t) = \beta \end{cases}$$

(see Fig. 1). A point τ such that $\mathcal{H}(g)(\tau) \neq \mathcal{H}(g)(\tau - 0)$ is called a *switching moment of* $\mathcal{H}(g)$.

We assume throughout that the following condition holds.

Condition 2.1 *The coefficient $K(x)$ in the boundary condition (2.3) and the weight function $m(x)$ satisfy*

$$\int_{\partial Q} K(x)\,d\Gamma > 0, \qquad \int_Q m(x)\,dx > 0. \tag{2.4}$$

Remark 2.1 From the physical viewpoint, the function $K(x)$ characterizes the density of the heating (or cooling) elements on the boundary and $m(x)$ characterizes the density of thermal sensors in the domain. Clearly, inequalities (2.4) hold in the physically relevant case $K(x) \geq 0$ for a.e. $x \in \partial Q$, $K(x) \not\equiv 0$, and $m(x) \geq 0$ for a.e. $x \in Q$, $m(x) \not\equiv 0$.

2.2 Functional Spaces and the Solvability of the Problem

For any Banach space B, denote by $C([a, b]; B)$ $(a < b)$ the space of B-valued functions continuous on the segment $[a, b]$ with the norm

$$\|u\|_{C([a,b];B)} = \max_{t\in[a,b]} \|u(t)\|_B$$

and by $L_2((a, b); B)$ the space of L_2-integrable B-valued functions with the norm

$$\|u\|_{L_2((a,b);B)} = \left(\int_a^b \|u(t)\|_B\, dt\right)^{1/2}.$$

We introduce the anisotropic Sobolev space $H^{2,1}(Q \times (a, b)) = \{v \in L_2((a, b); H^2) : v_t \in L_2((a, b); L_2)\}$ with the norm

$$\|v\|_{H^{2,1}(Q\times(a,b))} = \left(\int_a^b \|v(\cdot,t)\|_{H^2}^2\, dt + \int_a^b \|v_t(\cdot,t)\|_{L_2}^2\, dt\right)^{1/2}.$$

Taking into account the results of the interpolation theory (see, e.g., [21, Chap. 1, Sects. 1–3, 9], we make the following remarks.

Remark 2.2 The continuous embedding $H^{2,1}(Q \times (a, b)) \subset C([a, b], H^1)$ takes place. Furthermore, for any $v \in H^{2,1}(Q \times (a, b))$ and $\tau \in [a, b]$, the trace $v|_{t=\tau} \in H^1$ is well defined and is a bounded operator from $H^{2,1}(Q \times (a, b))$ to H^1.

Remark 2.3 Consider two functions $v_1 \in H^{2,1}(Q \times (a, b))$ and $v_2 \in H^{2,1}(Q \times (b, c))$, where $a < b < c$. Let $v(\cdot, t) = v_1(\cdot, t)$ for $t \in (a, b)$ and $v(\cdot, t) = v_2(\cdot, t)$ for $t \in (b, c)$. Then $v \in H^{2,1}(Q \times (a, c))$ if and only if $v_1|_{t=b} = v_2|_{t=b}$.

Definition 2.1 A function $v(x, t)$ is called a solution of problem (2.1)–(2.3) in Q_T with the initial data $\varphi \in H^1$ if $v \in H^{2,1}(Q_T)$ and v satisfies Eq. 2.1 a.e. in Q_T and conditions (2.2), (2.3) in the sense of traces.

Definition 2.2 We say that $v(x, t)$ $(t \ge 0)$ is a *solution of problem* (2.1)–(2.3) *in* Q_∞ if it is a solution in Q_T for all $T > 0$.

The following result about the solvability of problem (2.1)–(2.3) is proved in [14, Theorem 2.2].

Theorem 2.1 *Let* $\varphi \in H^1$ *and* $\|\varphi\|_{H^1} \le R$ ($R > 0$ *is arbitrary*). *Then there exists a unique solution* v *of problem* (2.1)–(2.3) *in* Q_∞ *and the following holds for any* $T > 0$.

1. *One has*

$$\|v(\cdot,t)\|_{H^1} \le c_0\|v\|_{H^{2,1}(Q_T)} \le c_1(\|\varphi\|_{H^1} + \|K\|_{H^{1/2}}), \tag{2.5}$$

where $c_0 = c_0(T) > 0$ *and* $c_1 = c_1(T) > 0$ *do not depend on* φ *and* R;

2. *The interval* $(0, T]$ *contains no more than finitely many switching moments* $t_1 < t_2 < \dots < t_J$ *of* $\mathcal{H}(\hat{v})$. *Moreover,*

$$t_i - t_{i-1} \le t^* + \frac{2(\beta-\alpha)}{m_0 K_0}, \quad i = 1, 2, \dots, \tag{2.6}$$

where t^* *depends on* m *and* R *but does not depend on* $\varphi, T, \alpha, \beta$;

$$t_i - t_{i-1} \ge \tau^*, \quad i = \begin{cases} 1, 2, \dots, J & \text{if } \hat{\varphi} \le \alpha \text{ or } \hat{\varphi} \ge \beta, \\ 2, 3, \dots, J & \text{if } \alpha < \hat{\varphi} < \beta, \end{cases} \tag{2.7}$$

where

$$\tau^* = \text{const}\frac{(\beta-\alpha)^2}{\|m\|_{L_2}^2} \tag{2.8}$$

with const > 0 *depending on* R *rather than on* $m, \varphi, T, \alpha, \beta$; *in* (2.6) *and* (2.8), $t_0 = 0$.

2.3 Reduction to Infinite-Dimensional Dynamical System

Due to Theorem 2.1, the study of the solutions of problem (2.1)–(2.3) with hysteresis can be reduced to the study of the solutions of parabolic problems without hysteresis by considering the time intervals between the switching moments t_i.

Thus, if $\mathcal{H}(\hat{v})(t) \equiv 1$, then problem (2.1)–(2.3) takes the form

$$v_t(x,t) = \Delta v(x,t) \quad ((x,t)\in Q_T), \tag{2.9}$$

$$v(x,0) = \varphi(x) \quad (x \in Q), \tag{2.10}$$

$$\left.\frac{\partial v}{\partial \nu}\right|_{\Gamma_T} = K(x) \quad ((x,t)\in \Gamma_T). \tag{2.11}$$

If $\mathcal{H}(\hat{v})(t) \equiv -1$, one should replace $K(x)$ by $-K(x)$ in (2.11).

Definition 2.3 A function $v(x,t)$ is called a solution of problem (2.9)– (2.11) in Q_T if $v \in H^{2,1}(Q_T)$ and v satisfies Eq. 2.9 a.e. in Q_T and conditions (2.10), (2.11) in the sense of traces.

It is well known that there is a unique solution $v \in H^{2,1}(Q_T)$ of problem (2.9)–(2.11).

Now we give a convenient representation of solutions of problem (2.9)–(2.11) in terms of the Fourier series with respect to the eigenfunctions of the Laplacian.

Let $\{\lambda_j\}_{j=0}^{\infty}$ and $\{e_j(x)\}_{j=0}^{\infty}$ denote the sequence of eigenvalues and the corresponding system of real-valued eigenfunctions (infinitely differentiable in $\overline{Q}$) of the spectral problem

$$-\Delta e_j(x) = \lambda_j e_j(x) \quad (x\in Q), \qquad \left.\frac{\partial e_j}{\partial \nu}\right|_{\partial Q} = 0. \tag{2.12}$$

It is well known that $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \ldots$, $e_0(x) \equiv (\text{mes}\, Q)^{-1/2} > 0$, and the system of eigenfunctions $\{e_j\}_{j=0}^{\infty}$ can be chosen to form an orthonormal basis for L_2. Then, the functions $e_j/\sqrt{\lambda_j+1}$ form an orthonormal basis for H^1.

Remark 2.4 In what follows, we will use the well-known asymptotics for the eigenvalues $\lambda_j = Lj^{2/n} + o(j^{2/n})$ as $j \to +\infty$ ($L > 0$ and n is the dimension of Q).

Any function $\psi \in L_2$ can be expanded into the Fourier series with respect to $e_j(x)$, which converges in L_2:

$$\psi(x) = \sum_{j=0}^{\infty} \psi_j e_j(x), \qquad \|\psi\|_{L_2}^2 = \sum_{j=0}^{\infty} |\psi_j|^2, \tag{2.13}$$

where $\psi_j = \int_Q \psi(x) e_j(x)\,dx$. If $\psi \in H^1$, then the first series in (2.13) converges to ψ in H^1 and

$$\|\psi\|_{H^1}^2 = \sum_{j=0}^{\infty} (1+\lambda_j)|\psi_j|^2. \tag{2.14}$$

Denote

$$m_j = \int_Q m(x)e_j(x)\,dx, \qquad K_j = \int_{\partial Q} K(x)e_j(x)\,dx \qquad (j = 0, 1, 2, \ldots). \tag{2.15}$$

Note that $m_0, K_0 > 0$ due to Condition 2.1. We also note that K_j are not the Fourier coefficients of $K(x)$. However, the following is proved in [14]:

$$\sum_{j=1}^{\infty} \left(\frac{|K_j|^2}{\lambda_j^2} + \frac{|K_j|^2}{\lambda_j} \right) \le c\|K\|_{H^{1/2}}^2, \tag{2.16}$$

where $c > 0$ does not depend on K.

Remark 2.5 Using (2.14) and (2.16), we obtain the estimate

$$\sum_{j=1}^{\infty} |m_j K_j| \le c\|m\|_{H^1}\|K\|_{H^{1/2}},$$

which will often be used later on.

The numbers m_j and K_j play an essential role when one describes the thermocontrol problem in terms of an infinite-dimensional dynamical system. The following result is true (see [14, Lemma 2.2]).

Theorem 2.2 *Let $\varphi \in H^1$. Then the following assertions hold.*

1. *The solution v of problem (2.9)–(2.11) can be represented as the series*

$$v(x,t) = \sum_{j=0}^{\infty} v_j(t)e_j(x), \quad t \ge 0, \tag{2.17}$$

where $v_j(t) = \int_Q v(x,t)e_j(x)\,dx$ and $v_j(t)$ satisfy the Cauchy problem

$$\dot{v}_j(t) = -\lambda_j v_j(t) + K_j, \qquad v_j(0) = \varphi_j \qquad (\dot{} = d/dt,\ j = 0, 1, 2 \ldots). \tag{2.18}$$

The series in (2.17) converges in H^1 for all $t \ge 0$.

2. *The mean temperature $\hat{v}(t)$ is represented by the absolutely convergent series*

$$\hat{v}(t) = \sum_{j=0}^{\infty} m_j v_j(t), \quad t \ge 0, \tag{2.19}$$

which is continuously differentiable for $t > 0$.

Remark 2.6 In what follows, we will also use the explicit formulas for the solutions of Eq. 2.18

$$v_0(t) = \varphi_0 + K_0 t, \qquad v_j(t) = \left(\varphi_j - \frac{K_j}{\lambda_j} \right) e^{-\lambda_j t} + \frac{K_j}{\lambda_j}, \quad j = 1, 2, \ldots.$$

Formally, relations (2.18) can be obtained by multiplying (2.9) by $e_j(x)$, integrating by parts over Q, and substituting $v(x,t) = \sum_{j=0}^{\infty} v_j(t)e_j(x)$. The rigorous proof is given in [14].

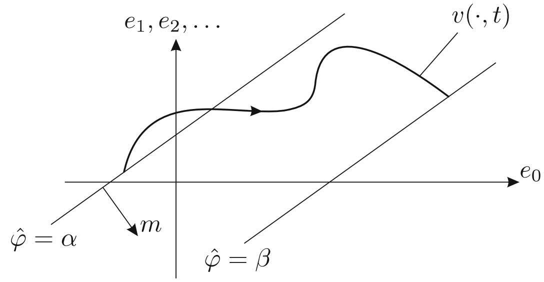


Fig. 2 The plane spanned by $e_0 = (1, 0, 0, \dots)$ and $m = (m_0, m_1, m_2, \dots)$

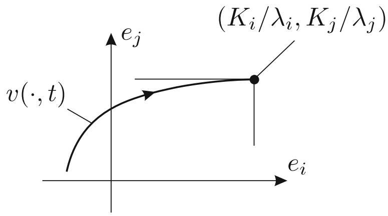


Fig. 3 The plane spanned by e_i and e_j, $i \neq j$, $i, j \geq 1$

A geometrical interpretation of the dynamics of $v_0(t), v_1(t), \dots$ is as follows. We choose the orthonormal basis in L_2 (which is orthogonal in H^1) consisting of the eigenfunctions $e_0, e_1, e_2, \dots$. Then, in the coordinate form, we have

$$e_0 = (1, 0, 0, 0, \dots), \quad e_1 = (0, 1, 0, 0, \dots), \quad e_2 = (0, 0, 1, 0, \dots), \quad \dots$$

and (cf. (2.17))

$$\varphi = (\varphi_0, \varphi_1, \varphi_2, \dots), \qquad v(\cdot, t) = (v_0(t), v_1(t), v_2(t), \dots).$$

Consider the plane going through the origin and spanned by the vector $e_0 = (1, 0, 0, \dots)$ and the vector $m = (m_0, m_1, m_2, \dots)$ (if they are parallel, i.e., $m_1 = m_2 = \dots = 0$, then we consider an arbitrary plane containing e_0). We note that the angle between the vectors m and e_0 is acute (their scalar product is equal to $m_0 > 0$). Clearly, the orthogonal projection of the hyperspace $\hat{\varphi} = \sum_{j=0}^{\infty} m_j \varphi_j = \alpha$ (or β) on this plane is a line (see Fig. 2).

Due to (2.18), $v_0(t)$ "goes" from the left to the right with the constant speed $K_0 > 0$, while $v_j(t)$ exponentially converge to K_j/λ_j (see Fig. 3).

Due to Theorem 2.2, the original problem (2.1)–(2.3) can be written as follows:

$$\begin{aligned} &\dot{v}_0(t) = \mathcal{H}(\hat{v})(t)K_0, \qquad v_0(0) = \varphi_0, \\ &\dot{v}_j(t) = -\lambda_j v_j(t) + \mathcal{H}(\hat{v})(t)K_j, \quad v_j(0) = \varphi_j \qquad (j = 1, 2 \dots). \end{aligned} \tag{2.20}$$

Equations (2.20) define an infinite-dimensional dynamical system for the functions $v_j(t)$. These functions are "coupled" via formula (2.19) for the mean temperature, which is the argument of the hysteresis operator $\mathcal{H}$.

2.4 Invariant Subsystem and "Guiding-Guided" Decomposition

In this subsection, we show that if some coefficients m_j vanish, then the system (2.20) has an invariant subsystem.

We introduce the sets of indices

$$\mathbb{J} = \{j \in \mathbb{N} : m_j \neq 0\}, \qquad \mathbb{J}_0 = \{j \in \mathbb{N} : m_j = 0\}.$$

Clearly, $\{0\} \cup \mathbb{J} \cup \mathbb{J}_0 = \{0, 1, 2, \dots\}$.

