Masters Thesis

Local well-poseness of a reaction-diffusion equation with hysteresis

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Contents

1 Introduction

This thesis deals with reaction-diffusion equations where the nonlinearity is a hysteresis operator defined at every spatial point in a domain $Q \subset \mathbb{R}^2$. Such systems were first introduced in [8] and [9] to model pattern formation in systems in which a non-diffusing substance interacts with diffusing substances via a hysteresis law. In these experiments a colony of bacteria (salmonella typhimurium, denoted B) is fixed to the surface of a petri dish however they have been denied an amino acid (denoted H) required for their growth. After adding the missing amino acid, the growing bacteria produce a growth inhibiting buffer (denoted G) as a by-product. Over the period of observation, the bacteria grow in a distinctive concentric ring pattern (see Figure 1a).

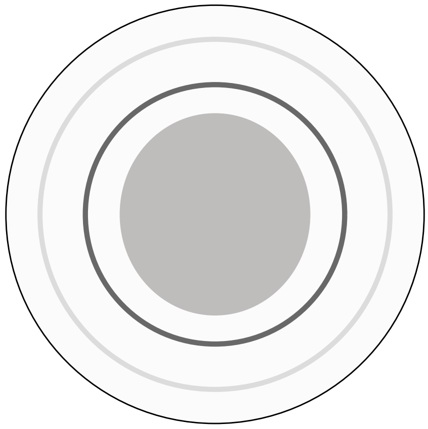
The system is modelled by the following differential equation where Q is an open disk in $\mathbb{R}^2$

$$\begin{cases} \dfrac{\partial B}{\partial t} = \alpha V B, \\ \dfrac{\partial H}{\partial t} = D_H \Delta H - \beta V B, \\ \dfrac{\partial G}{\partial t} = D_G \Delta G - \gamma V B, \end{cases} \tag{1.1}$$

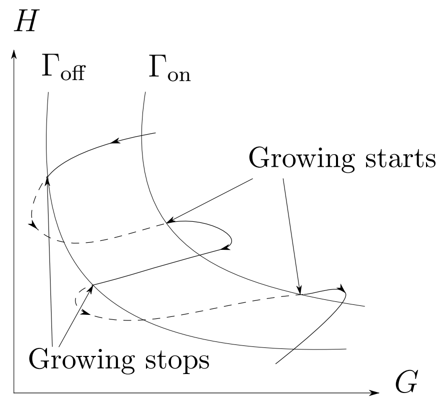
where α, β, γ are constants that are large compared to the diffusion rates D_H and D_G. Here $V(G, H, V)$ is a function describing the internal mechanism of a bacterium where V can take values 0 (no growth) or 1 (growth). The function V behaves according to a hysteresis law. In detail, one defines two curves Γ_{off} and Γ_{on} in the G, H plane, which divide the first quadrant into three regions (see Figure 1a). If (G, H) is below Γ_{off} then $V = 0$, if it is above Γ_{on} then $V = 1$ and if it between the two curves then V maintains its current value ($V = 0$, $V = 1$) and switches value when (G, H) crosses the appropriate threshold (Γ_{on}, Γ_{off} respectively). Indeed this operator has memory hence the dependency of V on itself in addition to its dependency on G, H.

Numerical simulations of problem (1.1) were able to reproduce the patterns observed in experiment, however questions on the existence and uniqueness of solutions, as well as their continuous dependence on initial data remain open. In this thesis we present the first well-posedness results where Q has dimension > 1 and V is discontinuous hysteresis, not a regularized analogue.

Such problems arise naturally as the singular limits of slow-fast systems,



(a) Bacteria density at end of experiment



(b) Sold lines = bacteria growing, dashed lines = bacteria not growing.

Figure 1: Bacterial Growth Model

where V is replaced by a "fast" ODE, for example

$$\varepsilon \frac{\partial V}{\partial t} = g(G, H, V)$$

As $\varepsilon \to 0$, V behaves like the hysteresis operator described above. In the case of the problem posed in [8] and [9], the parameter ε arises due to the difference in the rate at which the substances diffuse and the rate at which the bacteria respond to changes in the concentration of G and H. For a more detailed discussion of this connection see [5], the content of which will inform how we pose the question of uniqueness of solutions to the problems considered in this thesis.

The essential feature of these problems is that at any given time, V can take on different values at different points of Q, and that the switching rule is given by two distinct thresholds (as opposed to, for example, the Stefan problem which contains only one). If we consider Q divided into two regions, one where $V = 0$ and the other where $V = 1$, the boundaries between these regions are free boundaries. There movement is due to the time dependent switching behaviour of the hysteresis operator, which in turn occurs when V crosses the appropriate threshold. Thus we must simultaneously consider the free boundary itself and points in Q where the input function takes on a threshold value. The interaction of the free boundary and the level set of the threshold value will form the mathematical core of this thesis.

There are already many results on parabolic equations containing some form of regularized hysteresis. We will give a superficial explanation of some of these results.

One such example is the Preisach operator. A particular case thereof is described by the following integral

$$\mathcal{P}(g)(t) = \int_0^\infty \int_{-\infty}^\infty \rho(r,s)\overline{H}_{s-r,s+r}(\zeta_0(s), g)(t)\, dsdr \tag{1.2}$$

We will not explain this equation in detail, but choose only to highlight that $g : \mathbb{R} \to \mathbb{R}$ is an input, $\overline{H}_{s-r,s+r}$ is a hysteresis operator with thresholds $s-r$ and $s+r$ (a definition of $\overline{H}$ is given in (2.1)) and $\rho(r,s)$ a density that vanishes when r and s are large (see [2], example 2.1.6). In other words we are taking an average over a family of hysteresis operators, each of which have different thresholds. In [19], a theorem on weak solutions to the problem

$$\frac{\partial u}{\partial t} + \frac{\partial w}{\partial t} - \Delta u = f(\xi, t)$$

is formulated where $\xi \in Q \subset \mathbb{R}^n$ and w is a version of $\mathcal{P}$ which is defined at every spatial point. The original papers are [16] and [17].

Another model is to take multivalued hysteresis. Using the notation of the bacteria model, this would mean that in addition to $V = 0$ and $V = 1$ we allow that $V \in [0,1]$ when (G, H) is on the curve Γ_{off} or Γ_{on}. An example of well-posedness of a parabolic equation where $Q \subset \mathbb{R}$ is an interval with multivalued hysteresis is given in [1].

A summary of results concerning hysteresis in biological models can be found in [10], where a version of the bacteria problem where $\frac{\partial B}{\partial t}$ does not depend on B is formulated. The proof of well-posedness for this simpler problem can be found in [18].

Turning our attention to the non-regularized hysteresis, rigorous results are much more current. For an interval $Q \subset \mathbb{R}$, existence, uniqueness and continuous dependence on initial data were obtained obtained in [7] and [5] for a single equation and for systems in [6]. In [7], the authors considered the existence of solutions to the following problem,

$$\begin{aligned} u_t - \Delta u &= f(u, \mathcal{H}(\eta_0, u)), \\ u|_{t=0} &= \varphi, \\ \left.\frac{\partial u}{\partial \nu}\right|_{\partial Q_T} &= 0. \end{aligned} \tag{1.3}$$

where $\mathcal{H}(\eta_0, u)$ is a hysteresis operator defined at every spatial point. We will give a detailed definition of this operator in section 2, but loosely speaking one takes two real numbers $\alpha < \beta$ (the analogue of Γ_{off} and Γ_{on}) and two curves $H_1 : (\infty, \beta] \to \mathbb{R}$, $H_2 : [\alpha, -\infty)$. $\mathcal{H}(\eta_0, u)(\xi, t) = H_1(u(\xi, t))$

when $u(\xi,t) < \alpha$, $H_2(u(\xi,t))$ when $u(\xi,t) > \beta$ and $H_1(u(\xi,t))$ or $H_2(u(\xi,t))$ for $u(\xi,t) \in (\alpha,\beta)$ where the choice H_1 or H_2 depends on the prehistory of $u(\xi,t)$. The dependence on prehistory necessitates a choice of H_1 or H_2 for every point in Q at time $t = 0$, and we denote by $Q_j \subset Q$ the subset of points where $u(\xi,0)$ maps to the curve H_j.

The key assumption of the papers [5], [7] and [6] is that the initial data has non zero spatial derivative on the boundary of Q_1 and Q_2. In this thesis we will formulate a similar assumption, however we will have to contend with a further difficult, namely that the boundaries ∂Q_1, ∂Q_2 are codimension 1, which for for a domain in $\mathbb{R}^2$ requires additional topological constraints. The key additional assumption we will make is that at points $\xi \in \partial Q_j$ ($j = 1,2$), where $u(\xi,0)$ equals α or β, we stipulate that there is a neighbourhood of ξ such that ∂Q_j is the graph of a continuous function. Taking these two conditions as the definition of transverse initial data, we will prove the existence of solutions for (1.3). We will prove and existence of solutions for the same problem with $\mathcal{H}(\eta_0,u)$ in place of $f(u,\mathcal{H}(\eta_0,u))$ and under slightly different assumptions on $\mathcal{H}(\eta_0,u)$. The exact formulation of the two problems we consider are found in section 3, equation (3.4) - (3.6) and (3.7) - (3.9). The relationship between the two problems will be made clear in Remark 3.17.

The thesis is organised as follows. In Section 2 we introduce the hysteresis operator and, in particular, state the assumptions on H_1 and H_2 needed to prove existence, and the assumptions needed to prove uniqueness.

At the beginning of section 3 we introduce the necessary function spaces. One should note that the hysteresis output is discontinuous so we will pose our problem in a Sobolev space in such a way that the data of the problem is in an L_p space, not a Hölder space. Then we will define the notion of transverse initial data and make mention of some ways in which the assumption can fail. We conclude section 3 with a statement of the main results of the thesis, namely existence of solution to problem (3.4) - (3.6) and uniqueness of solutions to problem (3.7) - (3.9).

In section 4 we obtain the necessary apriori estimates for existence of solutions via a localization technique. The localization construction underpins the thesis. As alluded to above, we will consider points at which the free boundary (∂Q_1 in this case) and the level set of the threshold values (in this case α) coincide. We will cover all such points with a finite collection of open sets (A_i with $i = 1,\cdots,d$) and make a coordinate transform in each of these sets. We will obtain all apriori estimates in these new coordinates and will relate these estimates to the original problem in section 6.

The sets A_i will be constructed in such a way such that an initial data is transverse if and only if it admits such a construction (Lemma 4.3). The proof is straightforward but cumbersome, however it allows us to make the claim that there is a sufficiently small time such that *any* solution is transverse for this time. This will also tie into our formulation of the uniqueness of transverse solution via Lemma 7.4.

The sets A_i will also inform the time for which we attempt to solve problem (3.4) - (3.6), and the main method for doing so. This method involves giving an explicit formula for the free boundary in the set A_i. At the initial time, the free boundary is, by assumption, the graph of a continuous function. We will construct a solution sufficiently close to the initial data so that the free boundary remains the graph of a continuous function. In particular we will not perturb the free boundary to the extent that it leaves A_i. We will also see that because of this, as the time for which we solve goes to zero, it is still possible for the limiting function to be transverse, hence the tile *local* well-posedness.

This tacit assumption can give rise to solutions where the existence time goes to zero but the limiting solution is still transverse, hence this thesis is titled *local* well posedness.

Section 5 states some auxiliary propositions about linear and semilinear parabolic equations. The fact that we can choose time small enough such that any solution remains close to the initial data stems from a result on the uniform boundedness of solutions to theses auxiliary problems (Lemma 5.8, appendix A for the proof). We will also observe that the solution belongs to a Hölder space with a certain exponent.

Existence of solutions is proved in section 6. We begin the section by calculating the existence time T which will inform how we construct a Schauder fixed point argument at the end of the section. This T will also allow us to assert that outside of the sets A_i, the configuration of the hysteresis does not change. Then we solve a semilinear problem with nonlinearity $f(u, \mathcal{H}(\eta_0, u_0)$ where u_0 is known apriori. Crucially, we show that the map $u_0 \mapsto u$ is continuous. We conclude the section by proving the existence of solution to problem (3.4) - (3.6) via the Schauder fixed point theorem. The compactness will come from the fact that u belongs to a Hölder space with a larger exponent than the exponent for the Hölder space to which u_0 belongs.

We treat uniqueness of solutions in section 7. We first use the transversality assumption to prove that the integral of $\mathcal{H}(\eta_0, u)$ in the transverse direction is Lipschitz in the argument u (Lemma 7.2). We use this Lemma to prove that two transverse solution of problem (3.7) - (3.9) coincide for a small time (Lemma 7.4). The final result in the thesis is the uniqueness of transverse solutions to problem (3.7) - (3.9) (Theorem 3.19).

We conclude this introduction by mentioning some possible generalizations. Firstly, these results generalize to higher dimensions with only slight modification. Indeed the assumption that the free boundary is the graph of a continuous function does not explicitly need that it's domain be a subset $\mathbb{R}$. The entire argument should carry over to functions with a domain in $\mathbb{R}^{n-1}$. Moreover, the Green function inequalities we use to prove uniqueness remain valid with slight modification (see of Remark 7.5). To prove existence we use an apriori L_∞ bound, and such a technique is amenable to systems of equations (see Theorem 14.4 [14] where there is a framework for solving systems of equations given an apriori L_∞ bound). Weakening the continuous function topological assumption is the subject of future work. A possible tool is a set-based method. These have already been successfully applied for $Q = \mathbb{R}$ and hysteresis thresholds $\alpha = 0$, $\beta = +\infty$ [12].

