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SOLVABILITY OF THE BOUNDARY VALUE PROBLEM FOR SOME DIFFERENTIAL–DIFFERENCE EQUATIONS

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Abstract.

The problems of solvability and smoothness of generalized solutions to boundary value problems for differential–difference equations on a finite interval $(0, d)$ in not self-adjoint case were considered in [1]. The interest to these problems was arisen by their numerous applications as well as by a number of quite new properties they possess. For instance, the smoothness of generalized solutions to such problems may fail inside the interval $(0, d)$ even in the case of infinitely differentiable right hand side of the equation and remains only in some subintervals. In [1] necessary and sufficient conditions of Fredholmian solvability and smoothness of solutions to such problems on the whole interval were established in the case of non–integer d. In the case of integer d only sufficient conditions were obtained. The problem of obtaining necessary and sufficient conditions was formulated in [1] as an unsolved one. This paper is dedicated to the solution of this problem.

In section 1, the properties of difference operators in Sobolev spaces are considered. In section 2, the necessary and sufficient conditions of Fredholmian solvability (with index zero) of a boundary value problem for a differential–difference equations are established. In section 3, the smoothness of the generalized solutions is considered in terms of the index of the corresponding differential–difference operator.

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1. Difference operators in the spaces $L_2(\mathbf{R})$, $L_2(0, N+1)$, and in the Sobolev spaces $W^k(0, N+1)$. We consider the *difference* operator $R : L_2(\mathbf{R}) \to L_2(\mathbf{R})$ defined by the formula

$$(Rv)(t) = \sum_{j=-N}^{N} b_j y(t+j). \tag{1}$$

Here b_j are real numbers, N is a natural number.

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We introduce the operators

$$I_Q : L_2(0, N+1) \to L_2(\mathbf{R}), \quad P_Q : L_2(\mathbf{R}) \to L_2(0, N+1),$$
$$R_Q : L_2(0, N+1) \to L_2(0, N+1)$$

by the formulas

$$(I_Q v)(t) = \begin{cases} v(t) & (t \in (0, N+1)), \\ 0 & (t \notin (0, N+1)); \end{cases} \quad (P_Q v)(t) = v(t) \quad (t \in (0, N+1));$$
$$R_Q = P_Q R I_Q. \tag{2}$$

Here $Q = (0, N+1)$.

We denote $Q_s = (s-1, s) \quad (s = 1, \dots, N+1)$.

We introduce an isomorphism of the Hilbert spaces

$$U : L_2(\cup_s Q_s) \to L_2^{N+1}(Q_1)$$

by the formula

$$(Uv)_k(t) = v(t + k - 1) \quad (t \in Q_1,\ k = 1, \dots, N+1), \tag{3}$$

where $L_2^{N+1}(Q_1) = \prod_{k=1}^{N+1} L_2(Q_1)$.

Let R_1 be the matrix of order $(N+1) \times (N+1)$ with the elements $r_{ik} = b_{k-i} \quad (i, k = 1, \dots, N+1)$. Let R_2 be the matrix of order $N \times N$ obtained from R_1 by deleting the last column and the last row. We denote also by B_{ik} the cofactor of the element r_{ik} of the matrix R_1.

Consider the operator $R_{Q1} : L_2^{N+1}(Q_1) \to L_2^{N+1}(Q_1)$ defined by the formula $R_{Q1} = U R_Q U^{-1}$.

Now we shall formulate the next four Lemmas (proofs are given in [1], Chapter I, Section 2).

Lemma 1. *The operator R_{Q1} is the operator of multiplication by the matrix R_1.*

Lemma 2. *The spectrum of the operator R_Q coincides with the spectrum of the matrix R_1.*

Lemma 3. *The operator R_Q maps continuously $\mathring{W}^k(0, N+1)$ into $W^k(0, N+1)$ and, for all $v \in \mathring{W}^k(0, N+1)$,*

$$(R_Q v)^{(j)} = R_Q v^{(j)} \quad (j \le k). \tag{4}$$

Lemma 4. *Let* $\det R_1 \neq 0$ *and let* $R_Q v \in W^k(Q_i)$ *for* $i = 1, \dots, N+1$. *Then* $v \in W^k(Q_j)$ $(j = 1, \dots, N+1)$ *and*

$$\|v\|_{W^k(Q_j)} \leq c \sum_{i=1}^{N+1} \|R_Q v\|_{W^k(Q_i)},$$

where $c > 0$ *doesn't depend on* v.

We denote by $W^k_\gamma(0, N+1)$ the subspace of functions from $W^k(0, N+1)$ satisfying conditions

$$u^{(\mu)}(N+1) = \sum_{1 \leq i \leq N+1,\, i \neq m+1} \gamma_{1i} u^{(\mu)}(i-1), \tag{5}$$

$$u^{(\mu)}(m) = \sum_{1 \leq i \leq N,\, i \neq m} \gamma_{2i} u^{(\mu)}(i), \tag{6}$$

where m is a fixed number from the set $\{1, \dots, N\}$, γ_{1i} $(i = 1, \dots, N+1,\ i \neq m+1)$, γ_{2i} $(i = 1, \dots, N,\ i \neq m)$ are real numbers; $\mu = 0, \dots, k-1$; $k \geq 1$.

Hereinafter, we shall assume that the following conjecture is fulfilled.

Conjecture 1. *We assume that* $\det R_1 \neq 0,\ \det R_2 = 0$.

The other cases have been studied in [1], Chapter I.

Theorem 1. *There exist real numbers* γ_{1i} $(i = 1, \dots, N+1,\ i \neq m+1)$, γ_{2i} $(i = 1, \dots, N,\ i \neq m)$ *such that the operator* R_Q *maps* $\mathring{W}^k(0, N+1)$ *onto* $W^k_\gamma(0, N+1)$ *continuously and in a one–to–one manner.*

Proof. 1. At first we proof that there exist γ_{1i} $(i = 1, \dots, N+1,\ i \neq m+1)$, γ_{2i} $(i = 1, \dots, N,\ i \neq m)$ such that $R_Q(\mathring{W}^k(0, N+1)) \subset W^k_\gamma(0, N+1)$.

We denote by $R_1^1 (R_1^2)$ the matrix, obtained from R_1 by deleting the first (the last) column. Denote by $e_i\,(g_i)$ the i-th row of the matrix $R_1^1 (R_1^2)$.

The condition $\det R_2 = 0$ implies that $g_1, \dots, g_N$ are linearly dependent. Hence there exists a number m from the set $\{1, \dots, N\}$ such that the row g_m is a linear combination of the other ones

$$g_m = \sum_{1 \leq i \leq N,\, i \neq m} \gamma_{2i} g_i, \tag{7}$$

where γ_{2i} $(i = 1, \dots, N,\ i \neq m)$ are real numbers.