Note that, for any solution $v(x,t)$ of problem (2.1)–(2.3), we have (cf. (2.19))

$$\hat{v}(t) = \sum_{j \in \{0\}\cup\mathbb{J}} m_j v_j(t), \quad t \geq 0.$$

Therefore, the dynamics of $v_j(t)$, $j \in \{0\}\cup\mathbb{J}$, does not depend on the functions $v_j(t)$, $j \in \mathbb{J}_0$, and is described by the invariant dynamical system

$$\begin{aligned} &\dot{v}_0(t) = \mathcal{H}(\hat{v})(t)K_0, \qquad v_0(0) = \varphi_0, \\ &\dot{v}_j(t) = -\lambda_j v_j(t) + \mathcal{H}(\hat{v})(t)K_j, \quad v_j(0) = \varphi_j \qquad (j \in \mathbb{J}). \end{aligned} \tag{2.21}$$

The dynamics of $v_j(t)$, $j \in \mathbb{J}_0$, is described by the system

$$\dot{v}_j(t) = -\lambda_j v_j(t) + \mathcal{H}(\hat{v})(t)K_j, \qquad v_j(0) = \varphi_j \qquad (j \in \mathbb{J}_0), \tag{2.22}$$

where the hysteresis operator $\mathcal{H}$ depends only on the functions $v_j(t)$ from the system (2.21).

Definition 2.4 We say that the system (2.21) is *guiding*, while the system (2.22) is *guided* (by (2.21)).

In what follows, we will use the following notation. For any number φ_0 and (possibly, infinite-dimensional) vectors $\{\varphi_j\}_{j\in\mathbb{J}}$ and $\{\varphi_j\}_{j\in\mathbb{J}_0}$ ($\varphi_j \in \mathbb{R}$), we denote

$$\boldsymbol{\varphi} = \{\varphi_j\}_{j\in\mathbb{J}}, \qquad \tilde{\boldsymbol{\varphi}} = \{\varphi_j\}_{j\in\{0\}\cup\mathbb{J}}, \qquad \boldsymbol{\varphi}_0 = \{\varphi_j\}_{j\in\mathbb{J}_0}.$$

Thus, e.g., $\tilde{\mathbf{v}}(t)$ and $\mathbf{v}_0(t)$ will represent the solutions of the guiding system (2.21) and the guided system (2.22), respectively.

The above decomposition of the system (2.20) implies the corresponding decomposition of the phase space H^1:

$$H^1 = \mathbb{R} \times V \times V_0 = \tilde{V} \times V_0, \tag{2.23}$$

where the norms in V, $\tilde{V}$, and V_0 are given by

$$\begin{aligned} \|\boldsymbol{\varphi}\|_V &= \left(\sum_{j\in\mathbb{J}} (1+\lambda_j)|\varphi_j|^2\right)^{1/2}, \quad \|\tilde{\boldsymbol{\varphi}}\|_{\tilde{V}} = \left(\sum_{j\in\{0\}\cup\mathbb{J}} (1+\lambda_j)|\varphi_j|^2\right)^{1/2}, \\ \|\boldsymbol{\varphi}_0\|_{V_0} &= \left(\sum_{j\in\mathbb{J}_0} (1+\lambda_j)|\varphi_j|^2\right)^{1/2}. \end{aligned} \tag{2.24}$$

Further we show that Definition 2.4 is quite natural. In particular, we prove that if $\tilde{\mathbf{z}}(t)$ is a periodic solution of the guiding system (2.21), then there exists a periodic solution of the full system (2.20) of the form $(\tilde{\mathbf{z}}(t), \mathbf{z}_0(t))$. Moreover, the latter is stable if and only if $\tilde{\mathbf{z}}(t)$ is a stable periodic solution of the guiding system (2.21).

As an application of this result, assuming that the set $\mathbb{J}$ is finite, we will construct a periodic solution $z(x,t)$ with small period such that $\tilde{\mathbf{z}}(t)$ is an unstable periodic solution of the guiding system (2.21). Clearly, $z(x,t)$ will be unstable in this case, too.

3 Periodic Solutions

3.1 Conditional Existence of Periodic Solution

We begin with a definition of periodic solutions (with two switchings on the period) of problem (2.1), (2.3). Recall that the symbol ˆ refers to the "average" of the function (see Sect. 2.1).

Definition 3.1 A function $z(x,t)$ is called an (s,σ)*-periodic solution* (*with period* $T = s+\sigma$) *of problem* (2.1), (2.3) if there is a function $\psi \in H^1$ such that the following holds:

1. $\hat{\psi} = \alpha$,
2. $z(x,t)$ is a solution of problem (2.1)–(2.3) (in Q_∞) with the initial data ψ,
3. there are exactly two switching moments s and T of $\mathcal{H}(\hat{z})$ on the interval $(0,T]$ (such that $\hat{z}(s) = \beta$ and $\hat{z}(T) = \alpha$),
4. $z(x,T) = z(x,0)$ $(= \psi(x))$.

Definition 3.2 If $z(x,t)$ is an (s,σ)-periodic solution of problem (2.1), (2.3) and $T = s+\sigma$, then the sets

$$\Gamma = \{z(\cdot,t),\ t \in [0,T]\}, \qquad \tilde{\Gamma} = \{\tilde{\mathbf{z}}(t) : t \in [0,T]\}, \qquad \Gamma_0 = \{\mathbf{z}_0(t) : t \in [0,T]\}.$$

are called the *trajectories* of $z(x,t)$, $\tilde{\mathbf{z}}(t)$, and $\mathbf{z}_0(t)$, respectively.

We also consider two parts of the trajectory corresponding to the hysteresis value $\mathcal{H}(\hat{z}) = 1$ and -1:

$$\Gamma_1 = \{z(\cdot,t),\ t \in [0,s]\}, \qquad \Gamma_2 = \{z(\cdot,t),\ t \in [s,T]\}.$$

Similarly, one introduces the sets $\tilde{\Gamma}_j$ and Γ_{0j}, $j = 1,2$.

Remark 3.1 It follows from the definition of the hysteresis operator $\mathcal{H}$ and from Definition 3.1 that if $z(x,t)$ is an (s,σ)-periodic solution with period $T = s+\sigma$ of problem (2.1), (2.3), then

$$\mathcal{H}(\hat{z})(t) = 1, \quad t \in [0,s); \qquad \mathcal{H}(\hat{z})(t) = -1, \quad t \in [s,T).$$

Further in this section, we establish the connection between periodic solutions of the guiding system (2.21) and those of the full system (2.20). The definitions of (s,σ)-periodic solutions for the guiding system (2.21) and for the full system (2.20) are analogous to Definition 3.1.

The following theorem generalizes Theorem 4.4 in [13], where $m_0 \neq 0$ and $m_1 = m_2 = \cdots = 0$ (i.e., $\mathbb{J} = \varnothing$ and $\mathbb{J}_0 = \mathbb{N}$).

Theorem 3.1 *Let* $\tilde{\mathbf{z}}(t)$ *be an* (s,σ)*-periodic solution of the guiding system* (2.21). *Then there exists a unique function* $\mathbf{z}_0(t)$ *such that* $(\tilde{\mathbf{z}}(t), \mathbf{z}_0(t))$ *is an* (s,σ)*-periodic solution of the full system* (2.20) (*which generates an* (s,σ)*-periodic solution* $z(x,t)$ *of problem* (2.1), (2.3)).

Proof We recall that the spaces $\tilde{V}$ and V_0 form the decomposition of H^1 (cf. (2.23)).

We introduce a nonlinear operator $\mathbf{M}_T : V_0 \to V_0$ as follows. For any $\boldsymbol{\varphi}_0 \in V_0$, we consider the element $(\tilde{\mathbf{z}}(0), \boldsymbol{\varphi}_0) \in H^1$. By Theorem 2.1 and the invariance of the guiding system (2.21), there is a unique solution of the full system (2.20), which is of the form $(\tilde{\mathbf{z}}(t), \mathbf{v}_0(t)) \in H^1$. Clearly, $\mathbf{v}_0(t)$ is a solution of the guided system (2.22). We set

$$\mathbf{M}_T(\boldsymbol{\varphi}_0) = \mathbf{v}_0(T), \qquad T = s+\sigma.$$

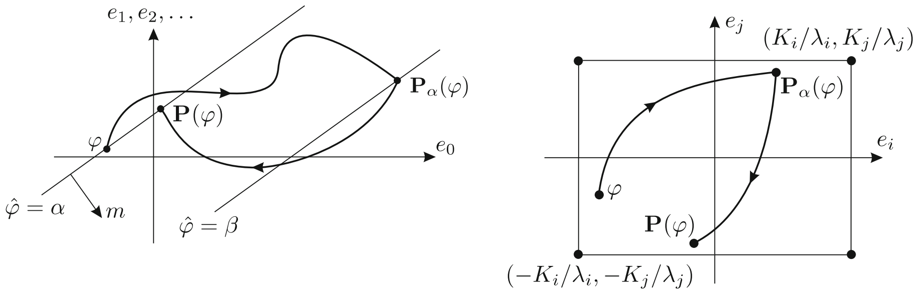


Fig. 4 The operators $\mathbf{P}_\alpha$ and $\mathbf{P} = \mathbf{P}_\beta \mathbf{P}_\alpha$ on the planes (e_0, m) and (e_i, e_j), $i \neq j$

We claim that $\mathbf{M}_T$ is a contraction map. Indeed, let $\boldsymbol{\varphi}_0^1, \boldsymbol{\varphi}_0^2 \in V_0$ and let $\mathbf{v}_0^1(t)$ and $\mathbf{v}_0^2(t)$ be the corresponding solutions of the guided system (2.22). Since the mean temperature is defined via $\tilde{\mathbf{z}}(t)$ and does not depend on $\mathbf{v}_0^1(t)$ and $\mathbf{v}_0^2(t)$, it follows that the difference $\mathbf{w}_0(t) = \mathbf{v}_0^1(t) - \mathbf{v}_0^2(t)$ satisfies the equations

$$\dot{w}_j(t) = -\lambda_j w_j(t), \qquad w_j(0) = \varphi_j^1 - \varphi_j^2 \qquad (j \in \mathbb{J}_0).$$

Therefore,

$$\|\mathbf{M}_T(\boldsymbol{\varphi}_0^1) - \mathbf{M}_T(\boldsymbol{\varphi}_0^2)\|_{V_0}^2 = \|\mathbf{w}_0(T)\|_{V_0}^2 = \sum_{j \in \mathbb{J}_0} (1 + \lambda_j) e^{-2\lambda_j T}$$
$$|\varphi_j^1 - \varphi_j^2|^2 \leq e^{-2\varkappa T} \|\boldsymbol{\varphi}_0^1 - \boldsymbol{\varphi}_0^2\|_{V_0}^2,$$

where $\varkappa = \min\limits_{j \in \mathbb{J}_0} \lambda_j > 0$.

Thus, $\mathbf{M}_T$ has a unique fixed point $\boldsymbol{\psi}_0 \in V_0$, which yields the desired (s, σ)-periodic solution $(\tilde{\mathbf{z}}(t), \mathbf{z}_0(t))$ of the full system (2.20). □

Further, we will study the connection between the stability and attractivity of solutions of the guiding system and the guided and full systems. To do so, we need to define the Poincaré maps of the respective systems.

3.2 The Poincaré Maps

In this subsection, we introduce the Poincaré maps for the full system (2.20) and for the guiding system (2.22). It is proved in [14] that the stability of a periodic solution of the full system follows from the stability of the corresponding fixed point of the Poincaré map. Therefore, we will concentrate on the properties of the Poincaré map.

We consider nonlinear operators (see Fig. 4)

$$\mathbf{P}_\alpha : \{\varphi \in H^1 : \hat{\varphi} < \beta\} \to \{\varphi \in H^1 : \hat{\varphi} = \beta\},$$
$$\mathbf{P}_\beta : \{\varphi \in H^1 : \hat{\varphi} > \alpha\} \to \{\varphi \in H^1 : \hat{\varphi} = \alpha\}$$

defined as follows.

Let $\varphi \in H^1, \hat{\varphi} < \beta$, and let $v(x, t)$ be the corresponding solution of problem (2.1)–(2.3) in (Q_∞). Due to Theorem 2.1, there exists the first switching moment t_1 such that $\hat{v}(t_1) = \beta$ and there are no other switchings on the interval $(0, t_1)$. In other words, the function $v^\alpha(x, t) := v(x, t)$ is a solution of the initial boundary-value problem on the interval $(0, t_1)$:

$$v_t^\alpha(x,t) = \Delta v^\alpha(x,t) \quad ((x,t) \in Q_{t_1}), \tag{3.1}$$
$$v^\alpha(x,0) = \varphi(x) \quad (x \in Q), \tag{3.2}$$
$$\frac{\partial v^\alpha}{\partial \nu} = K(x) \quad ((x,t) \in \Gamma_{t_1}). \tag{3.3}$$

We set $\mathbf{P}_\alpha(\varphi) = v^\alpha(\cdot, t_1)$.

The operator $\mathbf{P}_\beta$ is defined in a similar way. Let $\varphi \in H^1$ and $\hat{\varphi} > \alpha$. As before, there is a moment $\tau_2 > 0$ and a function $v^\beta(x,t)$ such that $v^\beta(x,t)$ is a solution of the problem

$$v_t^\beta(x,t) = \Delta v^\beta(x,t) \quad ((x,t) \in Q_{\tau_2}), \tag{3.4}$$
$$v^\beta(x,0) = \varphi(x) \quad (x \in Q), \tag{3.5}$$
$$\frac{\partial v^\beta}{\partial \nu} = -K(x) \quad ((x,t) \in \Gamma_{\tau_2}), \tag{3.6}$$

$\widehat{v^\beta}(\tau_2) > \alpha$ for $t < \tau_2$, and $\widehat{v^\beta}(\tau_2) = \alpha$. We set $\mathbf{P}_\beta(\varphi) = v^\beta(\cdot, \tau_2)$.

We introduce the *Poincaré map* for problem (2.1)–(2.3), or, equivalently, for the full system (2.20)

$$\mathbf{P} : \{\varphi \in H^1 : \hat{\varphi} < \beta\} \to \{\varphi \in H^1 : \hat{\varphi} = \alpha\},$$
$$\mathbf{P} = \mathbf{P}_\beta \mathbf{P}_\alpha.$$

We also introduce the operator (functional) $\mathbf{t}_1 : \{\varphi \in H^1 : \hat{\varphi} < \beta\} \to \mathbb{R}$ given by

$$\mathbf{t}_1(\varphi) = \text{the first switching moment of } \mathcal{H}(\hat{v}) \text{ for system (2.1)–(2.3).}$$

We will use the following result (see Remark 4.3 in [14]).