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Common Notation

We will define the relevant notation and conventions as we proceed, however here is a list of the important quantities

- $Q \subset \mathbb{R}^2$ is a domain with C^∞ boundary, with coordinates $\xi = (\xi_1, \xi_2)$.
- $Q_T = Q \times (0, T)$ with coordinates (ξ, t).
- $\partial Q_T = \partial Q \times (0, T)$.
- $B(\varepsilon_x, m) := \{(x, y) \in \mathbb{R}^2 \mid |x| \leq \varepsilon_x, |y| \leq \frac{2}{m}\}$ with coordinates (x, y).
- $I(\varepsilon_x) := [-\varepsilon_x, \varepsilon_x]$ with coordinates x.
- $I(\varepsilon_x)_T := [-\varepsilon_x, \varepsilon_x] \times [0, T]$ with coordinates (x, t).
- $B(\varepsilon_x, m_i)_T = B(\varepsilon_x, m_i) \times [0, T]$.
- $\alpha < \beta$ are the thresholds of the hysteresis.
- $\eta_0 : Q \to \{1, 2\}$ is the initial configuration of the hysteresis.
- $\mathcal{H}(\eta_0, u)$ is the spatially distributed hysteresis for the function u with initial configuration η_0. We will write $\mathcal{H}(\eta_0, u)(\xi, t)$ to refer to hysteresis on Q_T and $\mathcal{H}(\eta_0, u)(x, y, t)$ to refer to hysteresis on $B(\varepsilon_x, m)$.
- $\varphi : Q \to \mathbb{R}$ is the initial data of a reaction-diffusion equation.
- $\eta(\xi, t) \to \{1, 2\}$ is the configuration of the hysteresis $\mathcal{H}(\eta_0, u)(\xi, t)$
- $Q_j = \{\xi \in Q \mid \eta_0(\xi) = j\}$ for $j = 1, 2$.
- $\Gamma_X = \{\xi \in Q \mid \varphi(\xi) = X\}$.
- $\Gamma_{X,t} = \{\xi \in Q \mid u(\xi, t) = X\}$
- $q_j(t) = \{\xi \in Q \mid \eta(t) = j\}$
- $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ is a composition of a translation and a rotation.
- $A \subset \mathbb{R}^2$ will be a set such that $\psi(A) = [-\varepsilon_x, \varepsilon_x] \times [-\varepsilon_y, \varepsilon_y]$.
- $A_T = A \times (0, T)$
- $N_j = \{(\xi, t) \in \overline{Q_T} \mid \eta(\eta_0, u_0)(\xi, t) = j\}$

As a general rule we will use subscripts to denote a derivative, for example u_y, means the derivative in the direction y. We will often need to consider two functions and where possible we use the index j eg. u_j with $j = 1, 2$. In this case we will write $u_{j;y}$ for the derivative of u_j in the direction y.

Remark 1.1. The key technical result in this thesis is the construction of sets $A_i \subset Q$ which cover points where $\varphi(\xi) = \alpha$ and $\varphi(\xi) \in \partial Q_1$. These will have corresponding maps ψ_i such that $\psi_i(A_i) = B(\varepsilon_x^i, m_i)$. In subsection 4.1, we will explicitly highlight the fact that we have a family indexed over i. We will also highlight this briefly in Section 6 where, in order to prove existence we need to describe a function in the coordinates $(\xi, t) \in Q_T$. However, for the majority of sections 4 (apriori estimates) and 7 (uniqueness) we will omit the index i for convenience.

2 The Hysteresis Operator

2.1 Definitions

Let $Q \subset \mathbb{R}^2$ be an open connected set with C^∞ boundary. Fix some $T > 0$ and let $Q_T = Q \times (0, T)$.

Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. Consider two continuous functions

$$H_1 : (-\infty, \beta] \to \mathbb{R},$$

$$H_2 : [\alpha, \infty) \to \mathbb{R}.$$

(see Figure 2a) Let $C_r[0,T)$ be the space of functions $[0,T) \to \mathbb{R}$ that are continuous from the right. Let $\zeta_0 \in \{1,2\}$ (an initial configuration). Let $g \in C[0,T]$. Define $\zeta : \{1,2\} \times C[0,T] \to C_r[0,T)$, the configuration function, as follows.

Let $X_t = \{t' \in [0,t] \mid g(t') = \alpha \text{ or } g(t') = \beta\}$. We define

$$\zeta(0) = \begin{cases} 1 & \text{if } g(0) \le \alpha, \\ 2 & \text{if } g(0) \ge \beta, \\ \zeta_0 & \text{if } \alpha < g(0) < \beta, \end{cases}$$

and

$$\zeta(g, \zeta_0)(t) = \begin{cases} \zeta(0) & \text{if } X_t = \emptyset, \\ 1 & \text{if } X_t \ne \emptyset \text{ and } g(\max X_t) = \alpha, \\ 2 & \text{if } X_t \ne \emptyset \text{ and } g(\max X_t) = \beta. \end{cases}$$

For any ζ_0 and g define the hysteresis operator

$$\overline{H} : \{1,2\} \times C[0,T] \to C_r[0,T),$$

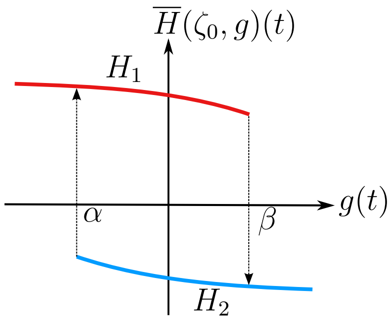
$$\overline{H}(\zeta_0, g)(t) = H_{\zeta(\zeta_0, g)}(g(t)). \tag{2.1}$$

Suppose that ζ_0 and g also depend on a parameter $\xi \in Q$. Denote them by (see Figure 2b)

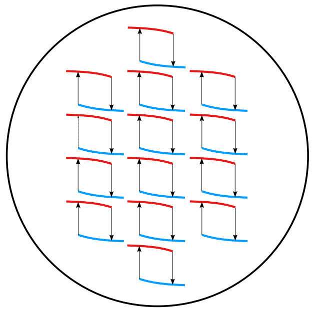
$$\eta_0 : Q \to \{1,2\},$$
$$u : Q_T \to \mathbb{R},$$
$$\varphi : Q \to \mathbb{R} \text{ where } \varphi = u|_{t=0}.$$

Definition 2.1. *φ and η_0 are* consistent *if for all $\xi \in Q$*

$$\eta_0(\xi) = \begin{cases} 1 & \text{if } \varphi(\xi) \le \alpha, \\ 2 & \text{if } \varphi(\xi) \ge \beta, \\ \in \{1,2\} & \text{if } \alpha < \varphi(\xi) < \beta. \end{cases}$$



(a) Scalar input.



(b) Spatially distributed

Figure 2: Comparison of scalar and spatially distributed hysteresis. In the spatially distributed case, ξ is a parameter.

Suppose η_0 and φ are consistent.

Definition 2.2. *The* spatial configuration function *is defined by*

$$\eta(\eta_0, u)(\xi, t) = \zeta(\eta_0(\xi), u(\xi, \cdot))$$

The spatially distributed hysteresis operator *is defined by*

$$\mathcal{H}(\eta_0, u)(\xi, t) = \overline{H}(\eta_0(\xi), u(\xi, \cdot))(t)$$

2.2 Conditions on the Hysteresis Branches

We make the same assumptions on H_1 and H_2 as in [5]. This is the natural assumption when considering hysteresis as the singular limit of a slow-fast system (one can consult [5] for further discussion in this direction). We will need Condition 2.3 to prove uniqueness of solutions, while we only need Condition 2.5 to prove existence of solutions.

Condition 2.3. *There is a $\sigma \in (0, 1]$ such that for any $U > 0$ there is a constant $M = M(U) > 0$ such that*

$$|H_1(u_1) - H_1(u_2)| \le \frac{M}{(\beta - u_1)^{1-\sigma} + (\beta - u_2)^{1-\sigma}} |u_1 - u_2| \, \forall u_1, u_2 \in [-U, \beta),$$

$$|H_2(u_1) - H_2(u_2)| \le \frac{M}{(u_1 - \alpha)^{1-\sigma} + (u_2 - \alpha)^{1-\sigma}} |u_1 - u_2| \, \forall u_1, u_2 \in (\alpha, U].$$

Remark 2.4. For $u > \beta$ we define $H_1(u) = H_1(\beta)$ and for $u < \alpha$ we define $H_2(u) = H_2(\alpha)$.

Condition 2.5. *The functions H_1 and H_2 are locally Hölder continuous with exponent σ.*

Remark 2.6. Functions H_1 and H_2 satisfying Condition 2.3 are locally Hölder continuous with exponent σ on $(\infty, \beta]$ and $[\alpha, \infty)$ respectively.

3 Setting of the Problem

3.1 Function spaces and embedding theorems

We introduce the following function spaces.

For $l > 0$, noninteger (with integer part $[l]$) we denote by $W_p^l(Q)$ the space consisting of all functions $u \in W_p^{[l]}(Q)$ with finite norm

$$\|u\|_{W_p^l(Q)} = \|u\|_{W_p^{[l]}(Q)} + \sum_{j=[l]} \left(\int_Q d\kappa \int_Q \frac{|D_\kappa^j u(\kappa) - D_\xi^j u(\xi)|^p}{|\kappa - \xi|^{2+p(l-[l])}} d\xi \right)^{\frac{1}{p}}. \tag{3.1}$$

where D^j is denotes the weak derivative with respect to the multi-index j (for more details see [11] pg.70) Let C^γ denote the standard Holder space and $W_p^{2,1}(Q_T)$ the anisotropic Sobolev space i.e. the space consisting of functions with weak derivatives $D_t^r D_\xi^s u$ where $2r + s \leq 2$. The space $W_p^{2,1}(Q_T)$ is endowed with the norm

$$\|u\|_{W_p^{2,1}(Q_T)} = \sum_{2r+s\leq 2} \|D_t^r D_\xi^s u\|_{L_p(Q_T)}.$$

When a function also depends on a time variable t we will use $C^{1,0}$ to indicate that it is continuously differentiable in space and continuous in time. We will make use of the following embedding theorems.

Lemma 3.1. *Let $\varphi \in W_p^{2-\frac{2}{p}}(Q)$, $0 \leq \gamma < 1 - \frac{4}{p}$, then for $j = 1, 2$ one has $\varphi, \varphi_{\xi_j} \in C^\gamma(\overline{Q})$ and*

$$\|\varphi\|_{C^\gamma(\overline{Q})} + \sum_{j=1,2} \|\varphi_{\xi_j}\|_{C^\gamma(\overline{Q})} \leq c\|\varphi\|_{W_p^{2-2/p}(Q)}, \tag{3.2}$$

where $c = c(p, \gamma)$

Proof. See [15] section 4.6.1 ☐

Lemma 3.2. *If $u \in W_p^{2,1}(Q_T)$ with $p > 4$, $0 \leq \gamma < 1 - \frac{4}{p}$ then $u, u_{\xi_j} \in C^\gamma(\overline{Q_T})$ and*

$$\|u\|_{C^\gamma(\overline{Q_T})} + \sum_{j=1,2} \|u_{\xi_j}\|_{C^\gamma(\overline{Q_T})} \leq c\|u\|_{W_p^{2,1}(Q_T)}, \tag{3.3}$$

where $c = c(p, \gamma, T)$

Proof. See [11], chapter 2, Lemma 3.3. □

Condition 3.3. *For the remainder of this paper fix*

$$p > 4 \text{ and } 0 < \gamma < 1 - \frac{4}{p},$$

i.e. p and γ satisfy Lemmas 3.1 and 3.2.

Let $\partial Q_T = \partial Q \times (0, T)$

Lemma 3.4. *Let $u \in W_p^{2,1}(Q_T)$. Then $u|_{t=0} \in W^{2-\frac{2}{p}}(Q)$ and*

$$\|u|_{t=0}\|_{W_p^{2-2/p}(Q)} \leq c\|u\|_{W_p^{2,1}(Q_T)}.$$

Moreover, for $j = 1, 2$ the trace of the weak spatial derivative $u_{\xi_j}|_{\partial Q_T}$ is well defined and belongs to $W_p^{1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p}}(\partial Q_T)$. Also

$$\|u_{\xi_j}|_{\partial Q_T}\|_{W_p^{1-1/p,1/2-1/2p}(Q_T)} \leq c\|u\|_{W_p^{2,1}(Q_T)}.$$

Where in both cases c does not depend on u but does depend on p and in the later case also on T.

Proof. See [11], chapter 2, lemma 3.4. □

The subspace of $W_p^{2-\frac{2}{p}}(Q)$ consisting of functions with the zero Neumann boundary conditions is a well defined subspace of $W_p^{p-\frac{2}{p}}(Q)$. We will denote it by $\mathcal{W}$ and equip it with the norm (3.1) (see [15] 4.3.3 and 4.4.1).

3.2 Transverse Initial Data

Given φ and η_0, define the sets

$$Q_j = \{\xi \in Q \mid \eta_0(\xi) = j\} \text{ for } j = 1, 2.$$

$$\Gamma_X = \{\xi \in Q \mid \varphi(\xi) = X\}.$$

For a Lebesgue measurable subset $Q' \subset Q$ let $\text{mes}(Q')$ denote its Lebesgue measure. Let $\text{int}(Q')$ denotes its topological interior.

Definition 3.5. *We say the function φ is transverse with respect to η_0 if*

(1) Q_1 *and* Q_2 *are measurable,* $\partial Q_1 \subset Q$, $\partial Q_2 = \partial Q_1 \cup \partial Q$ *and* $mes(\partial Q_1) = 0$

(2) $\varphi(\xi) < \beta$ *for* $\xi \in int(Q_1)$

(3) $\varphi(\xi) > \alpha$ *for* $\xi \in int(Q_2) \cup \partial Q$

(4) $\Gamma_\beta \cap \partial Q_1 = \emptyset$ *and if* $\xi \in \Gamma_\alpha \cap \partial Q_1$ *then there is a neighbourhood* A *of* ξ *and a map* ψ *such that (see Figure 3)*

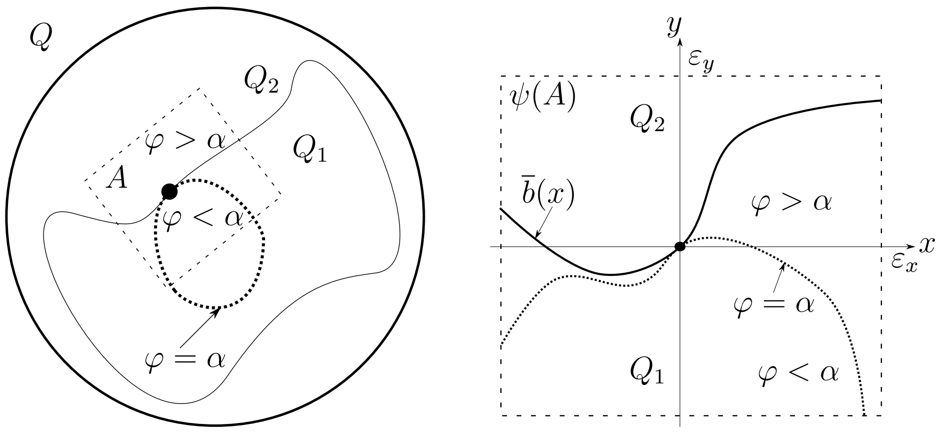
(i) ψ *is a composition of a translation and a rotation and*

$$\psi(A) = [-\varepsilon_x, \varepsilon_x] \times [-\varepsilon_y, \varepsilon_y], \psi(\xi) = (0,0)$$

(ii) There is a continuous function $\bar{b} : [-\varepsilon_x, \varepsilon_x] \to [-\varepsilon_y, \varepsilon_y]$ *such that the configuration function* $\eta_0 \circ \psi^{-1}$ *in* $\psi(A)$ *(which we denote by* $\eta_0(x,y)$*) is given by*

$$\eta_0(x,y) = \begin{cases} 1 & \textit{if } -\varepsilon_y \le y \le \bar{b}(x) \\ 2 & \textit{if } \bar{b}(x) < y \le \varepsilon_y \end{cases}$$

(iii) $(\varphi \circ \psi^{-1})_y(0,0) > 0$



(a) Coordinates $\xi \in Q$ (b) Coordinates (x,y) of $\psi(A)$.