It is easy to see that $e_{i+1} = g_i$ $(i = 1, \dots, N)$. Therefore, using (7), we get

$$e_{m+1} = \sum_{1\le i\le N,\, i\ne m} \gamma_{2i} e_{i+1}, \tag{8}$$

i.e.,

$$e_{m+1} = \sum_{2\le i\le N+1,\, i\ne m+1} \gamma_{2,i-1} e_i. \tag{9}$$

From non-singularity of the matrix R_1 it follows that the rows e_i $(i = 1, \dots N+1,\ i \ne m+1)$ form the basis in $\mathbf{R}^N$ and the rows g_j $(j = 1, \dots N+1,\ j \ne m)$ also form the basis in $\mathbf{R}^N$.

By Lemma (3), $R_Q(\mathring{W}^k(0, N+1)) \subset W^k(0, N+1)$. Thus (3), (7) and Lemma (1) implies that, for $v \in \mathring{W}^k(0, N+1)$ and $\mu = 0, \dots, k-1$,

$$\begin{aligned} (R_Q v)^{(\mu)}(m) &= (UR_Q v)^{(\mu)}_m(1) \\ &= (R_1 U v^{(\mu)})_m(1) = \sum_{1\le i\le N,\, i\ne m} \gamma_{2i}(R_1 U v^{(\mu)})_i(1) \\ &= \sum_{1\le i\le N,\, i\ne m} \gamma_{2i}(R_Q v)^{(\mu)}(i) \qquad (\mu = 0, \dots, k-1). \end{aligned} \tag{10}$$

Further,

$$\begin{aligned} (R_Q v)^{(\mu)}(N+1) &= (UR_Q v)^{(\mu)}_{N+1}(1) \\ &= (R_1 U v^{(\mu)})_{N+1}(1) = \sum_{s=1}^{N} r_{N+1,s}(Uv^{(\mu)})_s(1) \\ &= \sum_{s=1}^{N} r_{N+1,s}(Uv^{(\mu)})_{s+1}(0) = \sum_{s=2}^{N+1} r_{N+1,s-1}(Uv^{(\mu)})_s(0). \end{aligned} \tag{11}$$

And, in the same way,

$$\begin{aligned} (R_Q v)^{(\mu)}(i-1) &= (UR_Q v)^{(\mu)}_i(0) = (R_1 U v^{(\mu)})_i(0) \\ &= \sum_{s=2}^{N+1} r_{is}(Uv^{(\mu)})_s(0) \quad (i = 1, \dots, N+1). \end{aligned} \tag{12}$$

Since the rows e_i $(i = 1, \dots, N+1;\ i \ne m+1)$ form the basis in $\mathbf{R}^N$, it follows that

$$g_{N+1} = \sum_{1\le i\le N+1,\, i\ne m+1} \gamma_{1i} e_i,$$

i.e.,

$$(r_{N+1,1}, \dots, r_{N+1,N}) = \sum_{1\le i\le N+1,\, i\ne m+1} \gamma_{1i}(r_{i2}, \dots, r_{i,N+1}). \tag{13}$$

Now, using (11), (12), (13), we get

$$(R_Q v)^{(\mu)}(N+1) = \sum_{1\le i\le N+1,\, i\ne m+1} \gamma_{1i}(R_Q v)^{(\mu)}(i-1) \quad (\mu = 0,\dots,k-1). \quad (14)$$

Therefore, by virtue (10) and (14), $R_Q(\mathring{W}^k(0,N+1)) \subset W^k_\gamma(0,N+1)$.

2. Now let us prove the inverse inclusion

$$W^k_\gamma(0,N+1) \subset R_Q(\mathring{W}^k(0,N+1)).$$

Suppose $u \in W^k_\gamma(0,N+1)$. By virtue of Lemma (2), the operator $R_Q : L_2(0,N+1) \to L_2(0,N+1)$ has a bounded inverse $R_Q^{-1} : L_2(0,N+1) \to L_2(0,N+1)$. We shall show that $v = R_Q^{-1}u \in \mathring{W}^k(0,N+1)$.

By virtue of Lemma (4), $v \in W(Q_s)\,(s = 1,\dots,N+1)$. Therefore, to prove this theorem, it is sufficient to prove that

$$(Uv)^{(\mu)}_s(1-0) = (Uv)^{(\mu)}_{s+1}(0+0) \quad (s = 1,\dots,N),$$

$$(Uv)^{(\mu)}_1(0+0) = (Uv)^{(\mu)}_{N+1}(1-0) = 0.$$

Denote

$$\varphi^\mu_s = (Uv)^{(\mu)}_{s+1}(0+0) \quad (s = 0,\dots,N;\ \mu = 0,\dots,k-1);$$
$$\psi^\mu_j = (Uv)^{(\mu)}_j(1-0) \quad (j = 1,\dots,N+1;\ \mu = 0,\dots,k-1).$$

Since $R_Q v \in W^k(0,N+1)$, we have

$$(R_Q v)^{(\mu)}|_{t=i-0} = (R_Q v)^{(\mu)}|_{t=i+0} \quad (i = 1,\dots,N;\ \mu = 0,\dots,k-1).$$

Thus, for every $\mu = 0,\dots,k-1$, the functions φ^μ_s, ψ^μ_j satisfy the following conditions

$$\sum_{s=1}^{N+1} r_{i+1,s}\varphi^\mu_{s-1} = \sum_{s=1}^{N+1} r_{is}\psi^\mu_s \quad (i = 1,\dots,N). \quad (15)$$

Moreover, the function $R_Q v$ satisfies conditions (10), which can be rewritten in the form

$$\sum_{s=1}^{N+1} r_{ms}\psi^\mu_s = \sum_{1\le i\le N,\, i\ne m} \gamma_{2i} \sum_{s=1}^{N+1} r_{is}\psi^\mu_s \quad (16)$$

or in the form

$$\begin{aligned}\sum_{s=1}^{N+1} r_{m+1,s}\varphi^{\mu}_{s-1} &= \sum_{1\le i\le N,\, i\ne m} \gamma_{2i} \sum_{s=1}^{N+1} r_{i+1,s}\varphi^{\mu}_{s-1}\\ &= \sum_{2\le i\le N+1,\, i\ne m+1} \gamma_{2,i-1} \sum_{s=1}^{N+1} r_{is}\varphi^{\mu}_{s-1}.\end{aligned} \tag{17}$$

From conditions (16), (17) and (7), (9), we obtain

$$\begin{aligned}&\Big(r_{m,N+1} - \sum_{1\le i\le N,\, i\ne m} \gamma_{2i} r_{i,N+1}\Big)\ \psi^{\mu}_{N+1} = 0,\\ &\Big(r_{m+1,1} - \sum_{2\le i\le N+1,\, i\ne m+1} \gamma_{2,i-1} r_{i1}\Big)\ \varphi^{\mu}_{0} = 0.\end{aligned}$$

The factor preceding ψ^{μ}_{N+1} (φ^{μ}_0) is non–zero. Otherwise, we have $\det R_1 = 0$, which contradicts Conjecture (1). Hence $\psi^{\mu}_{N+1} = \varphi^{\mu}_0 = 0$.