Lemma 3.1 *Let $z(x,t)$ be a periodic solution of problem* (2.1), (2.3). *If*

$$\frac{d\hat{z}}{dt} \neq 0 \quad \textit{at the switching moments},$$

then the operators $\mathbf{P}_\alpha(\varphi), \mathbf{P}(\varphi)$, and $\mathbf{t}_1(\varphi)$ are continuously differentiable in a neighborhood of $\Gamma_1 \cap \{\varphi \in H^1 : \hat{\varphi} < \beta\}$. The operator $\mathbf{P}_\beta(\varphi)$ is continuously differentiable in a neighborhood of $\Gamma_2 \cap \{\varphi \in H^1 : \hat{\varphi} > \alpha\}$.

We denote by $\tilde{\mathbf{E}} : H^1 \to \tilde{V}$ the orthogonal projector from H^1 onto $\tilde{V}$.

Similarly to the operators $\mathbf{P}_\alpha, \mathbf{P}_\beta, \mathbf{P}$, and $\mathbf{t}_1$, we introduce the operators $\tilde{\Pi}_\alpha, \tilde{\Pi}_\beta, \tilde{\Pi}$, and $\tilde{\mathbf{t}}_1$, respectively, corresponding to the invariant guiding system (2.21) and defined on the elements from $\tilde{V}$. Due to the invariance of (2.21), we have

$$\tilde{\Pi}_\alpha(\tilde{\boldsymbol{\varphi}}) = \tilde{\mathbf{E}}\mathbf{P}_\alpha(\tilde{\boldsymbol{\varphi}}, \boldsymbol{\varphi}_0), \quad \tilde{\Pi}_\beta(\tilde{\boldsymbol{\varphi}}) = \tilde{\mathbf{E}}\mathbf{P}_\beta(\tilde{\boldsymbol{\varphi}}, \boldsymbol{\varphi}_0), \quad \tilde{\Pi}(\tilde{\boldsymbol{\varphi}}) = \tilde{\mathbf{E}}\mathbf{P}(\tilde{\boldsymbol{\varphi}}, \boldsymbol{\varphi}_0), \quad \tilde{\mathbf{t}}_1(\tilde{\boldsymbol{\varphi}}) = \mathbf{t}_1(\tilde{\boldsymbol{\varphi}}, \boldsymbol{\varphi}_0)$$
$$\forall \tilde{\boldsymbol{\varphi}} \in \tilde{V}, \ \forall \boldsymbol{\varphi}_0 \in V_0.$$

We say that $\tilde{\Pi}$ is the *guiding Poincaré map*.

The following theorem shows that the stability of (exponential convergence to) a fixed point of the guiding Poincaré map $\tilde{\Pi}$ implies the stability of (exponential convergence to) the fixed point of the Poincaré map $\mathbf{P}$ of the full system.

First, we introduce some notation. Let $z(x,t)$ be an (s,σ)-periodic solution with period $T = s + \sigma$ of problem (2.1), (2.3) and $(\tilde{\mathbf{z}}(t), \mathbf{z}_0(t))$ the corresponding periodic solution of the full system (2.20). We denote

$$\tilde{\boldsymbol{\psi}} = \tilde{\mathbf{z}}(0), \qquad \boldsymbol{\psi}_0 = \mathbf{z}_0(0).$$

Let $v(x,t)$ be another solution of problem (2.1)–(2.3) such that $\hat{v}(0)=\alpha$, and let $(\tilde{\mathbf{v}}(t),\mathbf{v}_0(t))$ be the corresponding solution of the full system (2.20). The initial data will be denoted by

$$\tilde{\boldsymbol{\varphi}}=\tilde{\mathbf{v}}(0), \qquad \boldsymbol{\varphi}_0=\mathbf{v}_0(0)$$

and the consecutive switching moments by $t_1, t_2, \ldots$. We also set $t_0=0$.

Theorem 3.2 *Suppose that*

$$\frac{d\hat{z}}{dt}\neq 0 \quad \textit{at the switching moments.}$$

1. *For any $\delta_0>0$, there exists $\tilde{\delta}>0$ such that if $\tilde{\mathbf{v}}(t_i)$ remain in the $\tilde{\delta}$-neighborhood of $\tilde{\boldsymbol{\psi}}$ for even i and in the $\tilde{\delta}$-neighborhood of $\tilde{\mathbf{z}}(s)$ for odd i $(i=0,1,2,\ldots)$, then, for all $\boldsymbol{\varphi}_0$ in the δ_0-neighborhood of $\boldsymbol{\psi}_0$, $\mathbf{v}_0(t_i)$ remain in the δ_0-neighborhood of $\boldsymbol{\psi}_0$ for even i and in the δ_0-neighborhood of $\mathbf{z}_0(s)$ for odd i;*
2. *Let*

$$\|\tilde{\mathbf{v}}(t_i)-\tilde{\boldsymbol{\psi}}\|_{\tilde{V}}+\|\tilde{\mathbf{v}}(t_{i+1})-\tilde{\mathbf{z}}(s)\|_{\tilde{V}}\le \tilde{k}\tilde{q}^i, \qquad i=0,2,4\ldots, \tag{3.7}$$

for some $0<\tilde{q}<1$ and $\tilde{k}>0$ which do not depend on i. Then, for any neighborhood $\mathcal{V}_0$ of $\boldsymbol{\psi}_0$ and for all $\boldsymbol{\varphi}_0\in\mathcal{V}_0$,

$$\|\mathbf{v}_0(t_i)-\boldsymbol{\psi}_0\|_{V_0}+\|\mathbf{v}_0(t_{i+1})-\mathbf{z}(s)\|_{V_0}\le k_0 q_0^i, \qquad i=0,2,4\ldots, \tag{3.8}$$

where $0<q_0=q_0(\tilde{q})<1$ and $k_0=k_0(\tilde{k},\mathcal{V}_0,\tilde{q})>0$ do not depend on $\tilde{\boldsymbol{\varphi}}$ and $\boldsymbol{\varphi}_0$ in the corresponding neighborhoods.

In the proof of this theorem, we will use the following technical lemma.

Lemma 3.2 *Let a sequence $b_0, b_2, b_4, \ldots$ of nonnegative numbers satisfies the inequalities*

$$b_{i+2}\le \zeta b_i + k\nu^i, \qquad i=0,2,4,\ldots,$$

where $k>0$ and $0<\zeta,\nu<1$ do not depend on i. Then there are numbers $0<q=q(\zeta,\nu)<1$ and $c=c(\zeta,\nu,q)>0$ which do not depend on i such that

$$b_i\le (b_0+c)q^i, \qquad i=0,2,4,\ldots.$$

Proof Let $\gamma=\max(\zeta,\nu^2)$. Clearly, $0<\gamma<1$. Then

$$b_{i+2}\le \gamma b_i + k\gamma^{i/2}, \qquad i=0,2,4,\ldots.$$

Consider the sequence $c_i=b_i\gamma^{-i/2}$. It satisfies

$$c_{i+2}\le c_i+k\gamma^{-1}\le c_0+k_1 i,$$

where $k_1>0$ does not depend on c_i and i, which yields the desired estimate of b_i. □

Proof of Theorem 3.2

1. Let us prove assertion 1.

   1a. By assumption, $s=\tilde{\mathbf{t}}_1(\tilde{\boldsymbol{\psi}})$ is the first switching moment of $\mathcal{H}(\hat{z})$. Denote by $\tau=t_1=\tilde{\mathbf{t}}_1(\tilde{\boldsymbol{\varphi}})$ the first switching moment of $\mathcal{H}(\hat{v})$. By Lemma 3.1, $\tilde{\mathbf{t}}_1(\tilde{\boldsymbol{\varphi}})$ is continuously differentiable in a sufficiently small $\tilde{\delta}_1$-neighborhood of $\tilde{\boldsymbol{\psi}}$. Hence,

$$|\tau-s|\le k_1\|\tilde{\boldsymbol{\varphi}}-\tilde{\boldsymbol{\psi}}\|_{\tilde{V}}\le k_1\tilde{\delta}, \tag{3.9}$$

   where $k_1>0$ depends on $\tilde{\delta}_1$ but does not depend on $\tilde{\delta}\le\tilde{\delta}_1$.

1b. Using Remark 2.6 and the fact that, for any $\varepsilon > 0$, there is $d_\varepsilon > 0$ such that

$$(a+b)^2 \le (1+\varepsilon)a^2 + d_\varepsilon b^2,$$

we obtain

$$\begin{aligned}\|\mathbf{v}_0(\tau) - \mathbf{z}_0(s)\|^2_{V_0} &= \sum_{j\in\mathbb{J}_0}(1+\lambda_j)|v_j(\tau) - z_j(s)|^2 \\ &= \sum_{j\in\mathbb{J}_0}(1+\lambda_j)\left|\left(\varphi_j - \frac{K_j}{\lambda_j}\right)e^{-\lambda_j\tau} - \left(\psi_j - \frac{K_j}{\lambda_j}\right)e^{-\lambda_j s}\right|^2 \\ &\le (1+\varepsilon)\sum_{j\in\mathbb{J}_0}(1+\lambda_j)|\varphi_j - \psi_j|^2 e^{-2\lambda_j\tau} \\ &\quad + d_\varepsilon \sum_{j\in\mathbb{J}_0}(1+\lambda_j)\left|\psi_j - \frac{K_j}{\lambda_j}\right|^2 \left|e^{-\lambda_j\tau} - e^{-\lambda_j s}\right|^2. \end{aligned} \tag{3.10}$$

Now we fix $\varepsilon > 0$ such that

$$\zeta = (1+2\varepsilon)e^{-2\varkappa s} < 1, \tag{3.11}$$

where $\varkappa = \min_{j\in\mathbb{J}_0}\lambda_j > 0$. Further, taking into account (3.9), we choose $\tilde{\delta} > 0$ so small that

$$(1+\varepsilon)e^{-2\lambda_j\tau} \le (1+2\varepsilon)e^{-2\varkappa s}. \tag{3.12}$$

Combining (3.10), (3.11), and (3.12) yields

$$\begin{aligned}\|\mathbf{v}_0(\tau) - \mathbf{z}_0(s)\|^2_{V_0} \;&\le \zeta\|\boldsymbol{\varphi}_0 - \boldsymbol{\psi}_0\|^2_{V_0} + d_\varepsilon\sum_{j\in\mathbb{J}_0}(1+\lambda_j)\left|\psi_j - \frac{K_j}{\lambda_j}\right|^2 \\ &\quad \left|e^{-\lambda_j\tau} - e^{-\lambda_j s}\right|^2. \end{aligned} \tag{3.13}$$

Using (2.16), (2.24), and estimate (3.9), we deduce from (3.13)

$$\|\mathbf{v}_0(\tau) - \mathbf{z}_0(s)\|^2_{V_0} \le \zeta\delta_0^2 + k_2(\tilde{\delta}),$$

where $k_2(\tilde{\delta}) > 0$ and $k_2(\tilde{\delta}) \to 0$ as $\tilde{\delta} \to 0$. In particular, this implies that $\mathbf{v}_0(t_1) = \mathbf{v}_0(\tau)$ belongs to the δ_0-neighborhood of $\mathbf{z}_0(s)$, provided that $\tilde{\delta} = \tilde{\delta}(\delta_0)$ is sufficiently small.
In the same way, one can now show that $\mathbf{v}_0(t_2)$ belongs to the δ_0-neighborhood of $\boldsymbol{\psi}_0 = \mathbf{z}_0(T)$. By induction, we obtain assertion 1.

2. Now we prove that

$$\|\mathbf{v}_0(t_i) - \boldsymbol{\psi}_0\|_{V_0} \le k_0 q_0^i, \qquad i = 0, 2, 4\ldots. \tag{3.14}$$

The rest part of estimate (3.8) can be proved analogously.

2a. First, we assume that $\tilde{\mathbf{v}}(0)$ is in a sufficiently small $\tilde{\delta}$-neighborhood of $\tilde{\boldsymbol{\psi}}$. Then, similarly to (3.13), we have for even i

$$\begin{aligned}\|\mathbf{v}_0(t_{i+1}) - \mathbf{z}_0(s)\|^2_{V_0} \;&\le \zeta\|\mathbf{v}_0(t_i) - \boldsymbol{\psi}_0\|^2_{V_0} + d_\varepsilon\sum_{j\in\mathbb{J}_0}(1+\lambda_j)\left|\psi_j - \frac{K_j}{\lambda_j}\right|^2 \\ &\quad \left|e^{-\lambda_j\tau_{i+1}} - e^{-\lambda_j s}\right|^2, \end{aligned} \tag{3.15}$$

where $\tau_{i+1} = t_{i+1} - t_i = \tilde{\mathbf{t}}_1(\tilde{\mathbf{v}}(t_i))$. Using the differentiability of $\tilde{\mathbf{t}}_1$ and estimate (3.7), we have

$$|\tau_{i+1} - s| \le k_1 \|\tilde{\mathbf{v}}(t_i) - \tilde{\boldsymbol{\psi}}\|_{\tilde{V}} \le k_3 \tilde{q}^i \tag{3.16}$$

Due to (3.16), we can assume that $\tau_{i+1} \ge s/2$. Then, taking into account (3.16), we have

$$\left| e^{-\lambda_j \tau_{i+1}} - e^{-\lambda_j s} \right| \le \lambda_j e^{-\lambda_j s/2} |\tau_{i+1} - s| \le k_4 \tilde{q}^i.$$

Combining this inequality with (3.15), (2.24), and (2.16) yields

$$\|\mathbf{v}_0(t_{i+1}) - \mathbf{z}_0(s)\|^2_{V_0} \le \zeta \|\mathbf{v}_0(t_i) - \boldsymbol{\psi}_0\|^2_{V_0} + k_5 \tilde{q}^{2i}.$$

Making one more step and using the last inequality, we obtain

$$\begin{aligned} \|\mathbf{v}_0(t_{i+2}) - \boldsymbol{\psi}_0\|^2_{V_0} &\le \zeta \|\mathbf{v}_0(t_{i+1}) - \mathbf{z}_0(s)\|^2_{V_0} + k_5 \tilde{q}^{2i} \\ &\le \zeta(\zeta \|\mathbf{v}_0(t_i) - \boldsymbol{\psi}_0\|^2_{V_0} + k_5 \tilde{q}^{2i}) + k_5 \tilde{q}^{2i} \\ &\le \zeta^2 \|\mathbf{v}_0(t_i) - \boldsymbol{\psi}_0\|^2_{V_0} + k_6 \tilde{q}^{2i}. \end{aligned} \tag{3.17}$$

Due to Lemma 3.2, the latter inequality implies (3.14).