Figure 3: Local coordinates in a neighbourhood of a point $\xi \in \Gamma_\alpha \cap \partial Q_1$.

Remark 3.6. We can formulate part (4) of Definition 3.5 for points $\xi \in \Gamma_\beta \cap \partial Q_1$, by interchanging configurations 1 and 2 in item (ii). Under this modification, all the results presented in this thesis are still valid for initial data where $\Gamma_\beta \cap \partial Q_1 \neq \emptyset$, however for clarity of presentation, we will restrict our notion of transverse to initial data satisfying $\Gamma_\beta \cap \partial Q_1 = \emptyset$.

Remark 3.7. In the coordinates (x, y) of A, we will permit the slight abuse of notation by writing $(x, y) \in Q_j$ (for $j = 1, 2$) instead of $\psi^{-1}(x, y) \in Q_j$.

Condition 3.8. *φ is transverse with respect to η_0.*

Definition 3.9. *We say that $u \in C^{1,0}(Q_T)$ is transverse with respect to $\eta(\xi, t)$ if for every $t \in [0, T]$, $u(\cdot, t)$ is transverse with respect to $\eta(\cdot, t)$.*

We conclude this subsection with some remarks about the ways in which the transverse assumption can fail. Solid lines denote the free boundaries and dashed lines denote level sets.

Remark 3.10. Suppose that $\xi \in \Gamma_\alpha \cap \partial Q_1$ and that $\nabla\varphi(\xi) = 0$. Numerics of a discetized version (3.4) - (3.6) with hysteresis branches H_1, H_2 equal to constants, exhibits a pattern whereby certain points in the spatial discretization switch configuration and others do not. The ratio of switched to not-switched points depends on the ratio $H_1 \backslash H_2$ but does not depend on the size of the discretization. This suggests that the problem is not well posed if one only allows the hysteresis output to take values H_1 and H_2. Further discussion in this direction is beyond the scope of this paper.

Remark 3.11. Suppose that $\nabla\varphi(\xi) \neq 0$ and that the height ε_y of A is small. Then the separation of regions $\eta(t) = 1$ and $\eta(t) = 2$ can go to zero as $t \to t_1$ and hence the topology of $q_1(t_1)$, $q_(t_1)$ can be significantly different from the topology of Q_1 and Q_2. See Figure 4. Moreover this can still be considered transverse since the measure of the separation is now zero. This highlights that the theorems in this thesis are indeed local. We do not make claims about the behaviour of solutions as our existence time goes to zero.

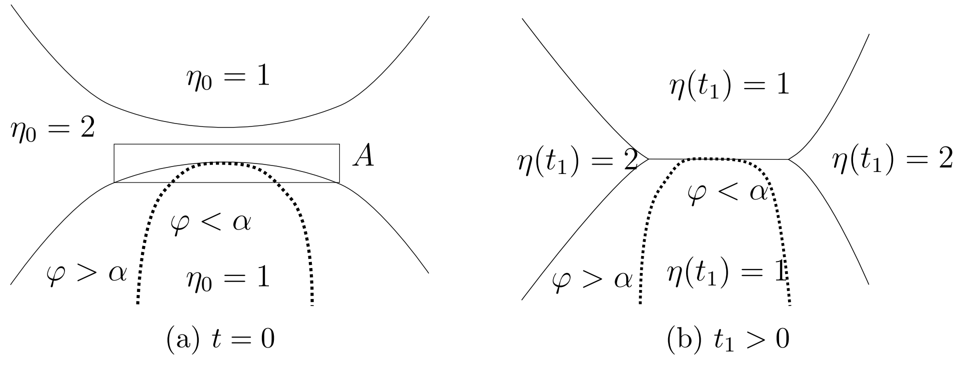


Figure 4: Two portions of the free boundary that are initially separated but touch at time t_1.

Remark 3.12. Consider Figure 5. Suppose that $\varphi_{\xi_1}(\xi) > 0$ and that the level set is the graph of a continuous function (in fact C^1 by the implicit function theorem) with domain ξ_2. Suppose the free boundary is the graph of a piecewise differentiable function with a cusp at the point ξ and with domain ξ_1.

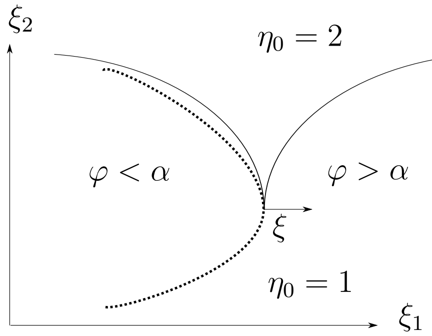


Figure 5: A point where the free boundary and level set are graphs relative to different choices of coordinates.

Then under a small perturbation of φ, it is possible for the level set to touch points on the left hand side of the cusp. Since the cusp point has measure zero, suppose one tacitly ignores this point and attempts to draw boxes A_i at all other points where the perturbation hits the right side of cusp. We require that the boxes are small enough so that the free boundary is still the graph of a continuous function i.e. the boxes cannot include points on the left side of the cusp (see Figure 6)

We will see later that we stipulate that the free boundary is never perturbed out of a box A_i, however the widths of the boxes in Figure 6 approach zero as one approaches the cusp, so we cannot solve in this case. This example highlights that even though both the boundary and the level set are graphs of continuous functions, it is necessary that they be graphs relative to the same choice of coordinates.

3.3 Main Results

Let $u : Q_T \to \mathbb{R}$. We will consider the existence of solutions to the following problem

$$u_t - \Delta u = f(u, \mathcal{H}(\eta_0, u)), \tag{3.4}$$

$$u|_{t=0} = \varphi, \tag{3.5}$$

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial Q_T} = 0. \tag{3.6}$$

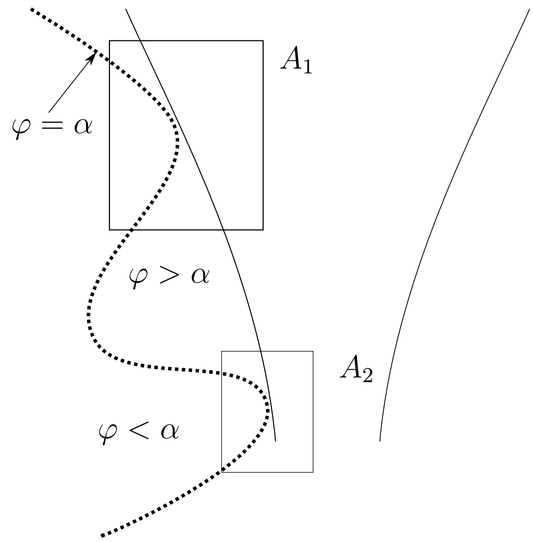


Figure 6: An attempt to cover the set $\Gamma_\alpha \cap \partial Q_1$ near a cusp point. Note that the width of A_2 is smaller than the width of A_1 and that one can expect the widths of the sets A_i to approach zero as one approaches the cusp.

Definition 3.13. *A solution to* (3.4) - (3.6) *is a function* $u \in W_p^{2,1}(Q_T)$ *such that* $\mathcal{H}(\eta_0, u)$ *is measureable,* (3.4) *is satisfied in* $L_p(Q_T)$ *and* u *satisfies* (3.5) *and* (3.6) *in terms of traces.*

Condition 3.14. *(Lipschitz continuity)*
For any bounded set $\Omega \subset \mathbb{R}^2$*, there is an* $L(\Omega)$ *such that*

$$|f(u_1, v_1) - f(u_2, v_2)| \leq L(|u_1 - u_2| + |v_1 - v_2|) \quad \forall (u_1, v_1), (u_2, v_2) \in \Omega$$

Condition 3.15. *(Dissipative)*
There is a U_0 *such that for all* $U > U_0$*,* $|u| \leq U$ *and* $j = 1, 2$

$$f(U, H_j(u)) < 0,\ f(-U, H_j(u)) > 0.$$

Remark 3.16. The simplest example of hysteresis branches and nonlinearity f satisfying these conditions is $f(u, \mathcal{H}(\eta_0, u)) = -\varepsilon u + \mathcal{H}(\eta_0, u)$ with hysteresis branches $H_1(u) \equiv h_1$ and $H_1(u) \equiv h_2$, where h_1, h_2 are arbitrary real numbers. In [7] it was shown that one can take $\varepsilon = 0$ if $h_1 > 0$ and $h_2 < 0$. We conjecture, but do not prove, that the same assertion is valid in a multidimensional domain.

We will also consider the uniqueness of solutions to the following problem

$$u_t - \Delta u = \mathcal{H}(\eta_0, u), \tag{3.7}$$

$$u|_{t=0} = \varphi, \tag{3.8}$$

$$\frac{\partial u}{\partial \nu} = 0. \tag{3.9}$$

A definition of a solution to (3.7) - (3.9) is analogous to Definition 3.13.

Remark 3.17. If one assumes the hysteresis branches satisfy Condition 2.3 and that f satisfies Condition 3.14, then problem (3.4) -(3.6) can be reduced to problem (3.7) - (3.9). Indeed by defining the hysteresis branches as $F_j(u) = f(u, H_j(u))$ one can see that F_j also satisfy Condition 2.3. In detail, let $U > 0$, and choose M' such that

$$(\beta - u_1)^{1-\sigma} + (\beta - u_2)^{1-\sigma} \leq M' \ \ \forall u_1, u_2 \in (-U, \beta],$$

then

$$\begin{aligned} |f(u_1, H_1(u)) - f(u_2, H_1(u_2))| &\leq |u_1 - u_2| + |H_1(u_1) - H_1(u_2)| \\ &\leq |u_1 - u_2| + \frac{M|u_1 - u_2|}{(\beta - u_1)^{1-\sigma} + (\beta - u_2)^{1-\sigma}} \\ &\leq \frac{(M + M')|u_1 - u_2|}{(\beta - u_1)^{1-\sigma} + (\beta - u_2)^{1-\sigma}}. \end{aligned}$$

Theorem 3.18. *(Local Existence)*

If Conditions 2.5, 3.14 and 3.15 are satisfied, then there is a $T > 0$ such that

(1) There is at least one solution of (3.4) - (3.6) *on Q_T.*
(2) Any solution of (3.4) - (3.6) *is transverse on Q_T.*

Theorem 3.19. *(Uniqueness)*

Assume that Condition 2.3 holds and let u_1, u_2 be two transverse solutions on Q_T. Then $u_1 = u_2$ on Q_T.

Remark 3.20. In particular Remark 3.17 implies that if Condition 2.3 holds then there exists a unique solution to (3.4) - (3.6).

4 Local apriori Estimates

4.1 Localization

Before proceeding we introduce the notation

$$B(\varepsilon_x, m) := \{(x, y) \in \mathbb{R}^2 \mid |x| \leq \varepsilon_x, |y| \leq \frac{2}{m}\}.$$

$$I(\varepsilon_x) := [-\varepsilon_x, \varepsilon_x].$$

Definition 4.1. *Let m be a positive integer. We say the pair $(\varphi, \eta_0) \in E_m$ if there is a collection (indexed over $i = 1, \cdots, d$ where d does not depend on m) of sets A_i, maps ψ_i, numbers $\varepsilon_i > 0$ and numbers $m_i > 0$ satisfying*

(1) $m_i \leq m$ *and for each* i, $|\alpha - \beta| \geq \frac{1}{4m_i^2}$.

(2) Q_1 *and* Q_2 *are measurable,* $\partial Q_1 \subset Q$, $\partial Q_2 = \partial Q_1 \cap \partial Q$ *and* $mes(\partial Q_1) = 0$.

(3) $\varphi(\xi) < \beta - \frac{1}{m^2}$ *on* $\overline{Q_1}$ *and* $\varphi(\xi) > \alpha + \frac{1}{m^2}$ *on* $(Q_2 \cup \partial Q) \setminus \bigcup_{i=1}^d A_i$

(4) The maps ψ_i *satisfy the following (where we write* $\varphi^i = \varphi \circ \psi_i^{-1}$*)*

  (i) $\{int(A_i)\}_{i=1}^d$ *is an open cover of* $\Gamma_\alpha \cap \partial Q_1$.

  (ii) Each ψ_i *is a composition of a translation and a rotation and* $\psi_i(A_i) = B(\varepsilon_x^i, m_i)$

  (iii) There is a continuous function $\overline{b}^i : [-\varepsilon_x, \varepsilon_x] \to [-\frac{2}{m_i}, \frac{2}{m_i}]$ *such that* $\eta_0^i = \eta_0 \circ \psi_i^{-1}$ *is given by*

$$\eta_0^i(x, y) = \begin{cases} 1 & \text{if } -\frac{2}{m_i} \leq y \leq \overline{b}^i(x), \\ 2 & \text{if } \overline{b}^i(x) < y \leq \frac{2}{m_i}. \end{cases}$$

  (iv) If $\varphi^i(x, y) \in [\alpha - \frac{1}{m_i^2}, \alpha + \frac{1}{m_i^2}]$ *then* $\varphi_y^i(x, y) \geq \frac{1}{m_i}$ *(see Figure 7)*

  (v) $\forall x \in I(\varepsilon_x^i)$ *there exists a unique* $a^i(x) \in [-\frac{1}{m_i}, \frac{1}{m_i}]$ *such that* $\varphi^i(x, a^i(x)) = \alpha$ *and* $a^i(x) \leq \overline{b}^i(x)$.

(5) $\|\varphi\|_{W_p^{2-2/p}(Q)} \leq m$

Remark 4.2. In Definition 4.1, note that if $\Gamma_\beta \cap \partial Q_1 \neq \emptyset$ then we would have to formulate sets A_i with the roles of configurations 1, 2 and the thresholds α, β reversed. In this case we would also need $\varphi(\xi) < \beta - \frac{1}{m^2}$ on $Q_1 \setminus \cup_{i=1}^d \cup A_i$.

Before proving the next Lemma, we make some remarks about Definition 4.1. The general principle underlying the definition is that hysteresis should not change outside of the sets A_i and within A_i the only way hysteresis can change is when an input takes on the threshold value α.