Thus system (15) will have the form

$$\sum_{s=1}^{N} r_{i+1,s+1}\varphi^{\mu}_s = \sum_{s=1}^{N} r_{is}\psi^{\mu}_s \quad (i = 1, \dots, N).$$

Since $r_{i+1,s+1} = r_{is}$ and the m-th row of this system is a linear combination of the other ones, this system will have the form

$$\sum_{s=1}^{N} r_{is}\varphi^{\mu}_s = \sum_{s=1}^{N} r_{is}\psi^{\mu}_s \quad (i = 1, \dots, N;\ i \ne m). \tag{18}$$

Now, using the condition $\psi^{\mu}_{N+1} = \varphi^{\mu}_0 = 0$, we rewrite relations (14) in the following form:

$$\sum_{s=1}^{N} r_{N+1,s}\psi^{\mu}_s = \sum_{1\le i\le N+1,\, i\ne m+1} \gamma_{1i} \sum_{s=1}^{N} r_{i,s+1}\varphi^{\mu}_s. \tag{19}$$

The condition (13) implies that

$$\sum_{1\le i\le N+1,\, i\ne m+1} \gamma_{1i} \sum_{s=1}^{N} r_{i,s+1}\varphi^{\mu}_s = \sum_{s=1}^{N} r_{N+1,s}\varphi^{\mu}_s.$$

Thus, using the last relation and relation (19), we obtain

$$\sum_{s=1}^{N} r_{N+1,s}\varphi^{\mu}_s = \sum_{s=1}^{N} r_{N+1,s}\psi^{\mu}_s. \tag{20}$$

Combining (18) and (20), we get the system of N equations with N unknowns

$$\sum_{s=1}^{N} r_{is}(\varphi_s^\mu - \psi_s^\mu) = 0 \quad (i = 1, \dots, N+1;\ i \neq m). \tag{21}$$

The rows of system (21) coincide with the linearly independent rows $g_i\,(i = 1, \dots, N+1;\ i \neq m)$. Hence $\varphi_s^\mu - \psi_s^\mu = 0$, i.e., $\varphi_s^\mu = \psi_s^\mu$ $(s = 1, \dots, N;\ \mu = 0, \dots, k-1)$. We have thus proved that $W_\gamma^k(0, N+1) \subset R_Q(\mathring{W}^k(0, N+1))$. □

Remark 1. *It can be given the following equivalent definition of the subspace* $W_\gamma^k(0, N+1)$. $W_\gamma^k(0, N+1)$ *is the subspace of functions from* $W^k(0, N+1)$ *satisfying conditions*

$$u^{(\mu)}(0) = \sum_{1\le i\le N+1,\, i\ne m'} \gamma'_{1i} u^{(\mu)}(i),$$
$$u^{(\mu)}(m') = \sum_{1\le i\le N,\, i\ne m'} \gamma'_{2i} u^{(\mu)}(i),$$

where m' *is a fixed point from the set* $\{1, \dots, N\}$, $\gamma'_{1i}(i = 1, \dots, N+1,\ i \neq m')$, $\gamma'_{2i}(i = 1, \dots, N,\ i \neq m')$ *are real numbers;* $\mu = 0, \dots, k-1;\ k \geq 1$.

Let us introduce the sets

$$\begin{aligned} M &= \{u \in \mathring{W}^1(0, N+1) : R_Q u \in W^2(0, N+1)\}, \\ M_k &= \{u \in \mathring{W}^1(0, N+1) : u,\ R_Q u \in W^{k+2}(0, N+1)\} = \\ &= \{u \in M : u,\ R_Q u \in W^{k+2}(0, N+1)\}, \end{aligned}$$

where $k = 0, 1, \dots$

These sets will play the role of the domains of the corresponding differential–difference operators.

We denote by G_j^1 (G_j^2) the j-th column of the $N\times(N+1)$-matrix obtained from R_1 by deleting the first (last) row $(j = 1, \dots, N+1)$. Notice that Conjecture (1) implies that $G_1^1 \neq 0$, $G_{N+1}^2 \neq 0$.

The following lemma allows to find out the structure of the sets M_k.

Lemma 5. *For any* $n \geq 2$, *we have:*
(a) Suppose that G_1^1 *and* G_{N+1}^2 *are linearly independent. Then*

$$\{v \in M : v,\ R_Q v \in W^n(0, N+1)\} = \mathring{W}^n(0, N+1).$$

(b) Suppose that G_1^1 and G_{N+1}^2 are linearly dependent. Then

$$\begin{aligned}&\{v \in M : v,\ R_Q v \in W^n(0, N+1)\}\\ &\qquad = \{v \in M : R_Q v \in W^n(0, N+1),\ (Uv)_{l+1}^{(\mu)}(0+0) = (Uv)_l^{(\mu)}(1-0),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mu = 1, \dots, n-1\},\end{aligned}$$

where $l \in \{1, \dots, N\}$ is a number satisfying the following condition: determinant of the matrix with the elements r_{ij}, where $1 \le i, j \le N$, $i \ne m$, $j \ne l$, doesn't equal zero. (By virtue of the linearly independence of the rows g_i, $i = 1, \dots N$, $i \ne m$, there really exists such a point l.)

Proof. First let us prove (a).

The inclusion $\mathring{W}^n(0, N+1) \subset \{v \in M : v,\ R_Q v \in W^n(0, N+1)\}$ follows from Lemma (3). Let us prove the inverse inclusion.