2b. Now we take an arbitrary $\tilde{\mathbf{v}}(0)$ in the $\tilde{k}$-neighborhood of $\tilde{\boldsymbol{\psi}}$. Due to (3.7), there exists an even number I (which does not depend on $\tilde{\mathbf{v}}(0)$) such that $\tilde{\mathbf{v}}(t_I)$ is in the $\tilde{\delta}$-neighborhood of $\tilde{\boldsymbol{\psi}}$. Then the inequality in (3.14) holds for $i = I, I+2, I+4, \ldots$ due to part 2a of the proof.
Theorem 2.1 implies the existence of $\theta > 0$ (which depends on $\tilde{k}$ and $\mathcal{V}_0$ but does not depend on $\tilde{\mathbf{v}}(0)$ and $\mathbf{v}_0(0)$) such that $t_I \le \theta$. Furthermore, Theorem 2.1 implies that

$$\max_{t \in [0,\theta]} \|\mathbf{v}_0(t)\|_{V_0} \le k_7(\theta, \tilde{k}, \mathcal{V}_0) = k_8(\tilde{k}, \mathcal{V}_0).$$

Hence, the inequality in (3.14) holds for $i = 0, 2, 4, \ldots$. □

Remark 3.2 It follows from the proof of Lemma 3.2 that the convergence rate q is greater than $\gamma^{1/2} = \max(\zeta^{1/2}, \nu)$ but can be chosen arbitrarily close to this number.

Therefore, the convergence rate q_0 in estimate (3.8) is greater than $\max\left(e^{-\varkappa s}, e^{-\varkappa(T-s)}, \tilde{q}\right)$ but can be chosen arbitrarily close to this number.

3.3 Conditional Attraction and Stability of Periodic Solution

Let $z(x,t)$ and $v(x,t)$ be the same as above, but now we do not assume that $\hat{v}(0)$ is necessarily equal to α.

The following theorem shows that the convergence to the periodic orbit in the guiding system implies the convergence to the corresponding periodic orbit in the full system. Thus, we call the phenomenon in that theorem the *conditional attraction*.

For the trajectories, we will use the notation given in Sect. 3.1.

Theorem 3.3 *Suppose that*

$$\frac{d\hat{z}}{dt} \ne 0 \quad \textit{at the switching moments.}$$

Let

$$\begin{cases} \operatorname{dist}(\tilde{\mathbf{v}}(t), \tilde{\Gamma}_1) \le \tilde{k}\tilde{q}^t & \text{if } \mathcal{H}(\hat{v})(t) = 1, \\ \operatorname{dist}(\tilde{\mathbf{v}}(t), \tilde{\Gamma}_2) \le \tilde{k}\tilde{q}^t & \text{if } \mathcal{H}(\hat{v})(t) = -1, \end{cases}$$

for some $0 < \tilde{q} < 1$ *and* $\tilde{k} > 0$*. Then, for any bounded set* $\mathcal{V}_0$ *in* V_0*, there exist* $0 < q = q(\tilde{q}) < 1$ *and* $k = k(\tilde{k}, \mathcal{V}_0, \tilde{q}) > 0$ *such that, for all* $\boldsymbol{\varphi}_0 \in \mathcal{V}_0$ *and* $t \ge 0$*,*

$$\begin{cases} \operatorname{dist}(v(\cdot, t), \Gamma_1) \le kq^t & \text{if } \mathcal{H}(\hat{v})(t) = 1, \\ \operatorname{dist}(v(\cdot, t), \Gamma_2) \le kq^t & \text{if } \mathcal{H}(\hat{v})(t) = -1, \end{cases}$$

Proof

1. Suppose we have shown that

$$\|v(\cdot, t_i) - z(\cdot, 0)\|_{H^1} + \|v(\cdot, t_{i+1}) - z(\cdot, s)\|_{H^1} \le kq^i, \qquad i = 0, 2, 4 \ldots, \tag{3.18}$$

where $0 < q = q(\tilde{q}) < 1$ and $k = k(\tilde{k}, \mathcal{V}_0, \tilde{q}) > 0$. Then, using Lemma 3.1 and arguing as in the proof of Theorem 4.3 in [14], we complete the proof.
So, let us prove estimate (3.18).

2. Consider the intersection of the closure of $\tilde{\Gamma}_1$ with the set $\{\tilde{\boldsymbol{\varphi}} \in \tilde{V} : \hat{\varphi} = \beta\}$. This intersection consists of the single point $\tilde{\mathbf{z}}(s)$, where s is the switching moment of the periodic solution.
Since $\left.\frac{d\hat{z}}{dt}\right|_{t=s} \ne 0$, it follows from the implicit function theorem that there exist a number $L > 0$ and a sufficiently small number $d_0 > 0$ such that if

$$|\beta - \hat{z}(\tau)| \le d$$

for some $d \le d_0$ and $\tau \in [0, s]$, then

$$|\tau - s| \le Ld.$$

3. Consider $i = 1, 3, 5, \ldots$. By assumption, there exists $\tau \in [0, s]$ such that

$$\|\tilde{\mathbf{v}}(t_i) - \tilde{\mathbf{z}}(\tau)\|_{\tilde{V}} \le \tilde{k}\tilde{q}^{t_i}. \tag{3.19}$$

This inequality together with the Cauchy–Bunyakovskii inequality implies that there exists a constant $C_1 > 0$ such that

$$|\beta - \hat{z}(\tau)| = |\hat{v}(t_i) - \hat{z}(\tau)| \le C_1\tilde{k}\tilde{q}^{t_i}. \tag{3.20}$$

3a. First, we assume that $t_i \ge \theta$, where $\theta > 0$ is so large that $C_1\tilde{k}\tilde{q}^{\theta} \le d_0$ (d_0 is the number from part 2 of the proof). Then, due to part 2 of the proof, we have

$$|\tau - s| \le LC_1\tilde{k}\tilde{q}^{t_i}. \tag{3.21}$$

It was proved in [14, Lemma 4.6] that the periodic solution $\tilde{\mathbf{z}}(t)$ is uniformly Lipschitz-continuous on $[0, T]$, which (together with (3.21)) implies that

$$\|\tilde{\mathbf{z}}(\tau) - \tilde{\mathbf{z}}(s)\|_{\tilde{V}} \le C_2\tilde{k}\tilde{q}^{t_i} \tag{3.22}$$

for some $C_2 > 0$.
Estimates (3.19) and (3.22) yield

$$\|\tilde{\mathbf{v}}(t_i) - \tilde{\mathbf{z}}(s)\|_{\tilde{V}} \le \tilde{k}(1 + C_2)\tilde{q}^{t_i} \quad \text{for } t_i \ge \theta.$$

Taking into account (2.7), we have

$$\|\tilde{\mathbf{v}}(t_i) - \tilde{\mathbf{z}}(s)\|_{\tilde{V}} \le \tilde{k}_1 \tilde{q}_1^i \quad \text{for } t_i \ge \theta, \tag{3.23}$$

where $\tilde{k}_1 \ge \tilde{k}$ and $0 < \tilde{q}_1 < 1$.

3b. For $t_i \le \theta$, we have

$$\|\tilde{\mathbf{v}}(t_i) - \tilde{\mathbf{z}}(s)\|_{\tilde{V}} \le \operatorname{dist}(\tilde{\mathbf{v}}(t_i), \tilde{\Gamma}_1) \le \tilde{k}. \tag{3.24}$$

Combining (3.23) and (3.24) yields

$$\|\tilde{\mathbf{v}}(t_i) - \tilde{\mathbf{z}}(s)\|_{\tilde{V}} \le \tilde{k}_2 \tilde{q}_1^i \quad \text{for all } t_i.$$

Applying similar arguments to $\tilde{v}(t_{i+1})$ and $\tilde{z}(0)$ and using Theorem 3.2, we obtain (3.18). □

Now we discuss the phenomenon of *conditional stability*. When studying the stability of periodic solutions, one considers its small neighborhood. When doing so, one has to take into account the initial state of the hysteresis operator.

Definition 3.3 An (s,σ)-periodic solution $z(x,t)$ of problem (2.1), (2.3) is *stable* if, for any neighborhoods $\mathcal{U}_1$ of Γ_1 and $\mathcal{U}_2$ of Γ_2 in H^1, there exist neighborhoods $\mathcal{V}_1$ of Γ_1 and $\mathcal{V}_2$ of Γ_2 in H^1 such that if

$$\varphi \in \mathcal{V}_1,\ \hat{\varphi} < \beta \quad \text{or} \quad \varphi \in \mathcal{V}_2,\ \hat{\varphi} \ge \beta,$$

then the solution $v(x,t)$ of problem (2.1)–(2.3) in Q_∞ with the initial data φ satisfies for all $t \ge 0$:

$$\begin{cases} v \in \mathcal{U}_1 & \text{if } \ \mathcal{H}(\hat{v})(t) = 1, \\ v \in \mathcal{U}_2 & \text{if } \ \mathcal{H}(\hat{v})(t) = -1. \end{cases}$$

An (s,σ)-periodic solution is *unstable* if it is not stable.

Definition 3.4 An (s,σ)-periodic solution $z(x,t)$ of problem (2.1), (2.3) is *uniformly exponentially stable* if it is stable and there exist neighborhoods $\mathcal{W}_1$ of Γ_1 and $\mathcal{W}_2$ of Γ_2 in H^1 and numbers $0 < q < 1$ and $k > 0$ such that if

$$\varphi \in \mathcal{W}_1,\ \hat{\varphi} < \beta \quad \text{or} \quad \varphi \in \mathcal{W}_2,\ \hat{\varphi} \ge \beta,$$

then the solution $v(x,t)$ of problem (2.1)–(2.3) in Q_∞ with the initial data φ satisfies

$$\begin{cases} \operatorname{dist}(v(\cdot,t),\Gamma_1) \le kq^t & \text{if } \ \mathcal{H}(\hat{v})(t) = 1, \\ \operatorname{dist}(v(\cdot,t),\Gamma_2) \le kq^t & \text{if } \ \mathcal{H}(\hat{v})(t) = -1 \end{cases}$$

for all $t \ge 0$ uniformly with respect to φ.

Let $\tilde{\mathbf{z}}(t)$ be an (s,σ)-periodic solution of the guiding system (2.21). Then, by Theorem 3.1, there exists a unique function $\mathbf{z}_0(t)$ such that $(\tilde{\mathbf{z}}(t), \mathbf{z}_0(t))$ is an (s,σ)-periodic solution of the full system (2.20). We denote by $z(x,t)$ the corresponding (s,σ)-periodic solution of problem (2.1), (2.3).

Theorem 3.4 *Suppose that*

$$\frac{d\hat{z}}{dt} \ne 0 \quad \textit{at the switching moments.}$$

Then the following assertions are equivalent.

1. *The periodic solution $z(x,t)$ of problem* (2.1), (2.3) *is stable* (*uniformly exponentially stable*).
2. *The periodic solution $\tilde{\mathbf{z}}(t)$ of the guiding system* (2.21) *is stable* (*uniformly exponentially stable*).
3. *The element $\tilde{\mathbf{z}}(0)$ is a stable* (*uniformly exponentially stable*) *fixed point of the Poincaré map $\tilde{\Pi}$.*

Proof Implication 1 ⇒ 2 is obvious.

Implication 2 ⇒ 3 is proved similarly to the proof of Theorem 3.3.

To prove implication 3 ⇒ 1, one should use Lemma 3.1 and Theorem 3.2 and argue as in the proof of Lemma 4.7 and Theorem 4.4 in [14]. □

4 Symmetric Periodic Solutions

4.1 Preliminary Considerations

It was noted in [14] that any (s,σ)-periodic solution possesses a certain symmetry, provided that it is unique. In fact a much stronger result holds, namely, we show that any (s,σ)-periodic solution possesses symmetry.

We underline that the results in the previous sections did not depend on the symmetry of periodic solutions, but the results of this section do. In particular, by exploiting the symmetry, we give an algorithm for finding *all* periodic solutions with two switchings on the period. Using their explicit form, we will study their stability.

Definition 4.1 An (s,σ)-periodic solution $z(x,t)$ of problem (2.1), (2.3) is called *symmetric* if $z_j(0) = -z_j(s)$, $j = 1, 2, \ldots$.

Lemma 4.1 *Let $z(x,t)$ be an (s,σ)-periodic solution of problem* (2.1), (2.3). *Then $s = \sigma$ and $z(x,t)$ is symmetric.*

Proof Let $\psi(x) = z(x,0) = z(x,s+\sigma)$ and $\xi(x) = z(x,s)$.

By Remark 2.6,

$$\xi_0 = \psi_0 + K_0 s, \tag{4.1}$$

$$\xi_j = \left(\psi_j - \frac{K_j}{\lambda_j}\right) e^{-\lambda_j s} + \frac{K_j}{\lambda_j}, \quad j \geq 1. \tag{4.2}$$

Applying Remark 2.6 (with K_j replaced by $-K_j$), we conclude that

$$\psi_0 = \xi_0 - K_0\sigma, \tag{4.3}$$

$$\psi_j = \left(\xi_j + \frac{K_j}{\lambda_j}\right) e^{-\lambda_j \sigma} - \frac{K_j}{\lambda_j}, \quad j \geq 1. \tag{4.4}$$

Equalities (4.1) and (4.3) imply that $s = \sigma$. Summing up (4.2) and (4.4) and taking into account that $s = \sigma$, we see that

$$\psi_j + \xi_j = (\psi_j + \xi_j)e^{-\lambda_j s}, \quad j \geq 1.$$

Hence, $\xi_j = -\psi_j$, and $z(x, t)$ is symmetric. □

Remark 4.1 Lemma 4.1 shows that the period (and the second switching time) of any (s, σ)-periodic solution is uniquely determined by the first switching time and vice versa. Therefore, we will say *"T-periodic solution"* or just *"periodic solution"* instead of saying "symmetric (s, s)-periodic solution with period $T = 2s$".

In [14], it was shown that there is a number $\delta_1 \geq 0$ such that if $\beta - \alpha > \delta_1$, then there exists a periodic solution of problem (2.1), (2.3). Furthermore, there is a number $\delta_2 \geq \delta_1$ such that if $\beta - \alpha > \delta_2$, then there exists a unique periodic solution of problem (2.1), (2.3); moreover, it is stable, and is a global attractor. Both numbers δ_1 and δ_2 depend on Q, m, and K.

In this section, we will formulate a sufficient condition which may hold for arbitrarily small $\beta - \alpha$ and still provides the existence of (symmetric) periodic solutions. We will show that these solutions may be both stable and unstable.

Lemma 4.2 *Let $z(x, t)$ be a solution of problem (2.1)–(2.3) with the initial data ψ, $\hat{\psi} = \alpha$, and let $s > 0$ be the first switching moment of $\mathcal{H}(\hat{z})$. If*

$$z_j(s) = -\psi_j, \quad j = 1, 2, \ldots,$$

then $z(x, t)$ is a (symmetric)$2s$-periodic solution of problem (2.1), (2.3).