- Part (2) encodes the fact that hysteresis cannot change outside of A_i.
- The condition $|\alpha - \beta| \geq \frac{1}{4m_i^2}$ ensures that under a small perturbation of φ, it is not possible for point where $\eta_0(x, y) = 1$ to reach the threshold β and thus switch to configuration 2.
- $\text{mes}(Q_1) = 0$ will allow to decompose an integral over Q in the following way.

$$\int_Q g(\xi) = \int_{Q_1 \setminus \cup A_i} g(\xi) d\xi + \int_{Q_2 \setminus \cup A_i} g(\xi) d\xi + \sum_i \int_{A_i} g(\xi) d\xi$$

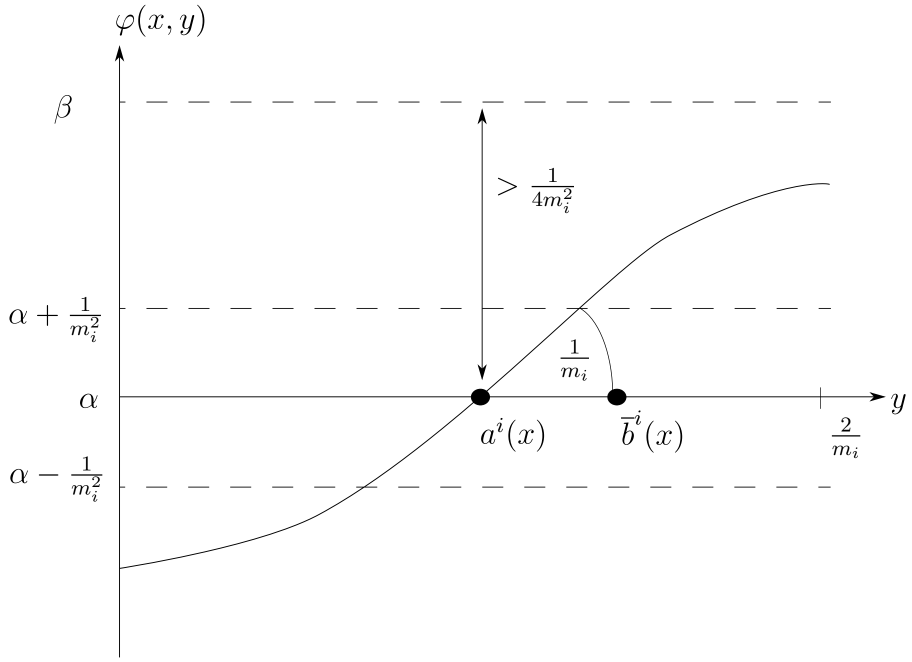


Figure 7: A cross section of interval $[-\frac{2}{m_i}, \frac{2}{m_i}]$ for a fixed $x \in [-\varepsilon_x^i, \varepsilon_x^i]$.

This will be critical when we want to obtain L_p estimates on the entire domain Q.

- The assumption $\varphi_y(x,y) \geq \frac{1}{m_i}$ when $\varphi(x,y) \in [\alpha - \frac{1}{m_i^2}, \alpha + \frac{1}{m_i^2}]$ is the most important technical assumption of the thesis. It will play a key role in all Lemma 4.6 which forms the groundwork for the apriori estimates we obtain in this section. It will also be crucial in proving uniqueness of solutions, specifically in Lemma 7.2.

Lemma 4.3. *If (φ, η) satisfies Condition 3.8, then there exists an m such that $(\varphi, \eta_0) \in E_m$. Conversely, if $(\varphi, \eta_0) \in E_m$ then (φ, η_0) is transverse.*

Proof. Let φ be transverse with respect to η_0. To begin with, we address part (4) of Definition 4.1. Let $\xi \in \Gamma_\alpha \cap \partial Q_1$. In what follows we will consider the quantities ψ and $\bar{b}$ from part (4) of Definition 3.5, however, in order to construct the functions ψ_i and $\bar{b}^i$ it will be necessary to decrease the side lengths of the set $A = [-\varepsilon_x, \varepsilon_x] \times [-\varepsilon_y, \varepsilon_y]$. When we replace ε_y by a smaller quantity $0 < \frac{2}{m_i} \leq \varepsilon_y$, $\bar{b}^i(x)$ will be given by the rule (See Figure 8)

$$\bar{b}^i(x) = \begin{cases} \frac{2}{m_i} & \text{if } \bar{b}(x) > \frac{2}{m_i}, \\ \bar{b}(x) & \text{if } -\frac{2}{m_i} \leq \bar{b}(x) \leq \frac{2}{m_i}, \\ -\frac{2}{m_i} & \text{if } \bar{b}(x) < -\frac{2}{m_i}. \end{cases} \tag{4.1}$$

This procedure constructs functions a ψ_i and $\overline{b}^i$ at every point $\xi \in \Gamma_\alpha \cap \partial Q_1$ that satisfy items (ii) and (iii) of part (4) of Definition 4.1. It remains to construct the function a^i and to correctly choose m_i and ε_x^i.

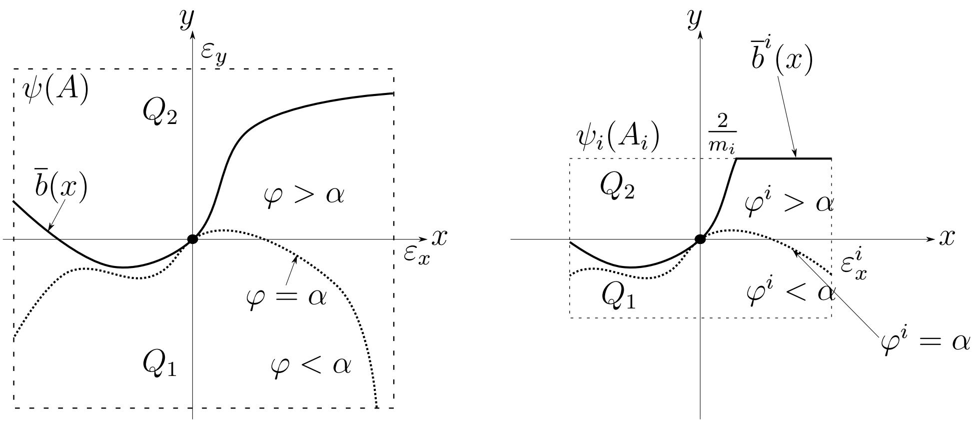


Figure 8: A point $\xi \in \Gamma_\alpha \cap \partial Q_1$ before and after reducing the side lengths of $\psi(A)$.

Continuing the proof pertaining to part (4), let $\xi \in \Gamma_\alpha \cap \partial Q_1$ and let $\varepsilon_x^i, \varepsilon_y^i, \overline{b}^i, \psi_i$ be defined as in part (4) of Definition 3.5, where i is treated as an index. Recall that $\psi_i(\xi) = (0,0)$ and $\varphi_y^i(0,0) > 0$.

First we will specify the number m_i which determines the height of A_i. Because $\varphi_y^i(0,0) > 0$ there is an m_i such that $\varphi_y^i(0,0) > \frac{1}{m_i}$. Increase m_i if necessary so that $\varphi_y^i(0,y) > \frac{1}{m_i}$ for $|y| \leq \frac{2}{m_i} \leq \varepsilon_y^i$. If necessary increase m_i again so that $|\alpha - \beta| \geq \frac{1}{4m_i^2}$.

Now that m_i is specified, we decrease ε_x^i. First observe that $\varphi(0,0) > 0$ so the implicit function theorem implies that there is a neighbourhood of $(0,0)$ and a unique function C^1 function a^i such that $\varphi(x, a^i(x)) = \alpha$. In particular we can choose ε_x^i small enough so that $a^i \in C^1(I(\varepsilon_x^i))$. Since $a^i(0) = 0$ and a^i is continuous, we can further decrease ε_x^i so that $a^i(x) \in [-\frac{1}{m_i}, \frac{1}{m_i}]$. That $a^i(x) \leq \overline{b}^i(x)$ follows from part (3) of Definition 3.5 and the observation that $y > \overline{b}^i(x)$ implies that $(x,y) \in \text{int}Q_2$. Since $\varphi_y^i(0,y) > \frac{1}{m_i}$ we can further decrease ε_x^i until $\varphi_y^i(x,y) > \frac{1}{m_i}$ in $B(\varepsilon_x^i, m_i)$ which is sufficient for item (iv).

Thus for every $\xi \in \Gamma_\alpha \cap \partial Q_1$, we have constructed an A_i and a $\overline{b}^i$ satisfying part (4).

This construction also has the property that $\xi \in \text{int}(A_i)$ so we can take a a finite subcover (indexed $i = 1, \cdots, d$) of the compact set $\Gamma_\alpha \cap \partial Q_1$ by sets A_i. Note that there is clearly an m such that $m > m_i$ for $i = 1, \cdots, d$.

Using item (2) of Definition 3.5 and the observation that $\Gamma_\beta \cap \partial Q_1 = \emptyset$, we know that there exists an m such that $\varphi(\xi) \leq \beta - \frac{1}{m^2}$ on $\overline{Q_1}$. Moreover for $\xi \in \text{int}\{Q_2\} \cup \partial Q$, $\varphi(\xi) > \alpha$ (part (iii) of Definition 3.5). This means that the only possibility for $\varphi(\xi) = \alpha$ with $\xi \in Q_2 \cup \partial Q$, is if $\alpha \in \Gamma_\alpha \cap \partial Q_1$. But the A_i cover $\Gamma_\alpha \cap \partial Q$. Hence there is an m such that $\varphi(\xi) > \alpha + \frac{1}{m^2}$ on $(Q_2 \cup \partial Q) \setminus \cup_{i=1}^d A_i$. Finally, we know that there is an m large enough so that $\|\varphi\|_{W_p^{2-2/p}(Q)} \leq m$, while part (2) of Definition 4.1 is identical to part (1) of Definition 3.5. Therefore it is possible to choose m such that $(\varphi, \eta_0) \in E_m$.

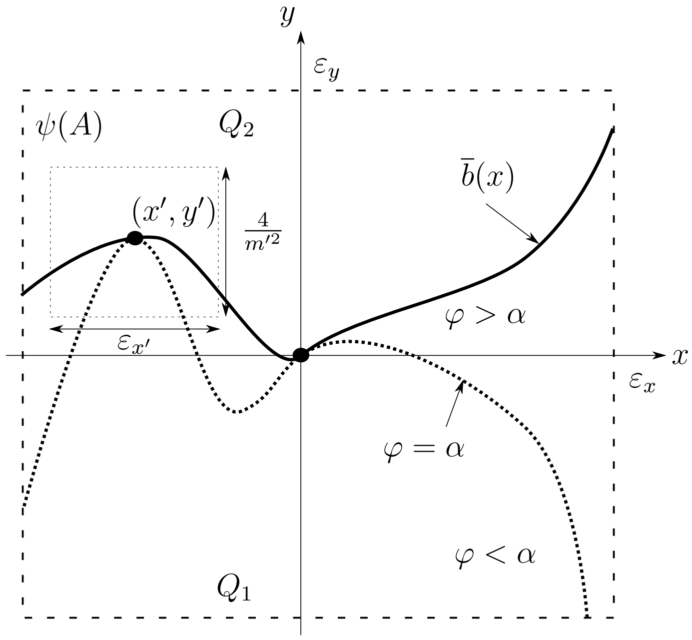


Figure 9: Constructing a neighbourhood around (x', y') that is contained in the original $\psi(A)$.

For the converse suppose that $(\varphi, \eta_0) \in E_m$ and note that we have just observed the equivalence of part (2) of Definition 4.1 and part (1) of Definition 3.5.

For part (2) of Definition 3.5 observe that $\varphi(\xi) < \beta - \frac{1}{m^2}$ on Q_1 so $\varphi(\xi) < \beta$ on $\text{int}(Q_1)$.

For part (3), observe that $\varphi(\xi) \geq \alpha + \frac{1}{m^2}$ on $(Q_2 \cup \partial Q) \setminus \cup_{i=1}^d A_i$, so in particular $\varphi(\xi) > \alpha$ on ∂Q. If $\xi \in A_i$, with $\psi_i(\xi) = (x, y)$ then $\xi \in \text{int}(Q_2)$ if and only if $y > \overline{b}^i(x)$, so part (3) follows from the assumption that $\overline{b}^i(x) \geq a^i(x)$.

If $\xi \in \Gamma_\alpha \cap \partial Q_1$ then there is an i such that $\xi \in \text{int}(A_i)$ with $\psi_i(\xi) = (x', y')$. Moreover $\varphi_y^i(x', y') > \frac{1}{m_i} > 0$. Choose a neighbourhood $(x', y') + B(\varepsilon_{x'}, m')$

$\subset B(\varepsilon_x^i, m_i)$ (see Figure 9)

Let $\theta_i = T_{x',y'} \circ \psi_i$ where $T_{x',y'}$ the translation $(x', y') \mapsto (x - x', y - y')$. Note that $\theta_i(\xi) = (0, 0)$. Let $\overline{b}^i_{(x',y')}(x) := \overline{b}^i(x + x') - y'$ and restrict $\overline{b}^i_{(x',y')}$ to the box $B(\varepsilon_{x'}, m')$ via the procedure specified in equation (4.1).

Then the set $\theta_i^{-1}(B(\varepsilon_{x'}, m'))$ and the functions θ_i and $\overline{b}^i_{x',y'}$ satisfy part (4) of Definition 3.5 $\square$

Remark 4.4. In what follows, we will consistently make reference to the behaviour of solutions in a box $B(\varepsilon_x^i, m_i)$, for some $i = 1, \cdots, d$ corresponding to a pair $(\varphi, \eta_0) \in E_m$. As a means of notational convenience, rather than repeating the definitions of ψ^i, $\overline{b}^i$, m_i etc. we will simply say $(\varphi, \eta_0) \in E_m$ and invite the reader to revisit the definition.

Remark 4.5. In what follows we will be concerned with estimates for functions defined on the sets A_i. In later sections, the value of m_i will be critical, therefore we will continue to write $B(\varepsilon_x^i, m_i)$. However for functions we will omit the i and write (using φ as an example) $\varphi(x, y)$ instead of φ^i i.e. we will use the coordinates (x, y) to indicate that we are making estimates on a set $\psi_i(A_i)$.

4.2 Spatial apriori estimates

Lemma 4.6. *Let $j = 1, 2$, $\omega_j, \omega_{j;y} \in C(B(\varepsilon_x^i, m_i))$. Let $(\varphi, \eta_0) \in E_m$ and suppose that*

$$\|\omega_j - \varphi\|_{C(B(\varepsilon_x^i, m_i))} + \|\omega_{j;y} - \varphi_y\|_{C(B(\varepsilon_x^i, m_i))} \leq \frac{1}{4m_i^2}. \tag{4.2}$$

Then

(1) For each $x \in [-\varepsilon_x^i, \varepsilon_x^i]$ the equation $\omega_j(x, y) = \alpha$ has exactly one root.

(2) If $\omega_j(x, y) = \alpha$ let $y = a_j(x)$, then

$$\|a_1 - a_2\|_{C(I(\varepsilon_x^i))} \leq 2m_i \|\omega_1 - \omega_2\|_{C(B(\varepsilon_x^i, m_i))}.$$

(3) $a_j(x) \in [-\frac{5}{4m_i}, \frac{5}{4m_i}] \subset (-\frac{2}{m_i}, \frac{2}{m_i})$.