Let $v \in \mathring{W}^1(0, N+1) \cap W^n(0, N+1)$, $R_Q v \in W^n(0, N+1)$. Then, using the notation of Theorem (1), for all $\mu = 1, \dots, n-1$, we obtain

$$\sum_{s=1}^{N+1} r_{i+1,s}\varphi_{s-1}^{\mu} = \sum_{s=1}^{N+1} r_{is}\psi_s^{\mu} \qquad (i = 1, \dots, N). \tag{22}$$

Regrouping the summands in (22) and noticing that $r_{i+1,s+1} = r_{is}$ ($1 \le i, s \le N$), we get

$$\sum_{s=1}^{N} r_{is}(\varphi_s^{\mu} - \psi_s^{\mu}) = -r_{i+1,1}\varphi_0^{\mu} + r_{i,N+1}\psi_{N+1}^{\mu} \qquad (i = 1, \dots, N). \tag{23}$$

Since $v \in W^n(0, N+1)$, we have $\varphi_s^{\mu} = \psi_s^{\mu}$ ($s = 1, \dots, N$). Hence

$$-r_{i+1,1}\varphi_0^{\mu} + r_{i,N+1}\psi_{N+1}^{\mu} = 0 \qquad (i = 1, \dots, N).$$

But the last relations are equivalent to the following:

$$-G_1^1\varphi_0^{\mu} + G_{N+1}^2\psi_{N+1}^{\mu} = 0.$$

Thus, by virtue of the linearly independence of G_1^1 and G_{N+1}^2, we have $\varphi_0^{\mu} = \psi_{N+1}^{\mu} = 0$. This implies that $v \in \mathring{W}^n(0, N+1)$.

Now let us prove (b). The inclusion

$$\begin{aligned}&\{v \in M : v,\ R_Q v \in W^n(0, N+1)\}\\ &\subset \{v \in M : R_Q v \in W^n(0, N+1),\ (Uv)_{l+1}^{(\mu)}(0+0) = (Uv)_l^{(\mu)}(1-0),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mu = 1, \dots, n-1\}\end{aligned}$$

is obviously.

Let us prove the inverse inclusion. Let

$$v \in \{v \in M : R_Q v \in W^n(0, N+1),\ (Uv)^{(\mu)}_{l+1}(0+0) = (Uv)^{(\mu)}_{l}(1-0),\ \mu = 1, \dots, n-1\}.$$

Note that it cannot be written "$v^{(\mu)}(l-0) = v^{(\mu)}(l+0)$" here and in the statement of the lemma because we don't know beforehand if the derivative of order μ for the function v belongs to the corresponding Sobolev space. Thus we have to write "$(Uv)^{(\mu)}_{l+1}(0+0) = (Uv)^{(\mu)}_{l}(1-0)$".

Since G^1_1 and G^2_{N+1} are linearly dependent, $G^1_1 \neq 0$ and $G^2_{N+1} \neq 0$, there exist non–zero real numbers α_1, α_2 such that

$$\alpha_1 G^1_1 + \alpha_2 G^2_{N+1} = 0. \tag{24}$$

Now we shall show that, in this case,

$$\alpha_1 (Uv)^{(\mu)}_{N+1}(1-0) + \alpha_2 (Uv)^{(\mu)}_{1}(0+0) \equiv \alpha_1 \psi^{\mu}_{N+1} + \alpha_2 \varphi^{\mu}_0 = 0. \tag{25}$$

Denote $w = R_Q v$. Since $(Uv)^{(\mu)}(t) = (R_1^{-1} U w^{(\mu)})(t) \quad (t \in (0,1))$, we can rewrite (25) in the form

$$\alpha_1 \sum_{i=1}^{N+1} \frac{B_{i,N+1}}{\det R_1} (Uw^{(\mu)})_i(1-0) + \alpha_2 \sum_{i=1}^{N+1} \frac{B_{i1}}{\det R_1} (Uw^{(\mu)})_i(0+0) = 0. \tag{26}$$

Since $B_{11} = B_{N+1,N+1} = \det R_2 = 0$, relation (26) has the form

$$\sum_{i=1}^{N} (\alpha_1 B_{i,N+1} + \alpha_2 B_{i+1,1}) w^{(\mu)}(i) = 0. \tag{27}$$

Then, analyzing $B_{i,N+1}$, $B_{i+1,1}$ and using (24), we see that $\alpha_1 B_{i,N+1} + \alpha_2 B_{i+1,1} = 0$ $(i = 1, \dots, N)$.

Therefore (27) is identical, i.e., (25) is valid for any

$$v \in \{v \in M : R_Q v \in W^n(0, N+1),\ (Uv)^{(\mu)}_{l+1}(0+0) = (Uv)^{(\mu)}_{l}(1-0),\ \mu = 1, \dots, n-1\}.$$

Further, we have (likewise (23))

$$\sum_{s=1}^{N} r_{is}(\varphi^{\mu}_s - \psi^{\mu}_s) = -r_{i+1,1}\varphi^{\mu}_0 + r_{i,N+1}\psi^{\mu}_{N+1} \qquad (i = 1, \dots, N). \tag{28}$$

By virtue (24), (25), system (28) will have the form

$$\sum_{s=1}^{N} r_{is}(\varphi^{\mu}_s - \psi^{\mu}_s) = 0 \qquad (i = 1, \dots, N). \tag{29}$$

Since $\varphi_l^\mu = \psi_l^\mu$ and the m-th row of (29) is a linear combination of the other ones, system (29) is equivalent to the following:

$$\sum_{1\le s\le N,\, s\ne l} r_{is}(\varphi_s^\mu - \psi_s^\mu) = 0 \qquad (i = 1,\dots,N;\ i \ne m). \tag{30}$$

Thus we have the system of $(N-1)$ equations with $(N-1)$ unknowns. Selection of point l implies non-singularity of the matrix of system (30). This system has a unique trivial solution. Hence, for any $\mu = 0,\dots,n-1$, we get $\varphi_s^\mu = \psi_s^\mu$ $(s = 1,\dots\ N)$. Therefore $v \in W^n(0, N+1)$ and thus Lemma (5) is proved. □

Let $R_Q^k : W^{k+2}(0, N+1) \to W^{k+2}(0, N+1)$ be a bounded operator defined by $\mathcal{D}(R_Q^k) = M_k$, $\quad R_Q^k v = R_Q v\, (v \in \mathcal{D}(R_Q^k))$, where $k \ge 0$.

THEOREM 2. *The operator* $R_Q^k\,(k \ge 0)$ *is Fredholm,* $\dim\ker(R_Q^k) = 0$,
$$\operatorname{codim}\operatorname{Im}(R_Q^k) = \begin{cases} 2(k+2), & \textit{if } G_1^1,\ G_{N+1}^2 \textit{ are linearly independent,}\\ k+3, & \textit{if } G_1^1,\ G_{N+1}^2 \textit{ are linearly dependent.}\end{cases}$$

Proof. Let G_1^1, G_{N+1}^2 be linearly independent. In this case, by virtue of Lemma (5), the domain M_k of the operator R_Q^k coincides with the space $\mathring{W}^{k+2}(0, N+1)$. By virtue of Theorem (1), the operator R_Q^k maps M_k onto $W_\gamma^{k+2}(0, N+1)$ in a one–to–one manner. This implies that $\dim\ker(R_Q^k) = 0$.