Proof

1. First, we show that there are no switchings for $t \in (s, 2s)$ and that the second switching occurs exactly for $t = 2s$. To do so, we have to show that $\hat{z}(t) > \alpha$, or, equivalently, $\hat{z}(s) - \hat{z}(t) < \beta - \alpha$ for $t \in (s, 2s)$. Using Remark 2.6 (with K_j replaced by $-K_j$) and the assumption that $z_j(s) = -\psi_j$, we have for $t \in (s, 2s)$

$$\begin{aligned} z_j(t) &= \left(z_j(s) + \frac{K_j}{\lambda_j}\right) e^{-\lambda_j(t-s)} - \frac{K_j}{\lambda_j} = \left(-\psi_j + \frac{K_j}{\lambda_j}\right) e^{-\lambda_j(t-s)} - \frac{K_j}{\lambda_j}, \\ &\quad j = 1, 2, \ldots, \\ z_0(t) &= z_0(s) - K_0(t-s) = \psi_0 + 2K_0 s - K_0 t. \end{aligned} \tag{4.5}$$

   Therefore, taking into account (2.19), we have

$$\begin{aligned} \hat{z}(s) - \hat{z}(t) &= m_0 K_0(t-s) + \sum_{j=0}^{\infty} m_j \left(\psi_j - \frac{K_j}{\lambda_j}\right)\left(e^{-\lambda_j(t-s)} - 1\right) \\ &= m_0 K_0 \sigma\theta + \sum_{j=0}^{\infty} m_j \left(\psi_j - \frac{K_j}{\lambda_j}\right)\left(e^{-\lambda_j\theta} - 1\right) = \hat{z}(\theta) - \hat{\psi}, \end{aligned} \tag{4.6}$$

   where $\theta = t - s \in (0, s)$. But $\hat{z}(\theta) - \hat{\psi} < \beta - \alpha$ for $\theta \in (0, s)$ and $\hat{z}(s) - \hat{\psi} = \beta - \alpha$ (because s is the first switching moment by assumption).
2. Now we show that $z(x, 2s) = \psi(x)$. Indeed, using (4.5) and the assumption that $z_j(s) = -\psi_j$, we obtain

$$\begin{aligned} z_j(2s) &= -\left[\left(\psi_j - \frac{K_j}{\lambda_j}\right) e^{-\lambda_j s} + \frac{K_j}{\lambda_j}\right] = -z_j(s) = \psi_j, \quad j = 1, 2, \ldots, \\ z_0(2s) &= \psi_0. \end{aligned}$$

□

4.2 Construction of Symmetric Periodic Solutions

Lemma 4.2 allows one to explicitly find all (s, s)-periodic solutions according to the following algorithm.

Step 1. For each $s > 0$, we find the (unique) $\psi_j = \psi_j(s)$ such that $v_j(s) = -\psi_j$ for $j = 1, 2, \dots$, assuming that $\mathcal{H}(\hat{v}) \equiv 1$ on the interval $[0, s)$. To do so, we solve the equation (cf. Remark 2.6)

$$\left(\psi_j - \frac{K_j}{\lambda_j}\right) e^{-\lambda_j s} + \frac{K_j}{\lambda_j} = -\psi_j,$$

which yields

$$\psi_j = \psi_j(s) = -\frac{K_j}{\lambda_j} \cdot \frac{1 - e^{-\lambda_j s}}{1 + e^{-\lambda_j s}}. \tag{4.7}$$

We note that $\psi_j(0) = 0$ and $\psi_j(s)$ monotonically decreases and tends to $-K_j/\lambda_j$ as $s \to +\infty$.

Step 2. We find the (unique) $\psi_0 = \psi_0(s)$ such that $\hat{\psi} = \alpha$. To do so, we solve the equation

$$m_0\psi_0 + \sum_{j=1}^{\infty} m_j \psi_j = \alpha,$$

which yields

$$\psi_0 = \psi_0(s) = \frac{1}{m_0}\left(\alpha - \sum_{j=1}^{\infty} m_j \psi_j(s)\right). \tag{4.8}$$

Note that the function ψ with the Fourier coefficients given by (4.7) and (4.8) belongs to H^1. This follows from (2.14) and (2.16).

Step 3. If the solution[1] $v(x, t) = v(x, t; s)$ of problem (2.9)–(2.11) with the initial data $\psi = \psi(s)$ is such that $\mathcal{H}(\hat{v})$ does not switch for $t < s$ and switches at the moment $t = s$, then, by Lemma 4.2, there exists a $2s$-periodic solution $z(x, t; s)$ (which coincides with $v(x, t; s)$ for $t \le s$).
The switching condition is

$$\sum_{j=0}^{\infty} m_j v_j(s) = \beta,$$

or, equivalently,

$$F(s) := m_0 K_0 s + 2\sum_{j=1}^{\infty} m_j \frac{K_j}{\lambda_j} \cdot \frac{1 - e^{-\lambda_j s}}{1 + e^{-\lambda_j s}} = \beta - \alpha. \tag{4.9}$$

To check that the switching does not occur before s, we note that, due to Remark 2.6, the mean temperature $\hat{v}(t; s)$ corresponding to the initial condition (4.7), (4.8) is given by

$$\hat{v}(t; s) = \sum_{j=0}^{\infty} m_j v_j(t; s) = \alpha + m_0 k_0 t + 2\sum_{j=1}^{\infty} m_j \frac{K_j}{\lambda_j} \cdot \frac{1 - e^{-\lambda_j t}}{1 + e^{-\lambda_j s}}.$$

[1] Here and further, we sometimes write s after the semicolon to explicitly indicate that the function depends on the chosen first switching time s as on a parameter.

Therefore, the condition $\hat{v}(t; s) = \beta$ is equivalent to

$$m_0 k_0 t + 2 \sum_{j=1}^{\infty} m_j \frac{K_j}{\lambda_j} \cdot \frac{1 - e^{-\lambda_j t}}{1 + e^{-\lambda_j s}} = \beta - \alpha.$$

Taking into account equality (4.9), we see that the condition $\hat{v}(t; s) = \beta$ is equivalent to the following:

$$H(t, s) := m_0 k_0 (t - s) + 2 \sum_{j=1}^{\infty} m_j \frac{K_j}{\lambda_j} \cdot \frac{e^{-\lambda_j s} - e^{-\lambda_j t}}{1 + e^{-\lambda_j s}} = 0. \tag{4.10}$$

Moreover, the fulfillment of the inequality $H(t, s) < 0$ for all $t \in (0, s)$ is necessary and sufficient for the absence of switching moments before the time moment s.

Definition 4.2 We will say that $F(s)$ and $H(t, s)$ are the *first* and the *second characteristic functions*, while (4.9) and (4.10) are the *first* and the *second characteristic equations*, respectively.

The first and the second characteristic equations will play a fundamental role in the description of periodic solutions and their bifurcation sets (see Theorems 4.1 and 4.2 below).

The following lemmas describe some properties of the characteristic functions.

Lemma 4.3

1. *$F(s)$ is continuous for $s \geq 0$ and analytic for $s > 0$,*
2. *$F(0) = 0$, $F(s)$ increases for all sufficiently large $s > 0$, and $\lim_{s \to +\infty} F(s) = +\infty$,*
3. *for each $\beta - \alpha > 0$, the first characteristic equation (4.9) has finitely many roots,*
4. *the positive zeroes of $F(s)$ are isolated and may accumulate only at the origin.*

Proof 1. The series in (4.9) is absolutely and uniformly convergent for $\operatorname{Re} s \geq 0$ due to the Cauchy–Bunyakovskii inequality and (2.16). Therefore, $F(s)$ is continuous for $s \geq 0$ and analytic for $s > 0$.

Assertion 2 is now straightforward.

To prove assertion 3, we note that, for $\beta - \alpha > 0$, the (positive) roots of the first characteristic equation (4.9) cannot accumulate at the origin. This follows by the continuity and the relation $F(0) = 0$. The roots cannot accumulate at infinity either (due to the monotonicity for large s). Therefore, all the roots belong to a compact separated from the origin. Now the analyticity for $s > 0$ implies assertion 3.

Assertion 4 follows from the analyticity of $F(s)$ for $s > 0$ and from the monotonicity for large s. □

Similarly, one can prove the following lemma.

Lemma 4.4

1. *$H(t, s)$ is continuous for $s \geq 0, 0 \leq t \leq s$,*
2. *for each $s > 0$, $H(t, s)$ is analytic in t for $t > 0$,*
3. *$H(0, s) = -F(s)$ and $H(s, s) \equiv 0$,*
4. *if $s > 0$ and $F(s) > 0$, then the second characteristic equation (4.10) has no more than finitely many roots in t for $t \in (0, s)$.*

Taking into account Lemmas 4.1, 4.3, and 4.4, we formulate the above algorithm as the following theorem (also mind Remark 4.1).

Theorem 4.1

1. *For a given* $\beta - \alpha > 0$*, there are no more than finitely many periodic solutions of problem* (2.1)*,* (2.3)*, which we denote* $z^{(1)}, \dots, z^{(N)}$*.*
2. *All the periodic solutions* $z^{(1)}, \dots, z^{(N)}$ *are symmetric.*
3. *If* $s_1, \dots, s_N$ *are half-periods of* $z^{(1)}, \dots, z^{(N)}$*, respectively, then* $s_1, \dots, s_N$ *are the roots of the first characteristic equation* (4.9)*.*
4. *Let* $s_{N+1}, \dots, s_{N_1}$ *be positive roots of the first characteristic equation* (4.9) *different from* $s_1, \dots, s_N$*. Then*
   (a) $H(t, s_j) < 0$ *for all* $t \in (0, s_j)$ *if* $j = 1, \dots, N$*,*
   (b) $H(t; s_j) = 0$ *for some* $t \in (0, s_j)$ *if* $j = N + 1, \dots, N_1$*.*

In particular, Theorem 4.1 implies that a positive root s_j of the first characteristic equation (4.9) "generates" a $2s_j$-periodic solution if and only if $H(t, s_j) < 0$ for all $t \in (0, s_j)$.

Now we will keep the domain Q and the functions $m(x)$ and $K(x)$ fixed, while allow the thresholds α and β vary. We will classify the existence of all periodic (i.e., (s, s)-periodic) solutions with respect to the parameter s and with respect to the parameter $\beta - \alpha$. By the existence of a periodic solution for a given $s > 0$ we mean that there exist numbers $\alpha < \beta$ (depending on s) such that problem (2.1), (2.3) with these α and β admits an (s, s)- or, equivalently, a $2s$-periodic solution.

First, we show that one can divide the positive s-semiaxis into intervals (whose union is denoted by L) in the following way. For every interval $L' \subset L$, either there are no $2s$-periodic solutions for all $s \in L'$ or there is exactly one $2s$-periodic solution for every $s \in L'$, which smoothly depends on s in L'. The complement S of the union L of all those intervals will consist of points of possible bifurcation with respect to s (half-period). It will be a compact set. Typically, S will consist of finitely many points (see Examples 4.1).

The compact set $\Sigma = F(S)$ will consist of points of possible bifurcation with respect to the parameter $\beta - \alpha$. This set divides the positive $(\beta - \alpha)$-semiaxis into open intervals (whose union is denoted by Λ). For $\beta - \alpha$ in an interval $\Lambda' \subset \Lambda$, the number of periodic solutions remains constant and they smoothly depend on $\beta - \alpha \in \Lambda'$ (see Example 4.1).

First, we introduce the set

$$S_0 = \{s > 0 : F(s) = 0\}.$$

Due to Lemma 4.3, the set S_0 consists of no more than countably many points, which may accumulate only at the origin.

To introduce the next set, we denote for $s > 0$

$$\tau(s) = \{t \in (0, s) : H(t, s) = 0\}. \tag{4.11}$$

By Lemma 4.4, $\tau(s)$ consists of finitely many roots of the equation $H(t, s) = 0$ on the interval $t \in (0, s)$, provided that $F(s) > 0$.

Consider the set

$$S_1 = \{s > 0 : F(s) > 0,\ \tau(s) = \varnothing \text{ and } H_t(t, s)|_{t=s} = 0\}.$$

Thus, S_1 consists of those s for which the corresponding trajectory $v(x, t; s)$ intersects the hyperplane $\hat{\varphi} = \beta$ for the first time at the moment s and touches it nontransversally at this moment. Note that any number $s \in S_1$ generates a $2s$-periodic solution.

Consider the set

$$S_2 = \{s > 0 : F(s) > 0, \tau(s) \neq \varnothing, \text{ and } H_t(t, s)|_{t=t'} = 0\ \forall t' \in \tau(s)\}.$$

Thus, S_2 consists of those s for which the corresponding trajectory $v(x, t; s)$ intersects the hyperplane $\hat{\varphi} = \beta$ for the first time before the moment s and touches it nontransversally at each of the intersection moments (before s). None of the numbers $s \in S_2$ generate a $2s$-periodic solution.

We also introduce the set

$$S_3 = \{s > 0 : F(s) > 0 \text{ and } F'(s) = 0\}.$$

We note that the set S_3 consists of no more than countably many isolated points which may accumulate only at the origin. This follows from the analyticity of $F(s)$ for $s > 0$ and from the monotonicity for large s.

Now we set

$$L = (0, \infty) \setminus \overline{S_0 \cup S_1 \cup S_2}.$$

and

$$\Sigma = F(S_1 \cup S_2 \cup S_3), \qquad \Lambda = (0, \infty) \setminus \overline{\Sigma}.$$

We note that the above sets S_i, L and Σ, Λ do not depend on s or $\beta - \alpha$. They only depend on m_j, K_j, and λ_j. We also note that the sets $S_0, \dots S_3$ and Σ are bounded. Indeed, S_0 and S_3 are bounded because $F(s)$ monotonically increases for sufficiently large s. Furthermore, it is proved in [14] that, for sufficiently large $\beta - \alpha$ (hence for sufficiently large s), the first switching moment for $v(x, t; s)$ is equal to s and $\left.\frac{d\hat{v}(t;s)}{dt}\right|_{t=s} > 0$. Therefore, S_2 and S_3 are also bounded. The boundedness of $S_0, \dots, S_3$ implies the boundedness of Σ.

Theorem 4.2

1. *Let L' be an open interval in L. Then either there are no $2s$-periodic solutions for all $s \in L'$ or, for any $s \in L'$, there is a unique $2s$-periodic solution $z(x, t; s)$ of problem (2.1), (2.3). Moreover, the initial value $z(x, 0; s)$ smoothly depends on $s \in L'$(in the H^1-topology).*
2. *Let Λ' be an open interval in Λ. Then the number of periodic solutions of problem (2.1), (2.3) remains constant for all $\beta - \alpha \in \Lambda'$. The initial values of those solutions and the first switching times continuously depend on $\beta - \alpha \in \Lambda'$(in the H^1-topology).*

Proof

1. Let L' be an open interval in L. For any $s \in L'$, we denote by $\hat{v}(t; s)$ the mean temperature corresponding to the initial condition (4.7), (4.8). We recall that

   $$\hat{v}(t; s) = \beta$$

   if and only if

   $$H(t, s) = 0.$$

   Fix an arbitrary $s' \in L'$. Then $s' \notin S_0$, i.e., $F(s') \neq 0$. If $F(s') < 0$, then $F(s) < 0$ for all $s \in L'$ (otherwise, $F(s) = 0$ for some $s \in L'$, but then $s \in S_0$, which is impossible). In this case, every $s \in L'$ does not generate a periodic solution.

   Assume that $F(s) > 0$.