Proof. We begin by making an observation about $\varphi(x, y)$ when $y \in [a(x) - \frac{1}{m_i}, a(x) + \frac{1}{m_i}]$. Suppose $a(x) \leq y \leq a(x) + \frac{1}{m_i}$ and $\varphi(x, y) < (y - \alpha)\frac{1}{m_i} + \alpha$. Then $\varphi(x, y) \leq \alpha + \frac{1}{m_i^2}$ which means $\varphi_y(x, y) > \frac{1}{m_i}$. But then the mean value theorem implies that

$$(y - a(x))\frac{1}{m_i} \leq \varphi(x, y) - \varphi(x, a(x)) < (y - a(x))\frac{1}{m_i}.$$

This contradiction proves that $\varphi(x,y) \geq (y - a(x)) \cdot \frac{1}{m_i} + \alpha$ when $y \in [a(x), a(x) + \frac{1}{m_i}]$ and one can similarly show that $\varphi(x,y) \leq (a(x) - y) \cdot \frac{1}{m_i} + \alpha$ when $y \in [a(x) - \frac{1}{m_i}, a(x)]$. In particular we have

$$\varphi(x, a(x) - \frac{1}{4m_i}) \leq -\frac{1}{4m_i^2} + \alpha,$$

$$\varphi(x, a(x) + \frac{1}{4m_i}) \geq \frac{1}{4m_i^2} + \alpha.$$

By the intermediate value theorem $\omega_j(x,y) = \alpha$ has a root $a_j(x) \in [a(x) - \frac{1}{4m_i}, a(x) + \frac{1}{4m_i}]$. Moreover $\omega_{j;y}(x,y) \geq \frac{1}{m_i} - \frac{1}{4m_i^2} > \frac{1}{2m_i}$. Since the y-derivative must be strictly positive at any such $a_j(x)$, $a_j(x)$ must be unique. This proves part (1) of the Lemma, while combining these observations with the fact that $\varphi(x, a(x)) \in [-\frac{1}{m_i}, \frac{1}{m_i}]$ (Definition 4.1, part (4), item (v)) implies part (3) of the Lemma.

For part (2) we fix some $x \in [-\varepsilon_x^i, \varepsilon_x^i]$.

Case: $\omega_j(x, a_j(x)) = \alpha$ for $j = 1, 2$. Without loss of generality assume that $a_1(x) \leq a_2(x)$. Let $y \in [a_1(x), a_2(x)]$. We have just shown that $\omega_{j;y}(x,y) > \frac{1}{2m_i}$. By the mean value theorem

$$\begin{aligned} |a_1(x) - a_2(x)| &\leq 2m_i |\omega_1(x, a_1(x)) - \omega_1(x, a_2(x))| \\ &= 2m_i |\omega_2(x, a_2(x)) - \omega_1(x, a_2(x))| \\ &\leq 2m_i \|\omega_1 - \omega_2\|_{C(B(\varepsilon_x^i, m_i))}. \end{aligned} \tag{4.3}$$

Case: $a_1(x) = a_2(x) = -\frac{2}{m_i}$ is trivial. In (4.3) take the supremum over all $x \in [-\varepsilon_x^i, \varepsilon_x^i]$. This gives part 2. of the lemma (note that due to the implicit function theorem, the a_j are continuous)

□

4.3 Temporal apriori estimates

Before proceeding we introduce the notation $I(\varepsilon_x^i)_T := [-\varepsilon_x^i, \varepsilon_x^i] \times [0,T]$ and $B(\varepsilon_x^i, m_i)_T = B(\varepsilon_x^i, m_i) \times [0,T]$.

Lemma 4.7. *Let $(\varphi, \eta_0) \in E_m$. Consider $u : B(\varepsilon_x^i, m_i)_T \to \mathbb{R}$. Assume that $u \in C^{1,0}(B(\varepsilon_x^i, m_i)_T)$ and for each $t \in [0,T]$,*

$$\|u(\cdot,\cdot,t) - \varphi\|_{B(\varepsilon_x^i, m_i)_T} + \|u_y(\cdot,\cdot,t) - \varphi_y\|_{B(\varepsilon_x^i, m_i)_T} \leq \frac{1}{4m_i^2}. \tag{4.4}$$

If $u(x,y,t) = \alpha$ let $y = a(x,t)$. Then $a(x,t)$ is unique, $a(x,t) \in C(I(\varepsilon_x^i)_T)$ and for any two u_1,u_2 satisfying these assumptions (with a_1 and a_2 defined analogously)

$$\|a_1 - a_2\|_{C(I(\varepsilon_x^i)_T)} \leq 2m_i \|u_1 - u_2\|_{C(B(\varepsilon_x^i, m_i)_T)}. \tag{4.5}$$

Proof. First note that $a(x,t)$ is unique and well defined due to Lemma 4.6. Let $(x_n,t_n) \to (x,t)$ in $I(\varepsilon_x^i)_T$. Since $a(x_n,t_n) \in [-\frac{5}{4m_i}, \frac{5}{4m_i}]$, (see part (3) of 4.6) there is a subsequence, not relabelled, and a $y \in [-\frac{5}{4m_i}, \frac{5}{4m_i}]$ such that $a(x_n,t_n) \to y$. Since u is continuous in Q_T and $(x_n, a(x_n,t_n), t_n) \to (x,y,t)$,

$$u(x_n, a(x_n,t_n), t_n) \to u(x,y,t),$$

$$u(x_n, a(x_n,t_n), t_n) = \alpha \Rightarrow u(x,y,t) = \alpha.$$

Since $a(x,t)$ is unique one has $y = a(x,t)$. One can always construct such a subsequence such that $a(x_n,t_n) \to a(x,t)$ therefore a is continuous in $I(\varepsilon_x^i)_T$.

For estimate (4.5), note that $I(\varepsilon_x^i)_T$ is compact so there is an $(x,t) \in I(\varepsilon_x^i)_T$ such that

$$\|a_1 - a_2\|_{C(I(\varepsilon_x^i)_T)} = |a_1(x,t) - a_2(x,t)|.$$

This choice of (x,t) also implies that

$$|a_1(x,t) - a_2(x,t)| = \|a_1(\cdot,t) - a_2(\cdot,t)\|_{C(I(\varepsilon_x^i))}.$$

By part (2) of Lemma 4.6,

$$|a_1(\cdot,t) - a_2(\cdot,t)\|_{C(I(\varepsilon_x^i))} \leq 2m_i \|u_1(\cdot,\cdot,t) - u_2(\cdot,\cdot,t)\|_{C(B(\varepsilon_x^i,m_i)}.$$

But we also have that

$$\|u_1(\cdot,\cdot,t) - u_2(\cdot,\cdot,t)\|_{C(B(\varepsilon_x^i,m_i)} \leq \|u_1 - u_2\|_{C(B(\varepsilon_x^i,m_i)_T)}.$$

Tying these inequalities together gives (4.5). □

Lemma 4.8. *Let $a \in C(I(\varepsilon_x^i)_T)$ and define*

$$\tilde{b}(x,t) = \max_{s\in[0,t]} a(x,s). \tag{4.6}$$

Then

(1) $\tilde{b} \in C(I(\varepsilon_x^i)_T)$

(2) If $a_j \in C(I(\varepsilon_x^i)_T)$ with $\tilde{b}_j$ defined analogously, then

$$\|\tilde{b}_1 - \tilde{b}_2\|_{C(I(\varepsilon_x^i)_T)} \leq \|a_1 - a_2\|_{C(I(\varepsilon_x^i)_T)}.$$

Proof. Let us prove item (1): Let $(x_n,t_n) \to (x,t)$. Then there exists a sequence $\delta_n > 0$ with $\delta_n \to 0$ such that $t_n \in [0, t+\delta_n]$ By definition, there exists $s_n \in [0, t+\delta_n]$ such that $\tilde{b}(x_n,t_n) = a(x_n,s_n)$ Choose a convergent subsequence, not relabelled, such that $s_n \to s$ with $s \in [0,t]$. Let $\tilde{b}(x,t) = a(x,s')$, with $s' \in [0,t]$. Since a is continuous

$$a(x_n, s_n) \to a(x, s)$$

$$a(x_n, s') \to a(x, s')$$

Since $a(x,s') = \tilde{b}(x,t)$, we have that $a(x,s') \geq a(x,s)$. We claim in fact that $a(x,s) = a(x,s')$. If this were the case then we would have

$$\lim_{n\to\infty} \tilde{b}(x_n, t_n) = \tilde{b}(x,t)$$

Hence, assume for contradiction that $a(x,s) < a(x,s')$. Then for sufficiently large n, $a(x_n,s_n) < a(x,s')$. But since $a(x_n,s') \to a(x,s')$, for sufficiently large n, $a(x_n,s_n) < a(x_n,s')$. But this would mean $\tilde{b}(x_n,t_n) < a(x_n,s')$ which in turn means $s' > t_n$ for all n and thus $s' = t$ Now note that $a(x_n,s_n) = \tilde{b}(x_n,t_n)$ implies that

$$a(x_n, s_n) \geq a(x_n, t_n).$$

Taking the limit on both sides $a(x,s) \geq a(x,t) = a(x,s')$; a contradiction.

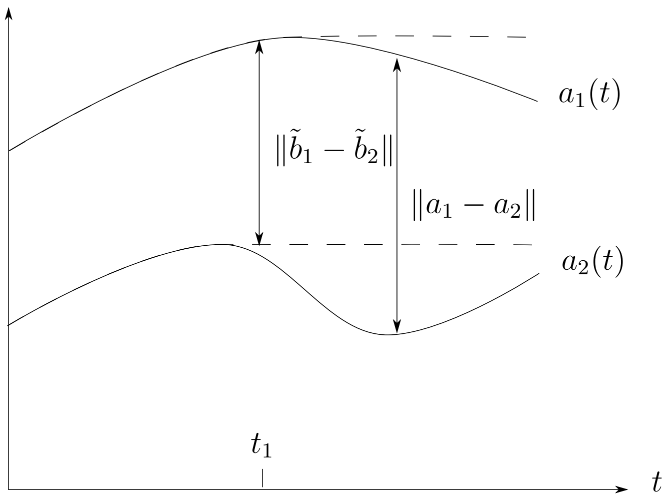


Figure 10: The maximum of $|a_1(t) - a_2(t)|$ occurs at a time bigger than or equal to the maximum of $|b_1(t) - b_2(t)|$.

For part (2) we will prove the claim that for every $x \in I(\varepsilon_x^i)$,

$$\|\tilde{b}_1(x,\cdot) - \tilde{b}_2(x,\cdot)\|_{C[0,T]} \leq \|a_1(x,\cdot) - a_2(x,\cdot)\|_{C[0,T]}$$

The result follows by taking the supremum over all $x \in I(\varepsilon_x^i)$.

Since x is fixed, we temporarily adopt the notation $a_j(t)$ with $a_j \in C[0,T]$ and $\tilde{b}_j(t) = \max_{s\in[0,t]} a_j(s)$ where $j = 1, 2$. We need to prove

$$\|\tilde{b}_1 - \tilde{b}_2\|_{C[0,T]} \leq \|a_1 - a_2\|_{C[0,T]} \tag{4.7}$$

As a special case of part 1 where the dependence on x is trivial, observe that $|\tilde{b}_1(t) - \tilde{b}_2(t)| \in C[0,T]$, so there exists a minimum value $t_1 \in [0,T]$ such that $\|\tilde{b}_1 - \tilde{b}_2\|_{C[0,T]} = |\tilde{b}_1(t_1) - \tilde{b}_2(t_1)|$ (see Figure 10). Without loss of generality assume that $\|\tilde{b}_1 - \tilde{b}_2\|_{C[0,T]} = \tilde{b}_1(t_1) - \tilde{b}_2(t_1)$. We claim that $\tilde{b}_1(t_1) = a_1(t_1)$. If not, then $t_1 \neq 0$ and there exists a t_2 such that $0 \leq t_2 < t_1$ and $a_1(t_2) = \tilde{b}_1(t_1)$. It's clear that $\tilde{b}_1(t_2) = a_1(t_2)$, otherwise $a_1(t_2) > \tilde{b}_1(t_1)$. Since $\tilde{b}_2$ is non-decreasing one obtains the inequality $\tilde{b}_1(t_1) - \tilde{b}_2(t_1) \leq \tilde{b}_1(t_2) - \tilde{b}_2(t_2)$. This implies that $\|\tilde{b}_1 - \tilde{b}_2\|_{C[0,T]} = \tilde{b}_1(t_2) - \tilde{b}_2(t_2)$ which contradicts the minimality of t_1. Now using that $a_2(t_1) \leq \tilde{b}_2(t_1)$ we obtain

$$\tilde{b}_1(t_1) - \tilde{b}_2(t_1) \leq a_1(t_1) - a_2(t_1).$$

By the choice of t_1 we obtain (4.7). □

Lemma 4.9. *Let $\tilde{b} \in C(I(\varepsilon_x^i)_T)$ be monotone increasing and $\bar{b} \in C(I(\varepsilon_x^i))$. Then the function $b(x,t) = \max\{\tilde{b}(x,t), \bar{b}(x)\}$ is continuous and if $\tilde{b}_1$, $\tilde{b}_2$ are also continuous monotone increasing with b_1, b_2 defined analogously,*

$$\|b_1 - b_2\|_{C(I(\varepsilon_x^i)_T)} \leq \|\tilde{b}_1 - \tilde{b}_2.\|_{C(I(\varepsilon_x^i)_T)} \tag{4.8}$$

Proof. For continuity observe that given $(x,t) \in I(\varepsilon_x^i)_T$ and $\varepsilon > 0$ one can choose $\delta > 0$ such $\|(x,t) - (x',t')\| < \delta$ implies that

$$\max\{\tilde{b}(x',t'), \bar{b}(x')\} \geq \max\{\tilde{b}(x,t) - \varepsilon, \bar{b}(x) - \varepsilon\} \geq b(x,t) - \varepsilon,$$
$$\max\{\tilde{b}(x',t'), \bar{b}(x')\} \leq \max\{\tilde{b}(x,t) + \varepsilon, \bar{b}(x) + \varepsilon\} \leq b(x,t) + \varepsilon.$$

For (4.8) consider a fixed $x \in [-\varepsilon_x^i, \varepsilon_x^i]$. If, for instance $\tilde{b}_1(x,t) \leq \bar{b}(x) \leq \tilde{b}_2(x,t)$, then

$$b_2(x,t) - b_1(x,t) = \tilde{b}_2(x,t) - \bar{b}(x) \leq \tilde{b}_2(x,t) - \tilde{b}_1(x,t).$$

The other cases can be checked similarly. Taking the supremum over all $(x,t) \in I(\varepsilon_x^i)_T$ gives (4.8).