Now let us find $\operatorname{codim}\operatorname{Im}(R_Q^k)$. We consider the equation

$$R_Q^k u = w \qquad (w \in W^{k+2}(0, N+1)). \tag{31}$$

Theorem (1) implies that equation (31) has a solution $u \in M_k = \mathring{W}^{k+2}(0, N+1)$ iff $w \in W_\gamma^{k+2}(0, N+1)$, i.e., iff w satisfies the conditions

$$\begin{aligned} w^{(\mu)}(N+1) &= \sum_{1\le i\le N+1,\, i\ne m+1} \gamma_{1i} w^{(\mu)}(i-1),\\ w^{(\mu)}(m) &= \sum_{1\le i\le N,\, i\ne m} \gamma_{2i} w^{(\mu)}(i) \qquad (\mu = 0,\dots,k+1). \end{aligned}$$

We introduce $2(k+2)$ linear functionals $F_{j\mu}\,(j = 0,1;\ \mu = 0,\dots,k+1)$ by the formulas

$$\begin{aligned} F_{0\mu}(w) &= w^{(\mu)}(N+1) - \sum_{1\le i\le N+1,\, i\ne m+1} \gamma_{1i} w^{(\mu)}(i-1),\\ F_{1\mu}(w) &= w^{(\mu)}(m) - \sum_{1\le i\le N,\, i\ne m} \gamma_{2i} w^{(\mu)}(i). \end{aligned} \tag{32}$$

By virtue of the trace theorem (for example, see [2]), $F_{j\mu}$ are continuous functionals over $W^{k+2}(0, N+1)$. It is not hard to check that $F_{j\mu}$ are linearly independent.

From the Riesz theorem it follows that $F_{j\mu}(w)=(w,f_{j\mu})_{W^{k+2}(0,N+1)}$, where $f_{j\mu}\in W^{k+2}(0,N+1)$ $(j=0,1;\ \mu=0,\dots,k+1)$ are linearly independent functions. This implies that $\operatorname{codim}\operatorname{Im}(R_Q^k)=2(k+2)$.

Now we consider the other case. Let G_1^1, G_{N+1}^2 be linearly dependent. Since $\mathcal{D}(R_Q^k)\subset \mathring{W}^1(0,N+1)$, R_Q maps $\mathring{W}^1(0,N+1)$ onto $W_\gamma^1(0,N+1)$ in a one–to–one manner, and $R_Q\supset R_Q^k$, it follows that $\dim\ker(R_Q^k)=0$.

Let us find $\operatorname{codim}\operatorname{Im}(R_Q^k)$. We consider the equation

$$R_Q^k v=w \qquad (w\in W^{k+2}(0,N+1)). \tag{33}$$

From Theorem (1) and Lemma (5), it follows that equation (33) has a solution $v\in M_k$ iff w satisfies the conditions

$$w(N+1)=\sum_{1\le i\le N+1,\, i\ne m+1}\gamma_{1i}w(i-1),\quad w(m)=\sum_{1\le i\le N,\, i\ne m}\gamma_{2i}w(i), \tag{34}$$

$$(Uv)_{l+1}^{(\mu)}(0+0)=(Uv)_l^{(\mu)}(1-0)\qquad (\mu=1,\dots,k+1). \tag{35}$$

Since

$$(Uv)_{l+1}^{(\mu)}(0+0)=\sum_{i=1}^{N+1}\frac{B_{i,l+1}}{\det R_1}(Uw^{(\mu)})_i(0+0),$$

$$(Uv)_l^{(\mu)}(1-0)=\sum_{i=1}^{N+1}\frac{B_{il}}{\det R_1}(Uw^{(\mu)})_i(1-0),$$

conditions (35) will have the form

$$\sum_{i=1}^{N+1}B_{i,l+1}w^{(\mu)}(i-1)=\sum_{i=1}^{N+1}B_{il}w^{(\mu)}(i).$$

And, after regrouping the summands, we obtain, for $\mu=1,\dots,k+1$,

$$B_{1,l+1}w^{(\mu)}(0)+\sum_{i=1}^{N}(B_{i+1,l+1}-B_{il})w^{(\mu)}(i)-B_{N+1,l}w^{(\mu)}(N+1)=0. \tag{36}$$

Thus a solution u of equation (33) belongs to M_k iff w satisfies conditions (34) and (36). Further, as above, we can introduce $k+3$ linear continuous functionals over $W^{k+2}(0,N+1)$, corresponding conditions (34), (36), and prove that they are linearly independent. (To prove it one can use the condition $B_{N+1,l}\ne 0$ which follows from Conjecture (1) and the condition on the point l.) And, as above, using the Riesz theorem, we get $\operatorname{codim}\operatorname{Im}(R_Q^k)=k+3$. □

2. The boundary value problem for the differential–difference equation with homogeneous boundary conditions. We consider the *differential–difference equation*

$$-(Rv)''(t) + (A_1 v)(t) = f_0(t) \qquad (t \in (0, N+1)) \tag{37}$$

with homogeneous boundary conditions

$$v(t) = 0 \qquad (t \in [-N, 0] \cup [N+1, 2N+1]). \tag{38}$$

Here $R : L_2(\mathbf{R}) \to L_2(\mathbf{R})$ is the difference operator defined by

$$(Rv)(t) = \sum_{j=-N}^{N} b_j v(t+j),$$

$b_j \in \mathbf{R}$; $N \in \mathbf{N}$; $A_1 : \mathring{W}^1(0, N+1) \to L_2(0, N+1)$ is a linear bounded operator; $f_0 \in L_2(0, N+1)$. One can easily reduce a differential–difference equation with non-homogeneous boundary conditions to differential–difference equation with homogeneous boundary conditions (see section 3). Therefore, without loss of generality, we can study the equation (37) with homogeneous boundary conditions (38).

Since the shifts $t \to t + j$ can map the points of the interval $[0, N+1]$ into the set $[-N, 0] \cup [N+1, 2N+1]$, we consider the boundary conditions for the equation (37) not only at the ends of the interval $[0, N+1]$, but also on the set $[-N, 0] \cup [N+1, 2N+1]$.