   Consider the sets $\tau(s)$ given by (4.11) for $s \in L'$. We claim that if $\tau(s') = \varnothing$, then $\tau(s) = \varnothing$ in a sufficiently small neighborhood of s'; if $\tau(s') \neq \varnothing$, then $\tau(s) \neq \varnothing$ in a sufficiently small neighborhood of s'. Indeed:

1a. Let $\tau(s') = \varnothing$. Suppose that there is a sequence s_i converging to s and a sequence $t_i \in (0, s_i)$ such that $H(t_i, s_i) = 0$. Taking a subsequence if needed, we can assume that $t_i \to t' \in (0, s']$. Thus, by continuity of $H(t, s)$, we have

$$H(t', s') = 0 \tag{4.12}$$

Since $\tau(s') = \varnothing$ and $s' \notin S_1$, we have $H_t(t, s')|_{t=s'} \neq 0$. Therefore, by the implicit function theorem and by the identity $H(s, s) \equiv 0$, it follows that, in a neighborhood of the point (s', s'), the only root (in t) of the equation $H(t, s) = 0$ is $t = s$. Hence, all t_i lie outside a fixed neighborhood of s', which means that $t' < s'$. Together with (4.12), this yields $\tau(s') \neq \varnothing$. This contradiction proves that $\tau(s) = \varnothing$ in a sufficiently small neighborhood of s'.

1b. Now let $\tau(s') \neq \varnothing$. Since $s' \notin S_2$, there is $t' < s'$ such that $H(t', s') = 0$ and $H_t(t, s')|_{t=t'} \neq 0$. By the implicit function theorem the equation $H(t, s) = 0$ admits a solution $t = t(s)$ in a neighborhood of s' such that $t' = t(s')$. By regularity, $t(s) < s$ if the neighborhood is small enough. Therefore, $\tau(s) \neq \varnothing$ in a sufficiently small neighborhood of s'.

To complete the proof of assertion 1, we choose an arbitrary compact interval in L', cover each point of it by the above neighborhood and take a finite subcovering.
The smooth dependence of the initial value of the periodic solution on $s \in L'$ follows from the explicit formulas (4.7) and (4.8).

2. Let Λ' be an open interval in Λ.
Fix an arbitrary $b' \in \Lambda'$. Since $b > 0$, Lemma 4.3 implies that the first characteristic equation $F(s) = b'$ has finitely many (say, N_1) positive roots $s'_1, \dots, s'_{N_1}$. Since $b' \notin \Sigma$, it follows that $s'_j \notin S_3$, i.e., $F'(s'_j) \neq 0$. Therefore, for b in a neighborhood of b', there exist exactly N_1 positive roots $s_1 = s_1(b), \dots, s_{N_1} = s_{N_1}(b)$ of the first characteristic equation $F(s) = b$, which smoothly depend on b.
Further, we assume that there are N ($N \leq N_1$) numbers $s'_1, \dots, s'_N$ for which the minimal root of the equation $H(t, s'_j) = 0$ on the interval $(0, s'_j)$ is equal to s'_j. As before, this means that s'_j generate $2s'_j$-periodic solutions for $j = 1, \dots, N$ and do not generate periodic solutions for $j = N + 1, \dots, N_1$ (cf. Theorem 4.1).
Since $b' > 0$ and $b' \notin \Sigma$, it follows that $s'_j \notin S_0 \cup S_1 \cup S_2$ ($j = 1, \dots, N_1$). Therefore, similarly to part 1 of the proof, for all b in a neighborhood of b', the numbers $s_j = s_j(b)$ generate $2s_j$-periodic solutions for $j = 1, \dots, N$ and do not generate periodic solutions for $j = N + 1, \dots, N_1$.
To complete the proof of assertion 2, we choose an arbitrary compact interval in Λ', cover each point b' of it by the above neighborhood and take a finite subcovering. □

Remark 4.2 Theorem 4.2 indicates the ways a new periodic solution may appear or an existing periodic solution may disappear, i.e., bifurcation occurs.

When varying the parameter s, bifurcation may occur only if $s \in S_0 \cup S_1 \cup S_2$.

1. The condition $s \in S_0$ implies that α and β coalesce.
2. The condition $s \in S_1$ corresponds to the tangential approach of the trajectory $v(x, t; s)$ to the hyperplane $\hat{\varphi} = \beta = \alpha + F(s)$. At the point s, the periodic solution exists. In the literature on switching (or hybrid) systems, such a bifurcation is usually called "grazing bifurcation". The corresponding Poincaré map will be discontinuous at this point.
3. The condition $s \in S_2$ also corresponds to the tangential approach of the trajectory $v(x, t; s)$ to the hyperplane $\hat{\varphi} = \beta = \alpha + F(s)$. However, at the point s, the periodic

solution does not exists. The switching occurs before the trajectory comes in the "symmetric" position. This bifurcation can also be called "grazing bifurcation".

When varying the parameter $\beta - \alpha > 0$, bifurcation may occur if a point $s \in F^{-1}(\beta - \alpha)$ belongs to S_1, S_2, or S_3.

Grazing bifurcation occurs on S_1 and S_2 as described above.

If $s \in S_3 \setminus (S_1 \cup S_2)$, then a new root of the first characteristic equation (4.9) may appear and then split into two roots (or two existing roots may merge into one and then disappear) as $\beta - \alpha$ crosses the value $F(s)$. If the first switching moment for $v(x, t; s)$ is equal to s (i.e., $H(t, s) < 0$ for $t < s$ or, equivalently, $\tau(s) = \varnothing$), then a new periodic solution will appear and then split into two (or the two existing periodic solutions will merge into one and then disappear). This corresponds to a fold bifurcation.

On the other hand, if the first switching moment for $v(x, t; s)$ is less than s (i.e., $H(t, s) = 0$ for some $t < s$ or, equivalently, $\tau(s) \neq \varnothing$), then no bifurcation happens.

We consider an example illustrating Theorems 4.1 and 4.2.

Example 4.1 Let Q be a one-dimensional domain, e.g., $Q = (0, \pi)$, cf. [8–11,23]. Let the boundary condition (2.3) be given by

$$v_x(0, t) = 0, \qquad v_x(\pi, t) = \mathcal{H}(\hat{v})(t).$$

From the physical point of view, these boundary conditions model a thermocontrol process in a rod with heat-insulation on one end and a heating (cooling) element on the other.

It is easy to find that

$$\lambda_0 = 0, \quad e_0 = \sqrt{\frac{1}{\pi}}, \quad K_0 = e_0(\pi) = \sqrt{\frac{1}{\pi}},$$
$$\lambda_j = j^2, \quad e_j(x) = \sqrt{\frac{2}{\pi}} \cos jx, \quad K_j = e_j(\pi) = (-1)^j \sqrt{\frac{2}{\pi}}, \quad j = 1, 2, \dots.$$

Let $m_0 = 2$, $m_1 = m_2 = 4$, and $m_3 = m_4 = \dots = 0$. Then the bifurcation diagram is depicted in Fig. 5.

Let $m_0 = 3.2$, $m_1 = m_2 = 4$, and $m_3 = m_4 = \dots = 0$. Then the bifurcation diagram is depicted in Fig. 6.

"Evolution" of periodic solutions with respect to the parameter $\beta - \alpha$ is visualized in Fig. 7.

In [14], it was shown that there exists a unique periodic solution if $\beta - \alpha$ is large enough. Moreover, it is stable and is a global attractor. To conclude this section, we prove that a periodic solution can also exist for arbitrarily small $\beta - \alpha$. Further, we will show that such a solution need not be stable.

Assume that the following condition holds.

Condition 4.1 *The functions $m \in H^1$ and $K \in H^{1/2}$ satisfy*

$$M := \sum_{j=0}^{\infty} m_j K_j = \int_{\partial Q} m(x) K(x)\, d\Gamma > 0.$$

The convergence of the sum follows from Remark 2.5. The equality follows from the definition of m_j and K_j. The essential requirement of Condition 4.1 is the positivity of the sum, or, equivalently, of the integral. From the physical viewpoint, this condition implies the presence of thermal sensors on a part of the boundary where the heating elements are.

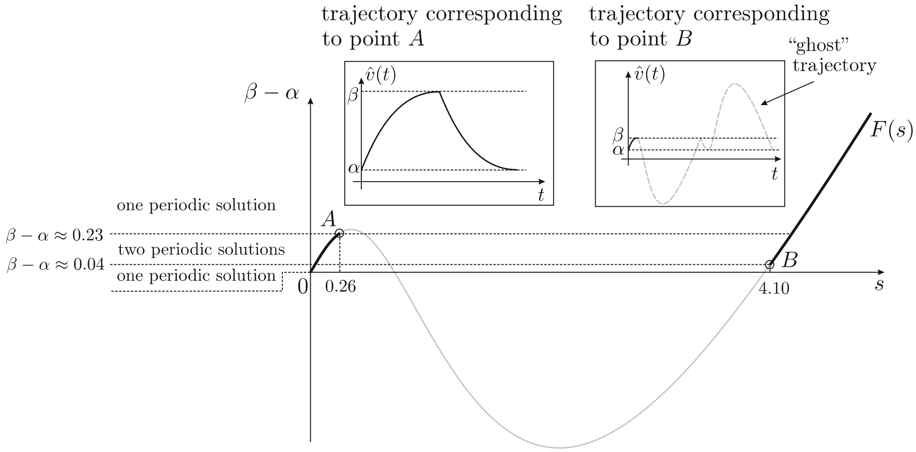


Fig. 5 Bifurcation diagram for $m_0 = 2$, $m_1 = m_2 = 4$, and $m_3 = m_4 = \cdots = 0$. For any $s > 0$, there exists a unique $2s$-periodic solution if the graph of F is bold at the point s and there are no $2s$-periodic solutions otherwise. For any $\beta - \alpha > 0$, there exist one or two periodic solutions depending on whether the horizontal line levelled at $\beta - \alpha$ intersects the bold part of the graph of F at one or two points, respectively. Point A ($s \approx 0.26$, $\beta - \alpha \approx 0.23$) on the graph corresponds to $s \in S_1$. There exists a corresponding $2s$-periodic solution, whose trajectory is tangent to the hyperplane $\hat{\varphi} = \beta$ at the moment s (see the *left* inset). Point B ($s \approx 4.10$, $\beta - \alpha \approx 0.04$) on the graph corresponds to $s \in S_2$; there does not exist a $2s$-periodic solution for this s. However, if one did not switch when $\hat{v}(t; s)$ tangentially intersected the hyperplane $\hat{\varphi} = \beta$ at the moment s, but switched only when $\hat{v}(t; s)$ intersected the hyperplane $\hat{\varphi} = \beta$ for the second time (at some moment $s_1 > s$), then the resulting trajectory would be $2s_1$ periodic. Such a trajectory is referred to as a "ghost" trajectory (see the *right* inset)

Theorem 4.3 *Let Condition 4.1 hold. Then there exist numbers $\omega > 0$ and $\sigma > 0$ such that, for any $\beta - \alpha \le \omega$, there exists a $2s$-periodic solution $z(x, t) = z(x, t; s)$ of problem (2.1), (2.3) such that $s \le \sigma$. On the interval $(0, \omega]$, the function $s = s(\beta - \alpha)$ is strictly monotonically increasing and $s \to 0$ as $\beta - \alpha \to 0$.*

Proof

1. By Condition 4.1, $F'(0) = \sum_{j=0}^{\infty} m_j K_j > 0$. Therefore, for sufficiently small $\beta - \alpha > 0$, the equation $F(s) = \beta - \alpha$ has a unique solution $s > 0$ in a small right-hand side neighborhood $(0, \sigma]$ of the origin. Clearly, the function $s = s(\beta - \alpha)$ possesses the properties from the theorem.
   Consider the solution $v(x, t) = v(x, t; s)$ of problem (2.1)–(2.3) with the initial data $\psi = \psi(s)$ defined in Steps 1–3 above.
   To complete the proof, it remains to show that $\hat{v}(t) = \hat{v}(t; s) < \beta$ for $t < s$ and apply Theorem 4.1.
2. Using representation (2.19), Remark 2.6, and formulas (4.7), we have for $t \le s$

$$\frac{d\hat{v}(t; s)}{dt} = m_0 K_0 + \sum_{j=1}^{\infty} m_j K_j \frac{2e^{-\lambda_j t}}{1 + e^{-\lambda_j s}} = M + \sum_{j=1}^{\infty} m_j K_j \left(\frac{2e^{-\lambda_j t}}{1 + e^{-\lambda_j s}} - 1 \right). \tag{4.13}$$

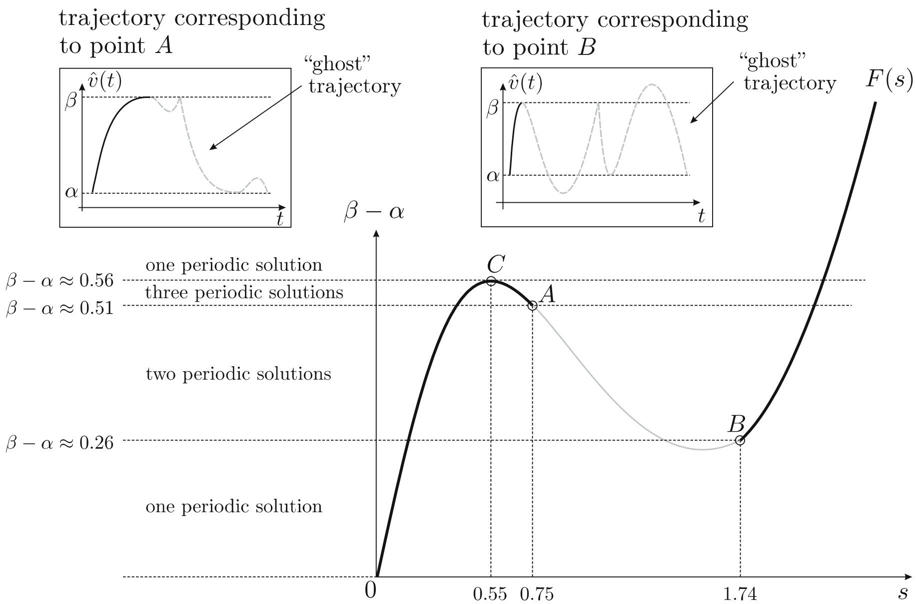


Fig. 6 Bifurcation diagram for $m_0 = 3.2, m_1 = m_2 = 4$, and $m_3 = m_4 = \cdots = 0$. For any $s > 0$, there exists a unique $2s$-periodic solution if the graph of F is bold at the point s and there are no $2s$-periodic solutions otherwise. For any $\beta - \alpha > 0$, there exist one, two, or three periodic solutions depending on whether the horizontal line levelled at $\beta - \alpha$ intersects the bold part of the graph of F at one, two, or three points, respectively. Points A ($s \approx 0.75$, $\beta - \alpha \approx 0.51$) and B ($s \approx 1.74$, $\beta - \alpha \approx 0.26$) on the graph correspond to $s \in S_2$. In each of these points, a $2s$-periodic solution does not exist. However, if one did not switch when $\hat{v}(t; s)$ tangentially intersected the hyperplane $\hat{\varphi} = \beta$ at the moment s, but switched only when $\hat{v}(t; s)$ intersected the hyperplane $\hat{\varphi} = \beta$ for the second time (at some moment $s_1 > s$), then the resulting trajectory would be $2s_1$ periodic. Such a trajectory is referred to as a "ghost" trajectory (see the insets). Point C ($s \approx 0.55$, $\beta - \alpha \approx 0.56$) corresponds to the fold bifurcation, where two periodic solutions merge into one and disappear as $\beta - \alpha$ increases and crosses the critical value ≈ 0.56

Using Remark 2.5, one can easily check that the absolute value of the series on the right-hand side is less than $M/2$ for sufficiently small s and $t \le s$. Therefore, $\hat{v}(t; s)$ is monotonically increasing until the first switching moment. Thus, the first switching occurs for $t = s$. □

We stress that Theorem 4.3 ensures the uniqueness of a periodic solution with a small first switching time s (hence small $\beta - \alpha$). However, the theorem does not forbid the existence of other periodic solutions with large period and large $\beta - \alpha$.