□

4.4 L_p apriori estimates

Remark 4.10. Let the initial configuration of the hysteresis (i.e. the function $\bar{b}$) be given by item (iii), part (4) of Definition 4.1. Let u and a be defined as in Lemma 4.7, $\tilde{b}$ as in Lemma 4.8 and b as in Lemma 4.9. Then $\tilde{b}(x,0) \le \bar{b}(x)$ and the configuration of the hysteresis $\eta(\eta_0,u)(x,y,t)$ on the set $B(\varepsilon_x^i, m_i)_T$ is given by

$$\eta(\eta_0,u)(x,y,t) = \begin{cases} 1 & \text{if } -\frac{2}{m_i} \le y \le b(x,t), \\ 2 & \text{if } b(x,t) < y \le \frac{2}{m_i}. \end{cases} \tag{4.9}$$

and the output of the hysteresis is given by

$$\mathcal{H}(\eta_0,u)(x,y,t) = \begin{cases} H_1(u(x,y,t)) & \text{if } -\frac{2}{m_i} \le y \le b(x,t), \\ H_2(u(x,y,t)) & \text{if } b(x,t) < y \le \frac{2}{m_i}. \end{cases} \tag{4.10}$$

In this case, the function $b(x,t)$ is playing the role of a free boundary.

Lemma 4.11. *For $j = 1,2$, consider functions $u_j : B(\varepsilon_x^i, m_i)_T \to \mathbb{R}$ and Suppose that $\|u_j\|_{C(B(\varepsilon_x^i,m_i)_T)} \le U$. Then*

$$\begin{aligned} \|\mathcal{H}(\eta_0,u_1) - \mathcal{H}(\eta_0,u_2)\|^p_{L_p(B(\varepsilon_x^i,m_i)_T)} \le CT(&\|u_1-u_2\|_{C(B(\varepsilon_x^i,m_i)_T)} \\ &+ \|u_1-u_2\|^{\sigma p}_{C(B(\varepsilon_x^i,m_i)_T)}). \end{aligned} \tag{4.11}$$

where $C = C(U,p,\sigma,m_i,\varepsilon_x^i) \ge 0$ but C does not depend on u.

Proof. Let $\bar{b}$ be defined as in item (iii), part (4) of Definition 4.1, u and a be defined as in Lemmas 4.7, $\tilde{b}$ as in Lemma 4.8 and b as in Lemma 4.9. As an intermediate step we will prove

$$\begin{aligned} \|\mathcal{H}(\eta_0,u_1) - \mathcal{H}(\eta_0,u_2)\|^p_{L_p(B(\varepsilon_x^i,m_i)_T)} \le C_1\varepsilon_x^i T\big(&\|b_1-b_2\|_{C(I(\varepsilon_x^i)_T)} \\ &+ \|u_1-u_2\|^{\sigma p}_{C(B(\varepsilon_x^i,m_i)_T)}\big), \end{aligned} \tag{4.12}$$

where C_1 depends on U,p and σ.

Consider

$$\int_0^T \int_{-\varepsilon_x^i}^{\varepsilon_x^i} \int_{-\frac{2}{m_i}}^{\frac{2}{m_i}} |\mathcal{H}(\eta_0,u_1) - \mathcal{H}(\eta_0,u_2)|^p(x,y,t)\,dy\,dx\,dt. \tag{4.13}$$

Fix a value of x and t and assume that $b_1(x,t) \le b_2(x,t)$. Then the inner integral in (4.13) is (omitting the arguments of the integral for convenience) equal to

$$\int_{-\frac{2}{m_i}}^{b_1(x,t)} |H_1(u_1) - H_1(u_2)|^p \, dy + \int_{b_1(x,t)}^{b_2(x,t)} |H_2(u_1) - H_1(u_2)|^p \, dy$$
$$+ \int_{b_2(x,t)}^{\frac{2}{m_i}} |H_2(u_1) - H_2(u_2)|^p \, dy. \tag{4.14}$$

Recalling that H_1 and H_2 are locally Holder continuous with exponent σ (see Remark 2.6) one has

$$\int_{-\frac{2}{m_i}}^{b_1(x,t)} |H_1(u_1) - H_1(u_2)|^p \leq C_1 \|u_1 - u_2\|^{\sigma p}_{C(B(\varepsilon_x^i, m_i)_T)}.$$

Observe that $C_1 = C_1(U, p, \sigma)$ and that one has the analogous estimate for $b_2(x,t) \leq y \leq \frac{2}{m_i}$. Next, we make the estimate

$$\int_{b_1(x,t)}^{b_2(x,t)} |H_2(u_1) - H_1(u_2)|^p \, dy \leq 2^p \|b_1 - b_2\|_{C(I(\varepsilon_x^i)_T)} \max_{|v| \leq U} \{|H_1(v)|, |H_2(v)|\}^p.$$

Integrating over $x \in [-\varepsilon_x^i, \varepsilon_x^i]$ and $t \in [0, T]$ gives the result.

Now combining Lemmas 4.7, 4.8, 4.8 and 4.9,

$$\begin{aligned} & \|\mathcal{H}(\eta_0, u_1) - \mathcal{H}(\eta_0, u_2)\|^p_{L_p(B(\varepsilon_x^i, m_i)_T)} \\ \leq \; & C_1 \varepsilon_x^i T \big(\|b_1 - b_2\|_{C(I(\varepsilon_x^i)_T)} + \|u_1 - u_2\|^{\sigma p}_{C(B(\varepsilon_x^i, m_i)_T)} \big) \\ \leq \; & C_1 \varepsilon_x^i T \big(\|\tilde{b}_1 - \tilde{b}_2\|_{C(I(\varepsilon_x^i)_T)} + \|u_1 - u_2\|^{\sigma p}_{C(B(\varepsilon_x^i, m_i)_T)} \big) \\ \leq \; & C_1 \varepsilon_x^i T (\|a_1 - a_2\|_{C(I(\varepsilon_x^i)_T)}) + \|u_1 - u_2\|^{\sigma p}_{C(B(\varepsilon_x^i, m_i)_T)} \\ \leq \; & 2 C_1 \varepsilon_x^i T m_i (\|u_1 - u_2\|_{C(B(\varepsilon_x^i, m_i)_T)} + \|u_1 - u_2\|^{\sigma p}_{C(B(\varepsilon_x^i, m_i)_T)} \big) \end{aligned} \tag{4.15}$$

Take $C = 2C_1 \varepsilon_x^i m_i$. □

5 Auxiliary Results

5.1 Linear parabolic equations in $W_p^{2,1}(Q_T)$

Let $T_0 \geq T$

Theorem 5.1. *Let $F \in L_p(Q_T)$ and $\varphi \in \mathcal{W}$. Consider the equation*

$$u_t - \Delta u = F(\xi, t), \tag{5.1}$$
$$u|_{t=0} = \varphi, \tag{5.2}$$
$$\left. \frac{\partial u}{\partial \nu} \right|_{\partial Q_T} = 0. \tag{5.3}$$

Then (5.1) - (5.3) *has a unique solution* $u \in W^{2,1}(Q_T)$ *that satisfies* (5.1) *in* $L_p(Q_T)$ *and* (5.2) *and* (5.3) *in terms of traces. It also satisfies the following estimates:*

$$\|u\|_{C^\gamma(\overline{Q_T})} + \sum_{j=1,2} \|u_{\xi_j}\|_{C^\gamma(\overline{Q_T})} \le c_1(\|F\|_{L_p(Q_T)} + \|\varphi\|_{W_p^{2-2/p}(Q)}), \tag{5.4}$$

$$\|u\|_{W_p^{2,1}(Q_T)} \le c(\|F\|_{L_p(Q_T)} + \|\varphi\|_{W_p^{2-2/p}(Q)}). \tag{5.5}$$

where c_1, c *depend on* p, γ *and* T_0 *but not on* u.

Remark 5.2. Though (5.4) can be thought of as a consequence of (5.5) (due to Lemma (3.2)), we highlight that (5.4) and the corresponding constant c_1 will play a central role in section 6 and 7.

Proof of Theorem 5.1. One can consult [11] for the proof of the theorem, however only inequality (5.5) is proved therein and moreover the constant c depends on T in their formulation. We will not prove the existence and uniqueness of solutions here, but we will prove inequality (5.4) and that both the constants c_1 and c in (5.4) and (5.5) can be chosen to depend on T_0 only.

Indeed let $F \in L_p(Q_T)$ and define F_0 by

$$F_0(\xi,t) = \begin{cases} F(\xi,t) & 0 \le t \le T \\ 0 & T < t \le T_0 \end{cases}$$

Let u_0 be the solution of (5.1) - (5.3) with nonlinearity F_0. Then

$$\begin{aligned} \|u\|_{W_p^{2,1}(Q_T)} \le \|u_0\|_{W_p^{2,1}(Q_T)} &\le c(T_0)(\|F_0\|_{L_p(Q_T)} + \|\varphi\|_{W_p^{2-\frac{2}{p}}(Q)}) \\ &= c(T_0)(\|F\|_{L_p(Q_T)} + \|\varphi\|_{W_p^{2-\frac{2}{p}}(Q)}) \end{aligned}$$

Hence we can take $c(T_0)$. If we denote the constant from Lemma (3.2) as $c'(T)$ then we can write

$$\begin{aligned} \|u\|_{C^\gamma(\overline{Q_T})} + \textstyle\sum_{j=1,2} \|u_{\xi_j}\|_{C^\gamma(\overline{Q_T})} &\le c'(T)\|u\|_{W_p^{2,1}(Q_T)} \\ &\le c'(T)c(T_0)(\|F\|_{W_p^{2,1}(Q_T)} + \|\varphi\|_{W_p^{2-\frac{2}{p}}(Q)}) \end{aligned}$$

Now one can repeat the same procedure for the constant $c'(T)c(T_0)$ to obtain the constant $c_1(T_0)$. $\square$

We will call solutions to (5.1) - (5.3) *strong solutions.*

5.2 Semilinear Parabolic Equations in $L_\infty(Q_T)$

Consider the equation

$$u_t - \Delta u = f_0(u,\xi,t), \tag{5.6}$$

$$u|_{t=0} = \varphi, \tag{5.7}$$

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial Q_T} = 0. \tag{5.8}$$

Definition 5.3. *We say that u is an $E_{\infty,T}$-mild solution of* (5.6) - (5.8) *for initial data $\varphi \in L_\infty(Q)$ on the interval $[0,T)$, if u is a measureable function on Q_T and satisfies*
(1) $u(\cdot,t) \in L_\infty(Q)$ for a.e. $t \in [0,T)$,
(2) $sup_{s\in(0,t)} \|u(\cdot,s)\|_{L_\infty(Q)} < \infty \ \forall t \in (0,T)$,
(3) $u(\cdot,t) = P(t)\varphi + \int_0^t P(t-s)(f_0(u(\cdot,s),\cdot,s)\,ds \ \forall t \in (0,T)$,
where P is a semigroup on $L_\infty(Q)$ (defined in Appendix C) and the integral is an absolutely convergent Bochner integral in $L_\infty(Q)$

Lemma 5.4. *Suppose that for every bounded set $\Omega \subset \mathbb{R} \times Q \times [0,\infty)$ the function $f_0(u,x,t)$ satisfies the following:*
(1) $|f_0(u,\xi,t)| \le L(\Omega) \ \forall (u,x,t) \in \Omega$
(2) $|f_0(u_1,\xi,t) - f_0(u_2,\xi,t)| \le L(\Omega)|u_1 - u_2| \ \forall (u_1,\xi,t),(u_2,\xi,t) \in \Omega$,
(3) For every fixed $u \in \mathbb{R}$, the function $f_0(u,\xi,t)$ is measurable in Q_T.
Then for each initial function $\varphi \in L_\infty(Q)$, there exists a $T \in (0,\infty]$ such that (5.6) - (5.8) *has a unique $E_{\infty,T}$ mild solution on the interval $[0,T)$.*

Definition 5.5. *We say that $T_1 \in (0,\infty)$ is a maximal existence time for the initial data φ if problem* (5.6) - (5.8) *has an E_{∞,T_1}-mild solution on the interval $[0,T_1)$ but for any $T' > T_1$, there is no E_{∞,T_1}-mild solution on the interval $[0,T')$.*

Lemma 5.6. *Assume that $\varphi \in L_\infty(Q)$ and the conditions of Lemma 5.4 are satisfied. Then there is a maximal existence time $T_1 \in (0,\infty]$ and problem* (5.6) - (5.8) *has a unique E_{∞,T_1}-mild solution on the interval $[0,T_1)$. If T_1 is finite then*

$$\lim_{t\to T_1} \|u(\cdot,t)\|_{L_\infty(Q)} = \infty$$

Proof. The proofs of Lemmas 5.4 and 5.6 are formulated as Theorem 1, pg.111 in [13]. □

Lemma 5.7. *Let f_0 satisfy the assumptions of Lemma 5.4 and u be the $E_{\infty,T}$-mild solution of* (5.6) - (5.8). *Then $F(\xi,t) = f_0(u,\xi,t) \in L_p(Q_T)$ and u coincides with the strong solution of* (5.1) -(5.3).

Proof. Since $u \in L_\infty(Q_T)$, the set $(u, \xi, t) \subset \mathbb{R} \times Q_T$ and so by part (1), $|f_0(u, \xi, t)|$ is bounded and hence $F(xi, t) = f_0(u, \xi, t) \in L_p(Q_T)$. Therefore there is a solution u' of (5.1) - (5.3) and one can show that $u = u'$, however the proof is beyond the scope of this thesis. □

Lemma 5.8. *Assume that u satisfies* (5.6) - (5.8) *and that $\varphi \in \mathcal{W}$ and $F(\xi, t) = f_0(u, \xi, t) \in L_p(Q_T)$. Further assume that there is some $U > 0$ such that*

(1) $f_0(\cdot, \xi, t) : \mathbb{R} \to \mathbb{R}$ is continuous at the points $\pm U$ uniformly with respect to $(\xi, t) \in Q_T$ i.e. for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|U - U'| < \delta$ implies that $|f_0(U, \xi, t) - f_0(U', \xi, t)| < \varepsilon$ where δ can be chosen independently of (ξ, t),

(2) $f_0(U, \xi, t) < 0$, $f_0(-U, \xi, t) > 0$ for all $(\xi, t) \in Q_T$,

(3) $\|\varphi\|_{C(\overline{Q})} < U$.

Then $\|u\|_{C(\overline{Q_T})} < U$

Proof. We refer the reader to Appendix A. □

6 Existence of Solutions

Remark 6.1. In this section we will continue to write functions (using u as an example) $u(x, y, t)$ as shorthand for $u \circ \psi_i^{-1}(x, y, t)$. We also introduce the notation $A_{i,T} = A_i \times [0, T]$

Fix some $T_0 \geq 0$. Consider $(\varphi, \eta_0) \in E_m$ and the quantity U_0 from Condition 3.15. Choose U so that

$$\|\varphi\|_{C(\overline{Q})} < U \text{ and } U_0 < U, \tag{6.1}$$

then define $V = \max_{|u| \leq U}\{|H_1(u)|, |H_2(u)|\}$ and

$$f_{U,T_0} = \left(\int_0^{T_0} \int_Q \max_{|u| \leq U, |v| \leq V} |f(u, v)|^p \, d\xi \, dt\right)^{\frac{1}{p}}. \tag{6.2}$$

Fix some $\lambda \in (0, \gamma)$ and let $c_{\lambda,\gamma}$ be an embedding constant

$$\|u\|_{C^\lambda(\overline{Q_{T_0}})} \leq c_{\lambda,\gamma} \|u\|_{C^\gamma(\overline{Q_{T_0}})}. \tag{6.3}$$

Note that the embedding constant c_λ, γ depends on the diameter of the domain, so, even though we have chosen Q_{T_0} in (6.3), the same constant is valid for the sub-domains Q_T with $T \leq T_0$.