Let $A_R : L_2(0, N+1) \to L_2(0, N+1)$ be the unbounded operator given by

$$\begin{aligned} \mathcal{D}(A_R) &= M = \{v \in \mathring{W}^1(0, N+1) : R_Q v \in W^2(0, N+1)\}, \\ A_R v &= -(R_Q v)''(t) + A_1 v \qquad (v \in \mathcal{D}(A_R)). \end{aligned}$$

Definition 1. *A function $v \in \mathcal{D}(A_R)$ is called a generalized solution to problem (37), (38) if $A_R v = f_0$.*

Theorem 3. *The operator A_R is Fredholm and $\operatorname{ind} A_R = 0$.*

To prove Theorem (3) we shall first consider the bounded operator $A : W^2(0, N+1) \cap W^1_\gamma(0, N+1) \to L_2(0, N+1)$ defined by the formula

$$Au = -u'' + A_1 R_Q^{-1} u.$$

Here we suppose that the space $W^2(0, N+1) \cap W^1_\gamma(0, N+1)$ has a topology of the space $W^2(0, N+1)$. Let us prove the following lemma.

LEMMA 6. *The bounded operator A is Fredholm and* $\operatorname{ind} A = 0$.

Proof. We introduce the bounded operator $A_2 : W^2(0, N+1) \cap W^1_\gamma(0, N+1) \to L_2(0, N+1)$ defined by the formula

$$A_2 u = u''(t).$$

Here we also suppose that the space $W^2(0, N+1) \cap W^1_\gamma(0, N+1)$ has a topology of $W^2(0, N+1)$.

Thus we have $A = -A_2 + A_1 R_Q^{-1}$. We show that the operator A_2 is Fredholm and $\operatorname{ind} A_2 = 0$.

It is clear that the homogeneous equation $A_2 u \equiv u''(t) = 0$ has a class of solutions $u(t) = c_1 t + c_2$ from $W^2(0, N+1)$. Therefore u belongs to $\ker(A_2)$ iff u satisfies conditions (5), (6) (for $\mu = 0$)

$$\begin{aligned} &c_1\Big[N+1-\sum_{2\le i\le N+1,\, i\ne m+1}\gamma_{1i}(i-1)\Big]+c_2\Big[1-\sum_{1\le i\le N+1,\, i\ne m+1}\gamma_{1i}\Big]=0,\\ &c_1\Big[m-\sum_{1\le i\le N,\, i\ne m}\gamma_{2i}i\Big]+c_2\Big[1-\sum_{1\le i\le N,\, i\ne m}\gamma_{2i}\Big]=0. \end{aligned} \tag{39}$$

Parallel with the homogeneous equation, we shall consider the non-homogeneous equation

$$A_2 v \equiv v''(t) = f(t) \qquad (f \in L_2(0, N+1)).$$

For any function $f \in L_2(0, N+1)$, there exists a class of solutions $v(t) = d_1 t + d_2 + \int\limits_0^t (t-\tau) f(\tau)\, d\tau$ from $W^2(0, N+1)$. Therefore v belongs to the domain of the operator A_2 iff v satisfies conditions (5), (6) (for $\mu = 0$)

$$\begin{aligned} &d_1\Big[N+1-\sum_{2\le i\le N+1,\, i\ne m+1}\gamma_{1i}(i-1)\Big]+d_2\Big[1-\sum_{1\le i\le N+1,\, i\ne m+1}\gamma_{1i}\Big]\\ &\quad=\sum_{1\le i\le N,\, i\ne m}\gamma_{1,i+1}\int\limits_0^i(i-\tau)f(\tau)\,d\tau-\int\limits_0^{N+1}(N+1-\tau)f(\tau)\,d\tau,\\ &d_1\Big[m-\sum_{1\le i\le N,\, i\ne m}\gamma_{2i}i\Big]+d_2\Big[1-\sum_{1\le i\le N,\, i\ne m}\gamma_{2i}\Big]\\ &\quad=\sum_{1\le i\le N,\, i\ne m}\gamma_{2i}\int\limits_0^i(i-\tau)f(\tau)\,d\tau-\int\limits_0^m(m-\tau)f(\tau)\,d\tau. \end{aligned} \tag{40}$$

It is clear that $\Phi_i(f) \equiv (f, \phi_i)_{L_2(0,N+1)} \equiv \int\limits_0^i (i-\tau) f(\tau)\, d\tau$ $(i = 1, \dots, N+1)$ are the linear continuous functionals over $L_2(0, N+1)$ (here $\phi_i(\tau) = (i-\tau) I(i-\tau)$, where $I(t) = 1,\ t \ge 0;\ I(t) = 0,\ t < 0$).

It is not hard to prove that the functionals $\Phi_i\,(i=1,\dots,N+1)$ are linearly independent. This implies that

$$F_1(f)=\sum_{1\le i\le N,\,i\ne m}\gamma_{1,i+1}\Phi_i(f)-\Phi_{N+1}(f),$$
$$F_2(f)=\sum_{1\le i\le N,\,i\ne m}\gamma_{2i}\Phi_i(f)-\Phi_m(f)$$

are also non-zero linearly independent continuous functionals over $L_2(0,N+1)$.

Thus system (40) will have the form

$$d_1[N+1-\sum_{1\le i\le N+1,\,i\ne m+1}\gamma_{1i}(i-1)]+d_2[1-\sum_{1\le i\le N+1,\,i\ne m+1}\gamma_{1i}]=F_1(f),$$

$$d_1[m-\sum_{1\le i\le N,\,i\ne m}\gamma_{2i}i]+d_2[1-\sum_{1\le i\le N,\,i\ne m}\gamma_{2i}]=F_2(f). \tag{41}$$

We analyse system (39) and system (41) simultaneously. Notice that the matrix of system (39) coincides with the matrix of system (41). Denote this matrix by $\mathcal{M}$. Let us consider three cases.

1. $\operatorname{Rank}(\mathcal{M})=2$. It is easy to see that we have $\dim\ker(A_2)=0$, $\operatorname{codim}\operatorname{Im}(A_2)=0$, i.e., $\operatorname{ind}A_2=0$.

2. $\operatorname{Rank}(\mathcal{M})=1$. Clearly, $\dim\ker(A_2)=1$. Using the Riesz theorem, we obtain $\operatorname{codim}\operatorname{Im}(A_2)=1$. Hence, in this case, we also have $\operatorname{ind}A_2=0$.

3. $\operatorname{Rank}(\mathcal{M})=0$. In this case, we see that $\dim\ker(A_2)=2$. Using again the Riesz theorem, we obtain $\operatorname{codim}\operatorname{Im}(A_2)=2$, i.e., $\operatorname{ind}A_2=0$.