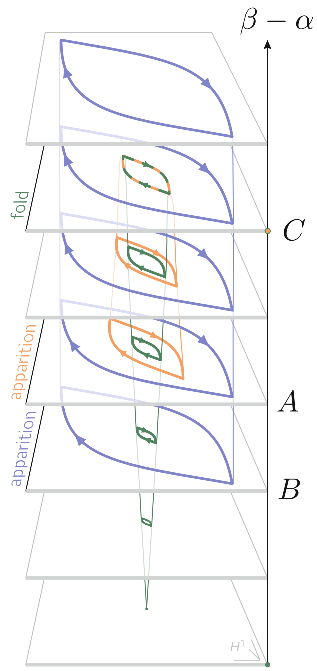
Remark 4.3 It is an open question whether one can choose the functions $m(x)$ and $K(x)$ and the parameters α and β in such a way that problem (2.1), (2.3) has no periodic solutions.

4.3 Stability of Periodic Solutions

In this section, we will show that the thermocontrol problem with hysteresis may admit unstable periodic solutions.

For simplicity, we assume that only finitely many Fourier coefficients m_j do not vanish (but see Remark 4.7).

Fig. 7 Visualization of "evolution" of periodic solutions with respect to the parameter $\beta-\alpha$ for $m_0=3.2$, $m_1=m_2=4$, and $m_3=m_4=\cdots=0$. For each $\beta-\alpha$, the horizontal plane represents the phase space H^1 with periodic solutions. Points A and B correspond to apparition (or termination) of periodic solutions, while point C corresponds to the fold bifurcation (cf. Fig. 6)



Condition 4.2 *There is* $N\geq 1$ *such that*

$$\mathbb{J}=\{m_0,m_1,\ldots,m_N\}.$$

Clearly, modifications needed if $\mathbb{J}$ consists of other Fourier coefficients m_j are trivial.

Remark 4.4 The fulfilment of Condition 4.2 implies that $m\in H^1$. Moreover, the sum in Condition 4.1 becomes finite:

$$\sum_{j=0}^{N} m_j K_j>0.$$

Remark 4.5 If $N=0$, i.e., $\mathbb{J}=\{m_0\}$, then it is easy to see that the (one-dimensional) guiding system (2.21) has a unique periodic solution for any α and β and this solution is uniformly exponentially stable. By Theorems 3.1 and 3.4, the same is true for the original problem (2.1), (2.3).

Assume that Condition 4.2 holds. Let $z(x,t)$ be a $2s$-periodic solution of problem (2.1), (2.3). Denote by $\tilde{\mathbf{z}}(t)=(z_0(t),\mathbf{z}(t))$ the corresponding $2s$-periodic solution of the guiding system (2.21). Let us study the map $\tilde{\Pi}_\alpha$ and the Poincaré map $\tilde{\Pi}$ (see Sect. 3) of the guiding system (2.21) in a neighborhood of $\tilde{\mathbf{z}}(0)$.

First of all, we consider the projections of these operators onto the N-dimensional space V (see (2.23)).

We consider the orthogonal projector

$$\mathbf{E}:\tilde{V}\to V$$

given by $\mathbf{E}\tilde{\boldsymbol{\varphi}}=\boldsymbol{\varphi}$, where

$$\tilde{\boldsymbol{\varphi}}=\{\varphi_j\}_{j=0}^N,\qquad \boldsymbol{\varphi}=\{\varphi_j\}_{j=1}^N.$$

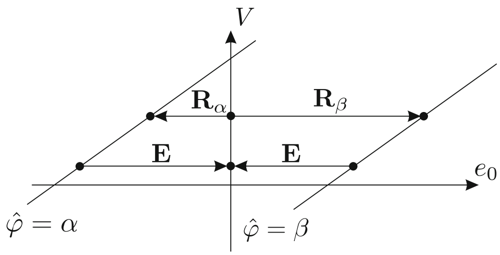


Fig. 8 The projection operator $\mathbf{E}$ and the lifting operators $\mathbf{R}_\alpha$ and $\mathbf{R}_\beta$

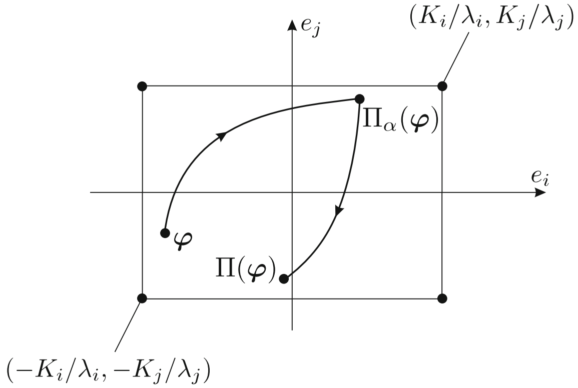


Fig. 9 The operators Π_α and $\Pi = \Pi_\beta \Pi_\alpha$ in the space $V = \mathrm{Span}(e_1, e_2, \dots, e_N)$

We also introduce the lifting operator

$$\mathbf{R}_\alpha : V \to \tilde{V}$$

given by

$$\mathbf{R}_\alpha(\boldsymbol{\varphi}) = \left(\frac{\alpha}{m_0} - \frac{1}{m_0} \sum_{k=1}^{N} m_k \varphi_k, \{\varphi_j\}_{j=1}^{N} \right).$$

Thus, $\mathbf{R}_\alpha \mathbf{E}(\tilde{\boldsymbol{\varphi}}) = \tilde{\boldsymbol{\varphi}}$ for $\tilde{\boldsymbol{\varphi}} \in \tilde{V}$ such that $\sum_{j=0}^{N} m_j \varphi_j = \alpha$, and $\mathbf{E}\mathbf{R}_\alpha(\boldsymbol{\varphi}) = \boldsymbol{\varphi}$ for $\boldsymbol{\varphi} \in V$ (see Fig. 8).

Denote by $\Pi_\alpha : V \to V$ the "projection" of $\tilde{\Pi}_\alpha$ onto V given by

$$\Pi_\alpha(\boldsymbol{\varphi}) = \mathbf{E} \tilde{\Pi}_\alpha \mathbf{R}_\alpha(\boldsymbol{\varphi}).$$

Similarly, one can define the operators $\mathbf{R}_\beta$ and Π_β.

The operators $\mathbf{E}$, $\mathbf{R}_\alpha$, and $\mathbf{R}_\beta$ are continuously (and even infinitely) differentiable. Therefore, the operators Π_α and Π_β are also continuously differentiable, provided so are $\tilde{\Pi}_\alpha$ and $\tilde{\Pi}_\beta$.

We introduce the operator $\Pi : V \to V$ by the formula

$$\Pi(\boldsymbol{\varphi}) = \mathbf{E} \tilde{\Pi} \mathbf{R}_\alpha(\boldsymbol{\varphi}).$$

The following property of Π is straightforward (see Fig. 9):

$$\Pi = \Pi_\beta \Pi_\alpha.$$

It is easy to see that the point $\boldsymbol{\psi} = \mathbf{z}(0)$ is a fixed point of the map Π acting in the N-dimensional space V.

In the formulation of the following results, we will use the following functions:

$$Q_j = Q_j(s) = \frac{2e^{-\lambda_j s}}{1 + e^{-\lambda_j s}}, \qquad Q = Q(s) = m_0 K_0 + \sum_{j=1}^{N} m_j K_j Q_j(s). \tag{4.14}$$

We note that, due to (4.13), we have at the switching moment s

$$\left.\frac{d\hat{z}(t)}{dt}\right|_{t=s} = Q(s). \tag{4.15}$$

In particular, this implies that $Q(s) \geq 0$.

Theorem 4.4 *Let Condition 4.2 hold, and let $z(x,t)$ be a $2s$-periodic solution of problem (2.1), (2.3). Assume that $Q(s) > 0$. Then $z(x,t)$ is stable (uniformly exponentially stable) if and only if the fixed point $\mathbf{z}(0)$ of the map Π is so.*

Proof Due to (4.15), we have $\left.\frac{d\hat{z}(t)}{dt}\right|_{t=s} > 0$. By symmetry, $\left.\frac{d\hat{z}(t)}{dt}\right|_{t=2s} < 0$. Now it remains to apply the formula $\tilde{\Pi}^i(\tilde{\boldsymbol{\varphi}}) = (\mathbf{R}_\alpha \Pi^i \mathbf{E})(\tilde{\boldsymbol{\varphi}})$ and Theorem 3.4. □

To study the stability of the point $\boldsymbol{\psi} = \mathbf{z}(0)$, we consider the derivative of Π at the point $\boldsymbol{\psi}$.

Lemma 4.5 *Let Condition 4.2 hold. If $Q(s) > 0$, then the operator $\Pi_\alpha : V \to V$ is differentiable in a neighborhood of $\boldsymbol{\psi} = \mathbf{z}(0)$ and the derivative*

$$D_{\boldsymbol{\psi}} \Pi_\alpha(\boldsymbol{\psi}) : V \to V$$

at the point $\boldsymbol{\psi} = \mathbf{z}(0)$ is given by

$$D_{\boldsymbol{\psi}} \Pi_\alpha(\boldsymbol{\psi})\boldsymbol{\varphi} = \sum_{j=1}^{N} e^{-\lambda_j s} \varphi_j e_j(x) + \frac{1}{Q(s)} \left(\sum_{k=1}^{N} m_k \left(1 - e^{-\lambda_k s}\right) \varphi_k \right) \sum_{j=1}^{N} K_j Q_j(s) e_j(x), \tag{4.16}$$

where $e_1(x), \ldots, e_N(x)$ form the basis in V and $Q_j(s)$ and $Q(s)$ are defined in (4.14).

Proof Since $Q(s) > 0$, it follows from (4.15) that $\left.\frac{d\hat{z}(t)}{dt}\right|_{t=s} > 0$. Therefore, applying Lemma 4.2 in [14], we have

$$D_{\boldsymbol{\psi}} \Pi_\alpha(\boldsymbol{\psi})\boldsymbol{\varphi} = \sum_{j=1}^{N} e^{-\lambda_j s} \varphi_j e_j(x) + \left(\left.\frac{d\hat{z}(t)}{dt}\right|_{t=s} \right)^{-1} \left(\sum_{k=1}^{N} m_k \left(1 - e^{-\lambda_k s}\right) \varphi_k \right)$$
$$\sum_{j=1}^{N} \lambda_j e^{-\lambda_j s} \left(\frac{K_j}{\lambda_j} - \psi_j \right) e_j(x).$$

Taking into account equalities (4.7), (4.14), and (4.15), we obtain the desired representation (4.16). □

Remark 4.6 Due to Lemma 4.5, the linear operator $D_{\boldsymbol{\psi}}\Pi_\alpha(\boldsymbol{\psi})$ is represented in the basis $e_1(x), \ldots, e_N(x)$ by the $(N \times N)$-matrix $\mathbf{A} = \mathbf{A}(s)$ of the form

$$\mathbf{A} = \begin{pmatrix} 1 - E_1 + S_1\sigma_1 & S_1\sigma_2 & S_1\sigma_3 & \ldots & S_1\sigma_N \\ S_2\sigma_1 & 1 - E_2 + S_2\sigma_2 & S_2\sigma_3 & \ldots & S_2\sigma_N \\ S_3\sigma_1 & S_3\sigma_2 & 1 - E_3 + S_3\sigma_3 & \ldots & S_3\sigma_N \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ S_N\sigma_1 & S_N\sigma_2 & S_N\sigma_3 & \ldots & 1 - E_N + S_N\sigma_N \end{pmatrix}, \quad (4.17)$$

where

$$E_j = E_j(s) = 1 - e^{-\lambda_j s}, \qquad S_j = S_j(s) = \frac{K_j Q_j(s)}{Q(s)}, \qquad \sigma_j = \sigma_j(s) = m_j E_j(s). \quad (4.18)$$

Note that $\mathbf{A}(0)$ is the identity matrix.

The following lemma results from Lemma 4.5 and from the symmetry of the periodic solution $z(x,t)$.

Lemma 4.6 *Let Condition 4.2 hold, and let $Q(s) > 0$. Then the operator $\Pi : V \to V$ is differentiable in a neighborhood of $\boldsymbol{\psi} = \mathbf{z}(0)$ and the derivative*

$$D_{\boldsymbol{\psi}}\Pi(\boldsymbol{\psi}) : V \to V$$

at the point $\boldsymbol{\psi} = \mathbf{z}(0)$ is given in the basis $e_1(x), \ldots, e_N(x)$ by the matrix $\mathbf{A}^2$, where $\mathbf{A}$ is defined in (4.17).

Denote the eigenvalues of the matrix $\mathbf{A} = \mathbf{A}(s)$ by $\mu_i = \mu_i(s)$, $i = 1, \ldots, N$.

The main result of this section is the following theorem. In particular, we will use it to construct unstable periodic solutions.

Theorem 4.5 *Let Condition 4.2 hold, and let $z(x,t)$ be a $2s$-periodic solution of problem (2.1), (2.3). Assume that $Q(s) > 0$. Then the following assertions are true.*

1. *All the eigenvalues μ_i of the matrix $\mathbf{A}$ satisfy $\mu_i \neq 1$.*
2. *If $|\mu_i| < 1$ for all $i = 1, \ldots N$, then the $2s$-periodic solution $z(x,t)$ of problem (2.1), (2.3) is uniformly exponentially stable.*
3. *If there is an eigenvalue μ_k such that $|\mu_k| > 1$, then the $2s$-periodic solution $z(x,t)$ of problem (2.1), (2.3) is unstable.*

Proof Assertion 1 follows from Lemma 4.7 below. It is known [16] that, under assumptions of items 2 and 3, a fixed point is, respectively, stable or unstable. By Theorem 4.4 and Lemma 4.6, this fact implies assertions 2 and 3. □

Corollary 4.1 *Let Conditions 4.1 and 4.2 hold. Then, for all sufficiently small $\beta - \alpha > 0$, there exists a $2s$-periodic solution $z(x,t)$ of problem (2.1), (2.3) and assertions 1–3 in Theorem 4.5 are true.*

Proof The existence of $z(x,t)$ follows from Theorem 4.3. Moreover, we have shown in the proof of Theorem 4.3 that $Q(s) > 0$ for all sufficiently small $\beta - \alpha > 0$, provided that Condition 4.1 holds. Thus, the hypothesis of Theorem 4.5 are true. Therefore, the conclusions are also true. □

Now we prove the following auxiliary result, which we have already used in the proof of Theorem 4.5.