Consider inequality (5.4). Let

$$c_2 := c_{\lambda,\gamma} c_1 (f_{U,T_0} + m). \tag{6.4}$$

Finally choose T such that

$$c_2 T^\lambda \leq \frac{1}{4m^2}. \tag{6.5}$$

Definition 6.2. *Define $P^\lambda(\overline{Q_T})$ as the set of all functions $u, u_{\xi_j} \in C^\lambda(\overline{Q_T})$ ($j = 1, 2$) equipped with the norm*

$$\|u\|_{P^\lambda(\overline{Q_T})} := \|u\|_{C^\lambda(\overline{Q_T})} + \sum_{j=1,2} \|u_{\xi_j}\|_{C^\lambda(\overline{Q_T})}.$$

Lemma 6.3. *$P^\lambda(\overline{Q_T})$ is a Banach space.*

Proof. It is clear that $\|\cdot\|_{P^\lambda(Q_T)}$ is a metric since $C^\lambda(\overline{Q_T})$ is, so one only needs to show that $P^\lambda(\overline{Q_T})$ is complete. Let $j = 1, 2$. For any Cauchy sequence in $P^\lambda(\overline{Q_T})$, say $\{u_n\}$, the sequence $\{u_{n;x_j}\}$ is also Cauchy in $C^\lambda(\overline{Q_T})$ and hence has a limit $u_{n;\xi_j} \to v_j$ in $C^\lambda(\overline{Q_T})$. Similarly, u has a limit $u_n \to u \in C^\lambda(\overline{Q_T})$. In particular both $\{u_n\}$ and $\{u_{n;\xi_j}\}$ converge uniformly to there respective limits and so u_{ξ_j} exists and $u_{\xi_j} = v_j$. We have also shown that the sequences converge in $C^\lambda(\overline{Q_T})$, and hence $u_n \to u$ in $P^\lambda(\overline{Q_T})$. □

Definition 6.4. *Define $R^\lambda(Q_T) \subset P^\lambda(\overline{Q_T})$ as the set of all functions such that*

(1) $|u(\xi, t)| \leq U \ \forall (\xi, t) \in Q_T$,
(2) $\|u\|_{C^\lambda(\overline{Q_T})} + \sum_{j=1,2} \|u_{\xi_j}\|_{C^\lambda(\overline{Q_T})} \leq c_2$, (where c_2 is given in (6.4))
(3) $u(\xi, 0) = \varphi(\xi)$.

Note that $R^\lambda(Q_T)$ is a closed convex set.

Lemma 6.5. *If $u_0 \in R^\lambda(Q_T)$ then $\mathcal{H}(\eta_0, u_0)$ is given by (4.10) on the set $A_{i,T}$ (in (x, y) coordinates) and by*

$$\mathcal{H}(\eta_0, u_0)(\xi, t) = \begin{cases} H_1(u_0(\xi, t)) & \text{if } \xi \in Q_1 \backslash \bigcup_i A_i, \\ H_2(u_0(\xi, t)) & \text{if } \xi \in Q_2 \backslash \bigcup_i A_i. \end{cases} \tag{6.6}$$

Moreover u is transverse with respect to $\eta(\eta_0, u)$ and

$$\begin{array}{ll} u(\xi, t) < \beta & \text{if } \xi \in Q_1 \backslash \bigcup_i A_i, \\ u(\xi, t) > \alpha & \text{if } \xi \in Q_2 \backslash \bigcup_i A_i. \end{array} \tag{6.7}$$

Proof. First we elaborate on the choice of T. Observe that for every $u_0 \in R^\lambda(Q_T)$ and $(\xi, t) \in Q_T$,

$$|u_0(\xi,t) - u_0(\xi,0)| + \sum_{j=1,2} |u_{0;\xi_j}(\xi,t) - u_{0;\xi_j}(\xi,0)| \\ \leq (\|u_0\|_{C^\lambda(\overline{Q_T})} + \sum_{j=1,2} \|u_{0;\xi_j}\|_{C^\lambda(\overline{Q_T})})t^\lambda.$$

Taking the supremum over all $\xi \in \overline{Q}$ and using Definition 6.4 and (6.5) one obtains

$$\|u_0(\cdot,t) - \varphi\|_{C(\overline{Q})} + \sum_{j=1,2} \|u_{0;\xi_j}(\cdot,t) - \varphi_{\xi_j}\|_{C(\overline{Q})} \leq c_2 T^\lambda \\ \leq \frac{1}{4m^2}. \tag{6.8}$$

Consider in some A_i, $u_0|_{A_i,T}$ and $\varphi|_{A_i}$. Since ψ_i is a composition of a translation and a rotation (see Definition 4.1, part (4), item (i)), both $u_0(\xi)|_{A_i,T}$ and $\varphi(\xi,t)|_{A_i}$ in the coordinates (ξ,t) of $A_{i,T}$ can be related to the corresponding functions $u_0(x,y,t)$ and $\varphi(x,y)$ in the coordinates (x,y,t) of $B(\varepsilon_x^i, m_i)_T$ in the following way.

$$\|u_0(\cdot,\cdot,t) - \varphi(\cdot,\cdot)\|_{C(B(\varepsilon_x^i,m_i))} \leq \|u_0(\cdot,t) - \varphi(\cdot)\|_{C(\overline{Q})},$$

$$\|u_{0;y}(\cdot,\cdot,t) - \varphi_y(\cdot,\cdot)\|_{C(B(\varepsilon_x^i,m_i))} \leq \sum_{j=1,2} \|u_{0;\xi_j}(\cdot,t) - \varphi_{\xi_j}(\cdot)\|_{C(\overline{Q})}.$$

In particular

$$\|u_0(\cdot,\cdot,t)-\varphi(\cdot,\cdot)\|_{C(B(\varepsilon_x^i,m_i))}+\|u_{0;y}(\cdot,\cdot,t)-\varphi(\cdot,\cdot)_y\|_{C(B(\varepsilon_x^i,m_i))} \leq \frac{1}{4m^2} \leq \frac{1}{4m_i^2}.$$

Therefore, in each set $A_{i,T}$, $u_0(x,y,t)$ and $\varphi(x,y)$ satisfy the assumptions of Lemma 4.7. By Remark 4.10, the output of the hysteresis on the set $A_{i,T}$ is given in the coordinates (x,y,t) by (4.10).

Moreover, using part (3) of Definition 4.1 combined with (6.8) one sees that for (ξ,t) with $\xi \in Q_1$

$$u_0(\xi,t) < \beta - \frac{1}{m^2} + \frac{1}{4m^2} < \beta \tag{6.9}$$

A similar assertion holds for $(Q_2 \cup \partial Q) \setminus \cup_i A_i$ and α. These two observations prove (6.6) and (6.7).

Turning our attention to the transversality claim, we have just shown that the hysteresis does not reach the threshold β on Q_1 or α on $Q_2 \cup \partial Q$. Therefore if we let $q_j(t) = \{\xi \in \overline{Q} \mid \eta(\eta_0, u)(\xi, t) = j\}$, then $q_1(t) \subset Q_1$ and $q_2(t) \subset (Q_2 \cup \partial Q)$.

In the set A_i, note that since $\varphi(x,y) < \beta - \frac{1}{m^2}$ for $-\frac{2}{m_i} \leq y \leq \overline{b}(x)$, we have that $u(x,y,t) < \beta$ on $-\frac{2}{m_i} \leq y \leq \overline{b}(x)$. If $\overline{b}(x) \leq y \leq b(x,t)$ then there is a time $t' \leq t$ such that $u(x,y,t) = \alpha$, thus $u(x,y,t) \in [\alpha - \frac{1}{4m_i^2}, \alpha + \frac{1}{4m_i^2}]$ and $u(x,y,t) < \beta$. This proves that $u(\xi,t) < \beta$ on $\overline{q_1(t)}$ which in turn implies that $\Gamma_{\beta,t} \cap \partial q_1(t) = \emptyset$ where $\Gamma_{\beta,t} = \{\xi \in Q \mid u(\xi,t) = \beta\}$. One can argue similarly (using the observation that $u(x,y,t) > \alpha$ for $b(x,t) < y \leq \frac{2}{m_i}$) to show that $u(\xi,t) > \alpha$ on $\operatorname{int}(q_2(t)) \cup \partial Q$. Thus we have shown parts (2) and (3) of Definition 3.5.

By referring to (4.9), we see that the set of $(x,y) \in B(\varepsilon_x^i, m_i)$ where $\eta(\eta_0,u)(x,y,t) = 1$ and $\eta(\eta_0,u)(x,y,t) = 2$ are both measurable and that the free boundary $b(x,t)$ is of measure zero. This, plus part (1) of Definition 3.5 for Q_1, Q_2 and ∂Q imply that $q_1(t)$, $q_2(t)$ and $\partial q_1(t)$ also satisfy part (1) of Definition 3.5.

It remains to consider points $\xi \in \Gamma_{\alpha,t} \cap \partial q_1(t)$. In this case, $\xi \in \operatorname{int}(A_i)$ for some $i = 1, \cdots, d$.

In the coordinates (x,y,t) on $A_{i,T}$ with $\psi_i(\xi) = (x',y')$, we have that $u_y^i(x',y',t) > 0$ (since $\varphi(x',y') \in [\alpha - \frac{1}{m_i^2}, \alpha + \frac{1}{m_i^2}]$ implies that $\varphi_y(x,y) > \frac{1}{m_i}$ which in turn implies that $u_y(x',y') > 0$). Now using a procedure almost identical that used in the proof of Lemma 4.3, we can translate by (x',y') to construct a box around ξ with corresponding map θ_i and free boundary $\overline{b}^i_{(x',y')}$. This gives part (4) of Definition 3.5 and thus completes the proof.

□

Theorem 6.6 (Solutions to the semilinear problem)**.**

For $u_0 \in R^\lambda(Q_T)$, let $f_0(u,\xi,t) = f(u, \mathcal{H}(\eta_0,u_0))(\xi,t)$. Then the following holds:

(1) The semilinear problem (5.6) - (5.8) *has a unique strong solution $u \in W_p^{2,1}(Q_T)$.*

(2) $u \in R^\gamma(Q_T)$

(3) Let $u_{0n} \in R^\lambda(Q_T)$ be a sequence with f_{0n} and u_n defined analogously. If $u_{0n} \to u_0$ in $P^\lambda(\overline{Q_T})$, then $u_n \to u$ in $P^\gamma(\overline{Q_T})$.

Proof. (1) Given $u_0 \in R^\lambda(Q_T)$ let us show that f_0 satisfies the assumptions of Lemma 5.4. Let $\Omega \subset \mathbb{R} \times Q_T$ be a bounded set. For part (1) note that $|u_0(\xi,t)| \leq U$, hence $\mathcal{H}(\eta_0,u_0)(\xi,t)$ is bounded. Since f is locally Lipschitz in both arguments (Condition 3.14), and u is bounded, $f_0(u,\xi,t) = f(u, \mathcal{H}(\eta_0,u_0)$ is also bounded. For part (2), let $(u,\xi,t), (v,\xi,t) \in \Omega$ then

$$\begin{aligned} |f_0(u,\xi,t) - f_0(v,\xi,t)| &= |f(u,\mathcal{H}(\eta_0,u_0)(\xi,t) - \\ &\quad f(v,\mathcal{H}(\eta_0,u_0)(\xi,t))| \\ &\le L(\Omega)|u-v| \end{aligned} \tag{6.10}$$

The constant L comes from the boundedness of u and v, and the local Lipschitz continuity of f. Observe that (ξ, t) only appears in the second argument of $f(\cdot,\cdot)$ i.e. we obtain an estimate that is uniform in $(\xi,t) \in \overline{Q_T}$.

For part (3) let u be fixed and observe that the real valued function $f(u,\cdot)$ is measurable since it is continuous. Compositions of measurable functions are measurable so it suffices to show that $\mathcal{H}(\eta_0,u_0) : \overline{Q_T} \to \mathbb{R}$ is measurable. To this end define the sets

$$N_j = \{(\xi,t) \in \overline{Q_T} \mid \eta(\eta_0,u_0)(\xi,t) = j\}.$$

Let $\chi \subset \mathbb{R}$ be measurable. Consider

$$\mathcal{H}(\eta_0,u_0)^{-1}(\chi) = ((H_1 \circ u_0)^{-1}(\chi) \cap N_1) \cup ((H_2 \circ u_0)^{-1}(\chi) \cap N_2).$$

$H_1 \circ u_0$ and $H_2 \circ u_0$ are continuous (hence measurable) so it remains to show that N_1 and N_2 are measurable. We begin with N_1. First note that by Lemma 6.5,

$$((Q_1 \times (0,T))\backslash A_{i,T} \subset N_1,$$

hence we need to show that $N_1 \cap A_{i,T}$ is measurable. Because ψ^i is continuous it suffices to show $N_1 \cap A_{i,t}$ is measurable in the coordinates (x,y,t) i.e.

$$A_{i,T} \cap N_1 = \{(x,y,t) \mid -\frac{2}{m_i} \le y \le b(x,t)\},$$

is measurable. We claim in fact that since $b(x,t)$ is continuous, this set is closed. In detail if $(x_n,y_n,t_n) \to (x,y,t)$ then $-\frac{2}{m_i} \le y_n \le b(x_n,t_n)$ and so $-\frac{2}{m_i} \le y \le b(x,t)$. For N_2 we write

$$A_{i,T} \cap N_2 = \{(x,y,t) \mid b(x,t) < y \le \frac{2}{m_i}\}.$$

It's closure is measurable and is formed by adding the points (x,y,t) such that $y = b(x,t)$. However the set $\{(x,y,t) \mid y = b(x,t)\}$ is itself closed because $b(x,t)$ is continuous. Hence $A_{i,T} \cap N_2$ is a measurable set modulo a measurable set, and is thus also measurable. Thus we have obtained part (3) of Lemma 5.4.

We can now apply Lemma 5.4. Let f be the unique E_{∞,T_1}-mild solution of (5.6) - (5.8). with maximal existence time T_1. Since the data is only defined up to T, we must have $T_1 \leq T$. By Lemma 5.7, f is also a strong solution of (5.1) - (5.3).