Thus we have proved that A_2 is Fredholm and $\operatorname{ind}A_2=0$.

It is not hard to check that the operator $A_1R_Q^{-1}: W^2(0,N+1)\cap W^1_\gamma(0,N+1)\to L_2(0,N+1)$ is bounded if the space $W^2(0,N+1)\cap W^1_\gamma(0,N+1)$ has a topology of $W^1(0,N+1)$. Therefore, by virtue of the compactness of the embedding operator from $W^2(0,N+1)$ into $W^1(0,N+1)$, the operator $A_1R_Q^{-1}: W^2(0,N+1)\cap W^1_\gamma(0,N+1)\to L_2(0,N+1)$ is compact if the space $W^2(0,N+1)\cap W^1_\gamma(0,N+1)$ has a topology of $W^2(0,N+1)$. Using the theorem about the compact perturbations (see [3], theorem 16.4), we have that the operator $A=-A_2+A_1R_Q^{-1}$ is Fredholm and $\operatorname{ind}A=0$. □

Now let us prove Theorem (3).

Proof of Theorem (3). The operator A_R can be presented as a composition $A_R=A\tilde{R}_Q$, where
$A: W^2(0,N+1)\cap W^1_\gamma(0,N+1)\to L_2(0,N+1)$ is given by

$$Au=-u''+A_1R_Q^{-1}u,$$

$\tilde{R}_Q : L_2(0, N+1) \to W^2(0, N+1) \cap W^1_\gamma(0, N+1) \subset W^2(0, N+1)$ is given by

$$\begin{aligned} \mathcal{D}(\tilde{R}_Q) &= \mathcal{D}(A_R) = M, \\ \tilde{R}_Q u &= R_Q u \qquad (u \in \mathcal{D}(\tilde{R}_Q)). \end{aligned}$$

By virtue of Lemma (6) and Theorem (1), the operators A and $\tilde{R}_Q$ are Fredholm and $\operatorname{ind} A = \operatorname{ind} \tilde{R}_Q = 0$. Hence the operator $A_R = A\tilde{R}_Q$ is also Fredholm and $\operatorname{ind} A_R = 0$ (see [3], theorem 12.2). □

3. Smoothness of generalized solutions to boundary value problem. It is known that the smoothness of generalized solutions of differential–difference equations can be broken even for infinitely differentiable right hand sides of equations. But there exists the following result.

THEOREM 4. *Let $f_0 \in W^k(0, N+1)$ and v be a generalized solution to boundary value problem (37), (38) such that $A_1 v \in W^k(0, N+1)$.*
Then $v \in W^{k+2}(Q_s),\ s = 1, \dots, N+1$.

Proof. The proof follows from Lemma (4). □

To obtain a smoothness of generalized solutions it is necessary to impose some additional conditions on right hand side of the equation (and on the boundary functions, in the case of non-homogeneous boundary conditions). Now we shall find out a type of these conditions for the case of the homogeneous boundary value problem.

We consider the bounded operator $A^k_R : W^{k+2}(0, N+1) \to W^k(0, N+1)$ given by

$$\begin{aligned} &\mathcal{D}(A^k_R) = M_k, \\ &A^k_R v = -(R_Q v)''(t), \end{aligned}$$

and the bounded operator $B^k_R : \mathring{W}^{k+2}(0, N+1) \to W^k(0, N+1)$ defined by the formula $B^k_R v = -(R_Q v)''(t)$.

Note that, by virtue of Lemma (5), A^k_R coincides with B^k_R if $G^1_1,\ G^2_{N+1}$ are linearly independent.

THEOREM 5. *The operator A^k_R $(k \geq 0)$ is Fredholm,* $\dim \ker(A^k_R) = 0$,

$$\operatorname{codim} \operatorname{Im}(A^k_R) = \begin{cases} 2(k+1), & \textit{if } G^1_1,\ G^2_{N+1} \textit{ are linearly independent,} \\ k+1, & \textit{if } G^1_1,\ G^2_{N+1} \textit{ are linearly dependent.} \end{cases}$$

Proof. First we prove that $\dim\ker(A_R^k)=0$. Let $v\in\ker(A_R^k)$. Then $(R_Qv)''(t)=0$. Hence $(R_Qv)(t)=c_1+c_2t$. Since $\det R_1\neq 0$, we obtain

$$v(t)=U^{-1}R_1^{-1}U(c_1+c_2t)\qquad (t\in(0,N+1)).$$

Thus a function v is piecewise linear on the interval $(0,N+1)$. Therefore $v\in W^2(0,N+1)\cap\mathring{W}^1(0,N+1)$ if and only if $v(t)=0$, i.e., $\dim\ker(A_R^k)=0$.

Let us present the operator A_R^k as a composition $A_R^k=A_2R_Q^k$. Here $R_Q^k:W^{k+2}(0,N+1)\to W^{k+2}(0,N+1)$ is the operator introduced in section 1, $A_2:W^{k+2}(0,N+1)\to W^k(0,N+1)$ is the bounded operator defined by the formula $(A_2v)(t)=-v''(t)$. It is obvious that A_2 is Fredholm and $\operatorname{ind}A_2=2$. Therefore, using Theorem (2) and the theorem about a composition of Fredholmian operators (see [3], theorem 12.2), we obtain the statement of Theorem (5). □

THEOREM 6. *The operator* B_R^k $(k\geq 0)$ *is Fredholm,* $\dim\ker(B_R^k)=0$, $\operatorname{codim}\operatorname{Im}(B_R^k)=2(k+1)$.

Proof. The idea of the proof is analogous to the previous proof. □

Now we shall generalize these results to the case of the boundary value problem with non-homogeneous boundary conditions.

We consider the differential–difference equation

$$-(Ry)''(t)+A_1y=f_0(t)\qquad (t\in(0,N+1)) \tag{42}$$

with non-homogeneous boundary conditions

$$\begin{cases} y(t)=f_1(t) & (t\in[-N,0]),\\ y(t)=f_2(t) & (t\in[N+1,2N+1]),\end{cases} \tag{43}$$

where

$$(Ry)(t)=\sum_{j=-N}^{N}b_jy(t+j),$$

$b_j\in\mathbf{R}$, N is a natural number; $A_1:W^1(-N,2N+1)\to L_2(0,N+1)$ is a linear bounded operator, $f=(f_0,f_1,f_2)\in\mathcal{W}(-N,2N+1)=L_2(0,N+1)\times W^1(-N,0)\times W^1(N+1,2N+1)$.