Lemma 4.7 *Let Condition 4.2 hold. If $Q(s) \neq 0$, then the eigenvalues μ_i of* **A** *satisfy*

$$\prod_{i=1}^{N}(\mu_i - 1) = (-1)^N \frac{m_0 K_0}{Q} \prod_{i=1}^{N} E_i,$$

where Q is defined in (4.14) *and E_i in* (4.18).

Proof Substituting $\sigma_j = m_j E_j$, we have

$$\prod_{i=1}^{N}(\mu_i - 1) = |\mathbf{A} - \mathbf{I}| = \prod_{i=1}^{N} E_i \cdot |\mathbf{B}|,$$

where

$$\mathbf{B} = \begin{pmatrix} S_1 m_1 - 1 & S_1 m_2 & S_1 m_3 & \dots & S_1 m_N \\ S_2 m_1 & S_2 m_2 - 1 & S_2 m_3 & \dots & S_2 m_N \\ S_3 m_1 & S_3 m_2 & S_3 m_3 - 1 & \dots & S_3 m_N \\ \dots & \dots & \dots & \dots & \dots \\ S_N m_1 & S_N m_2 & S_N m_3 & \dots & S_N m_N - 1 \end{pmatrix}$$

and $|\cdot|$ stands for the determinant of a matrix.

Let us compute the determinant of **B**:

$$|\mathbf{B}| = m_1 \begin{vmatrix} S_1 & S_1 m_2 & S_1 m_3 & \dots & S_1 m_N \\ S_2 & S_2 m_2 - 1 & S_2 m_3 & \dots & S_2 m_N \\ S_3 & S_3 m_2 & S_3 m_3 - 1 & \dots & S_3 m_N \\ \dots & \dots & \dots & \dots & \dots \\ S_N & S_N m_2 & S_N m_3 & \dots & S_N m_N - 1 \end{vmatrix} - \begin{vmatrix} S_2 m_2 - 1 & S_2 m_3 & \dots & S_2 m_N \\ S_3 m_2 & S_3 m_3 - 1 & \dots & S_3 m_N \\ \dots & \dots & \dots & \dots \\ S_N m_2 & S_N m_3 & \dots & S_N m_N - 1 \end{vmatrix}.$$

To find the determinant of the first matrix, we multiply its first column by m_j and subtract it from the jth column for all $j = 2, \dots, N$. As a result, we have

$$|\mathbf{B}| = (-1)^{N-1} S_1 m_1 - \begin{vmatrix} S_2 m_2 - 1 & S_2 m_3 & \dots & S_2 m_N \\ S_3 m_2 & S_3 m_3 - 1 & \dots & S_3 m_N \\ \dots & \dots & \dots & \dots \\ S_N m_2 & S_N m_3 & \dots & S_N m_N - 1 \end{vmatrix}.$$

Similarly decomposing the second determinant, we obtain (after finitely many steps)

$$|\mathbf{B}| = (-1)^{N-1}(S_1 m_1 + \dots + S_N m_N - 1) = (-1)^N \frac{m_0 K_0}{Q}.$$

□

Remark 4.7 Let us discuss modifications needed in the case of infinite set $\mathbb{J}$ in Condition 4.2. The construction of the maps Π_α, Π_β, Π is quite similar and the modifications are obvious. The conclusion of Theorem 4.4 with the modified map Π remains true.

Formula (4.16) for the Fréchet derivative $D_{\boldsymbol{\psi}} \Pi_\alpha(\boldsymbol{\psi})$ remains the same but the sums become infinite. Their convergence follows from Remark 2.4. Formally, the linear operator $D_{\boldsymbol{\psi}} \Pi_\alpha(\boldsymbol{\psi})$ can be represented as the matrix **A** (see (4.17)), which now becomes infinite-dimensional.

It is proved in [14] that the operators Π_α, Π_β, Π are compact. Therefore, the same is true for their Fréchet derivatives. In particular, this means that the spectrum of $D_{\boldsymbol{\psi}}\Pi_\alpha(\boldsymbol{\psi})$ consists of no more than countably many eigenvalues, which may accumulate only at the origin. Thus, assertions 2 and 3 in Theorem 4.5 remain true (possibly with $N = \infty$ in assertion 2).

4.4 Corollaries

In this subsection, we assume that Condition 4.1 holds and that $\beta - \alpha > 0$ and $s > 0$ are sufficiently small. Then $Q(s) > 0$ and a $2s$-periodic solution $z(x, t)$ exists. Using Theorem 4.5, we provide some explicit conditions of its stability or instability. Moreover, we will show that a periodic solution may have a saddle structure.

The case $N = 0$ is trivial (see Remark 4.5), so we begin with the case $N = 1$.

Corollary 4.2 *Let Condition 4.2 hold with $N = 1$. Then, for any $\beta - \alpha > 0$, there exists a unique periodic solution $z(x, t)$ of problem (2.1), (2.3). The solution $z(x, t)$ is uniformly exponentially stable.*

Proof

1. By using the explicit formulas (Rermark 2.6) for the trajectories, we see that, for any trajectory $v(x, t)$, the function $\hat{v}(t)$ either increases for all $t > 0$ or first decreases and than increases. In particular, this implies that $dv/dt > 0$ at the first switching moment.
2. One can directly verify that the first characteristic function

$$F(s) := m_0 K_0 s + 2m_1 \frac{K_1}{\lambda_1} \cdot \frac{1 - e^{-\lambda_1 s}}{1 + e^{-\lambda_1 s}}$$

   satisfies one of the two conditions:

   (a) $F(s) > 0$ and increases for all $s > 0$, or
   (b) there is $s^* > 0$ such that $F(s) < 0$ for $0 < s < s^*$ and $F(s) > 0$ and increases for all $s > s^*$.

   In both cases, the equation $F(s) = \beta - \alpha$ has exactly one positive root s_1.
3. Due to the observation in part 1 of the proof, the second characteristic function

$$H(t, s) := m_0 k_0 (t - s) + 2m_1 \frac{K_1}{\lambda_1} \cdot \frac{e^{-\lambda_1 s} - e^{-\lambda_1 t}}{1 + e^{-\lambda_1 s}} = 0$$

   satisfies the inequality $H(t, s_1) < 0$ for all $t < s_1$. Therefore, by Theorem 4.1, there is a unique $2s_1$ periodic solution of problem (2.1), (2.3).
3. To prove its stability, we note that the matrix $\mathbf{A}$ consists of one element μ_1. It satisfies (due to Lemma 4.7 or by direct computation)

$$\mu_1 = 1 - E_1 + S_1\sigma_1 = 1 - (1 - e^{-\lambda_1 s_1}) \frac{m_0 K_0}{Q},$$

   where $Q > 0$ due to (4.15) and the observation in part 1 of the proof. If we show that $\mu_1 \in (-1, 1)$, then the stability result will follow from Theorem 4.5.
   Clearly, $\mu_1 \neq 1$ for $s_1 > 0$. One can also show that $\mu_1 \neq -1$ for $s_1 > 0$. To do so, one can check for example that the equation $\mu_1 = -1$ uniquely determines $m_1 K_1$ as a function of the other parameters. Then substituting it into the formula for $F(s_1)$ yields the contradiction $F(s_1) < 0$.
   Since $\mu_1 \neq \pm 1$, $\mu_1 \in (-1, 1)$ for sufficiently large s_1, and μ_1 continuously depends on s_1, it follows that $\mu_1 \in (-1, 1)$ for any s_1. □

Now we consider the case $N = 2$.

Corollary 4.3 *Let Condition 4.2 hold with* $N = 2$*, and let*

$$M = m_0 K_0 + m_1 K_1 + m_2 K_2 > 0. \tag{4.19}$$

Then, for all sufficiently small $\beta - \alpha > 0$*, there exists a* $2s$*-periodic solution* $z(x, t)$ *of problem* (2.1)*,* (2.3) *uniquely determined by Theorem 4.3. If*

$$(M - m_1 K_1)\lambda_1 + (M - m_2 K_2)\lambda_2 < 0, \tag{4.20}$$

then $|\mu_1|, |\mu_2| > 1$ *and* $z(x, t)$ *is unstable for all sufficiently small* $\beta - \alpha > 0$*. If*

$$(M - m_1 K_1)\lambda_1 + (M - m_2 K_2)\lambda_2 > 0, \tag{4.21}$$

then $|\mu_1|, |\mu_2| < 1$ *and* $z(x, t)$ *is exponentially stable for all sufficiently small* $\beta - \alpha > 0$.

Proof

1. The matrix $\mathbf{A}$ is a (2×2)-matrix. Therefore, it has two eigenvalues μ_1 and μ_2, which are either both real or complex conjugate. Denote $\delta = \delta_{1,2} = \mu_{1,2} - 1$. Clearly, $\delta_{1,2}$ are the eigenvalues of $\mathbf{A} - \mathbf{I}$; hence, they are the roots of the quadratic equation

$$\delta^2 - \operatorname{tr}(\mathbf{A} - \mathbf{I})\delta + |\mathbf{A} - \mathbf{I}| = 0. \tag{4.22}$$

Let us compute $\operatorname{tr}(\mathbf{A} - \mathbf{I})$ and $|\mathbf{A} - \mathbf{I}|$. Due to (4.17) and (4.18),

$$\operatorname{tr}(\mathbf{A} - \mathbf{I}) = E_1 \left(-1 + \frac{m_1 K_1 Q_1}{Q} \right) + E_2 \left(-1 + \frac{m_2 K_2 Q_2}{Q} \right).$$

On the other hand, formulas (4.18) and (4.14) imply that $E_j = \lambda_j s + O(s^2)$, and $Q_j(s) = 1 + O(s)$, and $Q(s) = M + O(s)$. Therefore,

$$\begin{aligned} \operatorname{tr}(\mathbf{A} - \mathbf{I}) = s(\lambda_1 + O(s)) \left(-1 + \frac{m_1 K_1}{M} + O(s) \right) + s(\lambda_2 + O(s)) \\ \times \left(-1 + \frac{m_2 K_2}{M} + O(s) \right) = -(Ls + O(s^2)), \end{aligned} \tag{4.23}$$

where

$$L = M^{-1}((M - m_1 K_1)\lambda_1 + (M - m_2 K_2)\lambda_2).$$

Further, by Lemma 4.7,

$$\begin{aligned} |\mathbf{A} - \mathbf{I}| &= E_1 E_2 \frac{m_0 K_0}{Q} = s^2(\lambda_1 + O(s))(\lambda_2 + O(s)) \left(\frac{m_0 K_0}{M} + O(s) \right) \\ &= J^2 s^2 + O(s^3), \end{aligned} \tag{4.24}$$

where

$$J^2 = \lambda_1 \lambda_2 \frac{m_0 K_0}{M}.$$

It follows from (4.23) and (4.24) that Eq. 4.22 is equivalent to the following:

$$\delta^2 + (Ls + O(s^2))\delta + J^2 s^2 + O(s^3) = 0.$$

Thus,

$$\mu_{1,2} = 1 - \frac{Ls}{2} \pm \frac{s\sqrt{L^2 - 4J^2 + O(s)}}{2} + O(s^2).$$

2. If inequality (4.20) holds, then $L < 0$ and $\operatorname{Re}\mu_{1,2} > 1$ for all small $s > 0$.
   Assume that inequality (4.21) holds, i.e., $L > 0$. If $L^2 - 4J^2 + O(s) \geq 0$, then the eigenvalues $\mu_{1,2}$ are real and belong to the interval $(0, 1)$. If $L^2 - 4J^2 + O(s) < 0$, then $\mu_{1,2}$ are complex conjugate and
$$(\operatorname{Re}\mu_1)^2 + (\operatorname{Im}\mu_1)^2 = 1 - Ls + O(s^2) < 1,$$
   i.e., $|\mu_{1,2}| < 1$. □

Example 4.2 Consider the problem described in Example 4.1.

Let $m_0 > 0$, $m_1 = m_2 > 0$, and $m_3 = m_4 = \cdots = 0$. Then condition (4.19) holds. Therefore, condition (4.20), which implies the instability of the periodic solution for small s, takes the form
$$\frac{m_0}{m_1} < \sqrt{2}\frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1} = \frac{3\sqrt{2}}{5},$$
while condition (4.21), which implies the uniform exponential stability of the periodic solution for small s, takes the form
$$\frac{m_0}{m_1} > \sqrt{2}\frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1} = \frac{3\sqrt{2}}{5}.$$

Finally, we show that periodic solutions can be unstable for $N \geq 3$. Moreover, if N is odd, they may have a saddle structure.

Corollary 4.4 *Let Condition 4.2 hold with $N \geq 3$, and let*
$$M = \sum_{j=0}^{N} m_j K_j > 0. \tag{4.25}$$
Then, for all sufficiently small $\beta - \alpha > 0$, there exists a $2s$-periodic solution $z(x, t)$ of problem (2.1), (2.3) *uniquely determined by Theorem 4.3. If*
$$\sum_{j=1}^{N} (M - m_j K_j)\lambda_j < 0, \tag{4.26}$$
then $z(x, t)$ is unstable.

If we additionally assume that N is odd, then there is an eigenvalue of $D_\psi \Pi(\mathbf{z}(0))$ with real part greater than 1 and a real eigenvalue in the interval $(0, 1)$.

Proof

1. The matrix $\mathbf{A}$ is an $(N \times N)$-matrix. Due to (4.17), (4.18), and (4.26),
$$\sum_{j=1}^{N} \mu_j = \operatorname{tr}\mathbf{A} = N + \sum_{j=1}^{N} (S_j m_j - 1)E_j$$
$$= N - M^{-1}\sum_{j=1}^{N} (M - m_j K_j)\lambda_j s + O(s^2) > N$$
   for sufficiently small $s > 0$. Therefore, the real part of at least one eigenvalue is greater than 1. By Theorem 4.5, this implies the instability of $z(x, t)$.

2. Now we additionally assume that N is odd. By Lemma 4.7,

$$\prod_{j=1}^{N}(\mu_j - 1) < 0.$$

Since N is odd, the set of eigenvalues of $\mathbf{A}$ consists of an odd number of real eigenvalues $\mu_1, \ldots, \mu_L$ $(1 \le L \le N)$ and $(N - L)/2$ pairs of complex conjugate eigenvalues. Therefore,

$$\prod_{j=1}^{L}(\mu_j - 1) < 0.$$

Hence, there is at least one eigenvalue, e.g., μ_1, which is real and is less than 1. Taking into account that $\mu_j(0) = 1$ and $\mu_j(s)$ continuously depend on s, we see that $\mu_1 \in (0, 1)$. Applying Lemma 4.6, we complete the proof. □

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