We introduce the linear unbounded operator $\mathcal{L}:L_2(-N,2N+1)\to\mathcal{W}(-N,2N+1)$ with the domain $\mathcal{D}(\mathcal{L})=\{y\in W^1(-N,2N+1):P_QRy\in W^2(0,N+1)\}$ by the formula

$$\mathcal{L}y=\left(-(P_QRy)''+A_1y,\,y|_{(-N,0)},\,y|_{(N+1,2N+1)}\right).$$

Definition 2. *A function $y \in \mathcal{D}(\mathcal{L})$ is called a generalized solution to problem (42), (43) if* $\mathcal{L}y = (f_0, f_1, f_2)$.

To obtain the smoothness of the generalized solution in the interval $(-N,\, 2N+1)$ we suppose that $A_1 : W^{k+1}(-N,\, 2N+1) \to W^k(0, N+1)$ is a bounded operator and $f = (f_0, f_1, f_2) \in \mathcal{W}^k(-N,\, 2N+1) = W^k(0, N+1) \times W^{k+2}(-N, 0) \times W^{k+2}(N+1,\, 2N+1)$.

We consider the linear bounded operator $\mathcal{L}_B : W^{k+2}(-N,\, 2N+1) \to \mathcal{W}^k(-N,\, 2N+1)$ by the formula

$$\mathcal{L}_B y = \mathcal{L}y \qquad (y \in W^{k+2}(-N,\, 2N+1)).$$

Theorem 7. *The operator $\mathcal{L}_B$ is Fredholm and* $\operatorname{ind} \mathcal{L}_B = -2(k+1)$.

Proof. By virtue of the compactness of the imbedding operator from $W^{k+2}(-N,\, 2N+1)$ into $W^{k+1}(-N,\, 2N+1)$, we have that the operator $A_1 : W^{k+2}(-N,\, 2N+1) \to W^k(0, N+1)$ is compact. Therefore, by theorem 16.4, [3], it suffices to prove Theorem (7) in the case $A_1 = 0$.

Let us assume now that $A_1 = 0$

We introduce the function

$$\psi(t) = \begin{cases} f_1(t) & (t \in [-N, 0]), \\ f_2(t) & (t \in [N,\, 2N+1]), \\ \eta(t) \sum\limits_{i=0}^{k+1} f_1^{(i)}(0) t^i / i! + \eta(t-N-1) \sum\limits_{i=0}^{k+1} f_2^{(i)}(N+1)(t-N-1)^i / i! & \\ \qquad\qquad (t \in (0, N+1)), & \end{cases}$$

where $\eta \in \dot{C}^\infty(\mathbf{R})$, $\eta(t) = 1$ $(|t| < 1/4)$, $\eta(t) = 0$ $(|t| > 1/3)$. It is clear that $\psi \in W^{k+2}(-N,\, 2N+1)$. Denote $w = y - \psi \in W^{k+2}(0, N+1)$ $(y \in W^{k+2}(-N,\, 2N+1))$. We see that the equation $\mathcal{L}_B y = f$ $(f \in \mathcal{W}^k(-N,\, 2N+1))$ has a solution $y \in W^{k+2}(-N,\, 2N+1)$ iff w belongs to $\mathring{W}^{k+2}(0, N+1)$ and is a solution of the equation

$$B_R^k w = f_0 + (R\psi)''. \tag{44}$$

By Theorem (6), equation (44) has a solution if and only if

$$(f_0 + (R\psi)'', \varphi_j)_{W^k(0,N+1)} = 0 \qquad (j = 1, \dots, 2(k+1)), \tag{45}$$

where $\varphi_j \in W^k(0, N+1)$ are linearly independent functions.

From the trace theorem and the Riesz theorem it follows that conditions (45) will have the form

$$(f, G_j)_{\mathcal{W}^k(-N,\, 2N+1)} = 0 \qquad (j = 1, \dots, 2(k+1)), \tag{46}$$

where $f=(f_0,f_1,f_2)$, vector-valued functions $G_j=(\varphi_j,B_1\varphi_j,B_2\varphi_j)$ are linearly independent (here $B_1:W^k(0,N+1)\to W^{k+2}(-N,0)$, $B_2:W^k(0,N+1)\to W^{k+2}(N+1,\,2N+1)$ are linear bounded operators). Thus for $A_1=0$ the equation $\mathcal{L}_B y=f$ has a solution $y\in W^{k+2}(-N,\,2N+1)$ for $f\in\mathcal{W}^k(-N,\,2N+1)$ if and only if conditions (46) are fulfilled.

Furthermore, by Theorem (6), $\dim\ker(\mathcal{L}_B)=0$. □

If we demand the smoothness of the solution only in the interval $(0,N+1)$, we can weaken the conditions of orthogonality in some cases.

To formalize this statement we suppose that $A_1:W^1(-N,\,2N+1)\to W^k(0,N+1)$ is a compact operator. Let us introduce the unbounded operator $\mathcal{L}_A:W^1(-N,2N+1)\to W^k(0,N+1)\times W^1(-N,0)\times W^1(N+1,\,2N+1)$ with the domain $\mathcal{D}(\mathcal{L}_A)=\{y\in W^1(-N,\,2N+1):P_Qy,P_QRy\in W^{k+2}(0,N+1)\}$ by the formula

$$\mathcal{L}_A y=\mathcal{L}y\qquad(y\in\mathcal{D}(\mathcal{L}_A)).$$

Theorem 8. *The operator $\mathcal{L}_A$ is Fredholm and*

$$\operatorname{ind}\mathcal{L}_A=\begin{cases}-2(k+1), & \text{if } G_1^1,\,G_{N+1}^2 \text{ are linearly independent,}\\ -(k+1), & \text{if } G_1^1,\,G_{N+1}^2 \text{ are linearly dependent.}\end{cases}$$

Proof. The proof is analogous to the previous proof. The main distinction refers to the operator on left hand side of equation (44) to which we reduce boundary value problem (42), (43).

In this case, A_R^k takes the place of B_R^k. □

Remark 2. *Using Lemma (5), one can easily show that $\mathcal{D}(\mathcal{L}_A)=W^{k+2}(-N,\,2N+1)$ if $G_1^1,\,G_{N+1}^2$ are linearly independent. In this case, a generalized solution has a proper smoothness in the whole interval* $(-N,\,2N+1)$.

Thus we see that the smoothness of generalized solutions to the boundary value problem to differential–difference equations is not broken in the interval $(0,N+1)$ (in the interval $(-N,\,2N+1)$) if we impose not only the conditions of smoothness but also some conditions of orthogonality on the right hand side of the differential–difference equation and on the boundary functions.

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