All that remains to show in part (1) is that $T_1 = T$. Let us show the conditions of Lemma 5.8 are satisfied for some $T_2 < T_1$. Firstly recall that we have chosen U such that $U > U_0$ (Condition 3.15) and $\|\varphi\|_{C(\overline{Q})} < U$. For part (1) observe that by (6.10), $f_0(\cdot, \xi, t)$ is locally Lipschitz with Lipschitz constant independent of (ξ, t). In particular it is continuous at $U \in \mathbb{R}$ which gives part (1). Part (2) follows from Condition 3.15, therefore Lemma 5.8 implies that $\|u\|_{C(\overline{Q_{T_2}})} \leq U$. In particular,

$$\sup_{t\in(0,T_1)} \|u(\cdot,t)\|_{L_\infty(Q)} \leq U.$$

By Lemma 5.6, $T_1 < T$ would require a blow up in the L^∞ norm of $u(\cdot, t)$, but since this is not the case we conclude that $T_1 = T$.

(2) To show that $u \in R^\gamma(Q_T)$, we begin by noting that we have just shown item (1) of Definition 6.4. Since u is a solution to (5.1) - (5.3), item (3) of Definition 6.4 follows from Theorem 5.1 and Lemma 3.1 i.e. at $t = 0$ the trace is regular enough for us to say $u(\xi, 0) = \varphi(\xi)$, as opposed to the equality holding a.e.

For item (2) we use estimate (5.4) (which also implies that $u \in P^\gamma(Q_T)$) to calculate

$$\begin{aligned} \|u\|_{C^\gamma(\overline{Q_T})} + \sum_{j=1,2} \|u_{\xi_i}\|_{C^\gamma(\overline{Q_T})} &\leq c_1(\|F\|_{L_p(Q_T)} + \|\varphi\|_{W_p^{2-2/p}(Q)}), \\ &\leq c_1(f_{U,T_0} + m), \\ &\leq c_{\lambda,\gamma} c_1(f_{U,T_0} + m) = c_2, \end{aligned} \tag{6.11}$$

where we have used that $c_{\lambda,\gamma} \geq 1$ (indeed the norms of $C^\lambda(\overline{Q_T})$ and $C^\gamma(\overline{Q_T})$ coincide on constant functions).

3. Assume for contradiction that for a sequence u_{0n} satisfying the conditions of the theorem, there is an $\varepsilon > 0$ such that

$$\begin{aligned} \varepsilon < \|u_n - u\|_{P^\gamma(Q_T)} &= \|u_n - u\|_{C^\gamma(\overline{Q_T})} + \textstyle\sum_{j=1,2} \|u_{n;\xi_j} - u_{\xi_j}\|_{C^\gamma(\overline{Q_T})} \\ &\leq c\|u_n - u\|_{W_p^{2,1}(Q_T)} \end{aligned} \tag{6.12}$$

By (5.5)

$$\|u_n\|_{W_p^{2,1}(Q_T)} \leq c(f_{U,T_0} + m)$$

where c is independent of $\{u_n\}$. By compactness of the embedding of $W_p^{2,1}(Q_T)$ in $C(\overline{Q_T})$ we can extract a subsequence, not relabelled, that converges in $C(\overline{Q_T})$.

For u_n, u_m, their difference satisfies (5.1) with F replaced by $f_{0n}(u_n, \xi, t) - f_{0m}(u_m, \xi, t)$, along with the zero initial and the zero Neumann boundary conditions. Inequalitiy (5.5) gives

$$\|u_n - u_m\|_{W_p^{2,1}(Q_T)} \leq c\|f_{0n}(u_n, \xi, t) - f_{0m}(u_m, \xi, t)\|_{L_p(Q_T)} \tag{6.13}$$

Now using the local Lipschitz continuity of f (Condition 3.14) and the boundedness of u_n, u_m and $\mathcal{H}(\eta_0, u_0)$

$$\begin{aligned} \|u_n - u_m\|_{W_p^{2,1}(Q_T)} \leq & L(\|u_n - u_m\|_{L_p(Q_T)} + \\ & \|\mathcal{H}(\eta_0, u_{0n}) - \mathcal{H}(\eta_0, u_{0m})\|_{L_p(Q_T)}), \end{aligned} \tag{6.14}$$

Where L depends on $\|\mathcal{H}(\eta_0, u_{0_n})\|_{L^\infty(Q_T)}$ and $\|u_{0_n}\|_{L^\infty(Q_T)}$ and thus only on U.

We will treat this inequality in some detail. First note that since $u_{0n} \in R^\lambda(Q_T)$, Lemma 6.5 applies. This allows us to write (omitting the arguments of the integrals and referring to Figure 11)

$$\begin{aligned} & \int_{Q_T} |(\mathcal{H}(\eta_0, u_{0n}) - \mathcal{H}(\eta_0, u_{0m})|^p \, d\xi dt, \\ \leq & \int_0^T \int_{Q_1 \setminus \bigcup_i A_i} |H_1(u_{0n}) - H_1(u_{0m})|^p \, d\xi dt \\ & + \int_0^T \int_{Q_2 \setminus \bigcup_i A_i} |H_2(u_{0n}) - H_2(u_{0m})|^p \, d\xi dt \\ & + \sum_{i=1}^d \|\mathcal{H}(\eta_0, u_{0n}) - \mathcal{H}(\eta_0, u_{0m})\|^p_{L_p(A_{i,T})}. \end{aligned} \tag{6.15}$$

Using Condition 2.5 and Lemma 6.5 we can estimate two of the terms in (6.15) as follows

$$\int_0^T \int_{Q_1 \setminus \bigcup_i A_i} |H_1(u_{0n}) - H_1(u_{0m})|^p \, d\xi dt \leq CT|Q|\|u_{0n} - u_{0m}\|^{\sigma p}_{C(Q_T)}, \tag{6.16}$$

$$\int_0^T \int_{Q_2 \setminus \bigcup_i A_i} |H_2(u_{0n}) - H_2(u_{0m})| \, d\xi dt \leq CT|Q|\|u_{0n} - u_{0m}\|^{\sigma}_{C(Q_T)}, \tag{6.17}$$

where C depends on U and m (i.e. $\|u_{0_n}\|_{L^\infty(Q_T)} \leq U$ and by (6.9) u_{0_n} is separated from the relevant thresholds by $\frac{1}{4m^2}$). Next we estimate the functions $\mathcal{H}(\eta_0, u_{0n})$ in $L_p(A_i)$ by considering them as functions $\mathcal{H}(\eta_0, u_{0n})(x, y, t)$ in

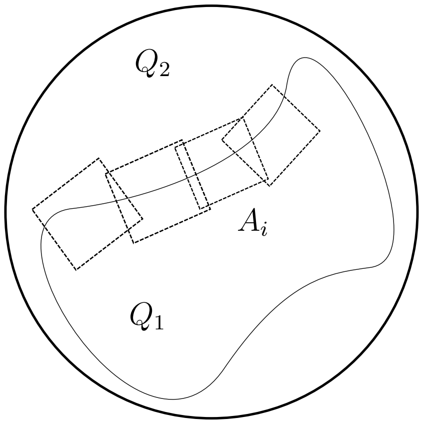


Figure 11: Boxes A_i covering a section of the boundary between Q_1 and Q_2.

the coordinates of $B(\varepsilon_x^i, m_i)_T$. By Lemma 4.11 and the associated inequality (4.11)

$$\begin{aligned}\|\mathcal{H}(\eta_0, u_{0n}) - \mathcal{H}(\eta_0, u_{0m})\|^p_{L_p(B(\varepsilon_x^i, m_i)_T)} \leq CT(&\|u_{0n} - u_{0m}\|_{C(B(\varepsilon_x^i, m_i)_T)} \\ &+ \|u_{0n} - u_{0m}\|^{\sigma p}_{C(B(\varepsilon_x^i, m_i)_T)}),\end{aligned} \tag{6.18}$$

where C depends on U and m. Combining (6.14), (6.15) (6.16), (6.17) and (6.18) we see that $\{u_n\}$ is Cauchy in $W_p^{2,1}(Q_T)$ and therefore has a limit which we denote by u'. Note that by Lemma 3.2, $u_n \to u'$ in $P^\gamma(\overline{Q_T})$. By replacing the quantities with subscript m by their respective quantities for u' and by applying Lemmas 6.5 and 4.11 as we did above, one can see that

$$f_0(u_n, \cdot, \cdot) \to f_0(u', \cdot, \cdot) \text{ in } L_p(Q_T).$$

Because u_n converges to u' in $W_p^{2,1}(Q_T)$ and satisfies (5.6) - (5.8) with f_{0n} in place of f, this means that u' satisfies

$$\begin{aligned} u'_t - \Delta u' &= f_0(u', \xi, t), \\ u'|_{t=0} &= \varphi, \\ \left.\frac{\partial u'}{\partial \nu}\right|_{t=0} &= 0. \end{aligned}$$

But this problem has a unique solution, namely u, hence $u = u'$. This contradiction concludes the proof of part 3.

□

proof of Theorem 3.18.

1. Given $(\varphi, \eta_0) \in E_m$, consider the set $R^\lambda(Q_T)$ defined in Definition 6.4. We will show that there is a $u \in R^\lambda(Q_T)$ that is a solution of (3.4) - (3.6). We refer the reader to the beginning of this section and Theorem 6.6 for the relevant notation. Given $u_0 \in R^\lambda(Q_T)$, let u be the solution to (5.6) - (5.8) with $f_0(u, \xi, t) = f(u, \mathcal{H}(\eta_0, u_0))$. By Theorem 6.6, part (2), $u \in R^\gamma(Q_T)$ and so by (6.11)

$$\begin{aligned}
&\|u\|_{C^\lambda(\overline{Q_T})} + \sum_{j=1,2} \|u_{\xi_j}\|_{C^\lambda(\overline{Q_T})}, \\
\le& c_{\lambda,\gamma}(\|u\|_{C^\gamma(\overline{Q_T})} + \sum_{j=1,2} \|u_{\xi_j}\|_{C^\gamma(\overline{Q_T})}), \\
\le& c_{\lambda,\gamma} c_1 (f_{U,T_0} + m) = c_2. \qquad (6.19)
\end{aligned}$$

Therefore $u \in R^\lambda(Q_T)$ and we can define a nonlinear operator

$$\mathcal{R} : R^\lambda(Q_T) \to R^\lambda(Q_T), \quad \mathcal{R}(u_0) = u.$$

This map is continuous by part (3) of Theorem 6.6 and the continuity of the embedding $P^\gamma(Q_T) \subset P^\lambda(Q_T)$. In part (2) of Theorem 6.6 it was shown that $\mathcal{R}(R^\lambda(Q_T)) \subset R^\gamma(Q_T)$. In particular $\mathcal{R}(R^\lambda(Q_T))$ can be treated as a bounded subset of $P^\gamma(\overline{Q_T})$ and therefore as a precompact subset of $P^\lambda(\overline{Q_T})$. The later observation follows from the compactness of the embedding $C^\gamma(\overline{Q_T}) \subset C^\lambda(\overline{Q_T})$. Recall that $R^\lambda(Q_T)$ is a closed convex set, so by the Schauder fixed point theorem, there is a function u such that $\mathcal{R}(u) = u$. It is immediate that u is a solution (3.4) - (3.6).

2. Assume that $u_0 \in W_p^{2,1}(Q_T)$ is a solution of (3.4) - (3.6). In order to show that u_0 is transverse we will show that for T satisfying (6.5), $u_0 \in R^\lambda(\overline{Q_T})$ and is therefore transverse by 6.5. If we let $f_0(u, xi, t) = f(u, \mathcal{H}(\eta_0, u_0))$, then by lemma 5.7, u_0 is also a solution to (5.6) - (5.8). One can now proceed as in the proof of part X of Theorem 6.6 to show that $\|u_0\|_{C(\overline{Q_T})} \le U$, with the observation that the terms $\mathcal{H}(\eta_0, u_0)$ in (6.10) cancel out and hence it is not necessary to assume apriori that $u_0 \in R^\lambda(\overline{Q_T})$.

Arguing as in part 1 we see that (6.19) is valid and hence $u_0 \in R^\lambda(Q_T)$. □

7 Uniqueness of Solutions

In this section let $T_0 > 0$ be fixed and assume Condition 2.3.

Remark 7.1. For Lemma 7.2, we will consider a function u satisfying Lemma 4.7 and the quantities a, b in 4.8 and 4.9 respectively. We will also need to recall the function $\overline{b}$ defining the initial configuration of the hysteresis as given by item (iii), part (4) of Definition 4.1. In this framework the configuration of the hysteresis $\eta(\eta_0, u)(x, y, t)$ on the set $B(\varepsilon_x^i, m_i)_T$ is given by (4.10).

Lemma 7.2. *For $j = 1, 2$, let $u_j : B(\varepsilon_x^i, m_i)_T \to \mathbb{R}$ satisfy the assumptions of Lemma 4.7 and further suppose that $\|u_j\|_{C(B(\varepsilon_x^i, m_i)_T} \leq U$. Then for all $(x, s) \in I(\varepsilon_x^i)_T$*

$$\int_{-\frac{2}{m_i}}^{\frac{2}{m_i}} |\mathcal{H}(\eta_0, u_1)(x, y, s) - \mathcal{H}(\eta_0, u_2)(x, y, s)| dy \leq C\|u_1 - u_2\|_{C(B(\varepsilon_x^i, m_i)_T)} \quad (7.1)$$

where C depends on m_i and U but does not depend on x, s or T.

Remark 7.3. Let us comment as to why we have formulated Lemma 7.2 in addition to Lemma 4.11. In what follows we will prove that u_1 and u_2 coincide. To this end we will find a continuous function $F(T)$ such that $F(T) \to 0$ and such that

$$\|u_1 - u_2\|_{C(\overline{Q_T})} \leq F(T)\|u_1 - u_2\|_{C(\overline{Q_T})}$$

If there did not exist a T such that $\|u_1 - u_2\|_{C(\overline{Q_T})} = 0$ then one could take T sufficiently small so that $F(T) < 1$ and so

$$\|u_1 - u_2\|_{C(\overline{Q_T})} < \|u_2 - u_2\|_{C(\overline{Q_T})},$$

which is clearly a contradiction. However if one only had the inequality

$$\|u_1 - u_2\|_{C(\overline{Q_T})} \leq F(T)\|u_1 - u_2\|^{\sigma}_{C(\overline{Q_T})},$$

then there is no contradiction for $F(T) < 1$. Therefore we must improve estimate (4.11) to the estimate (7.1). We shall see that in order to obtain this estimate, we need more than Condition 2.5 i.e. we must use Condition 2.3.

Proof of Lemma 7.2. Let $(x, s) \in I(\varepsilon_x^i)_T$ be fixed and assume that $b_1(x, s) \leq b_2(x, s)$. We will divide the interval $[-\frac{2}{m_i}, \frac{2}{m_i}]$ into several subsets. For convenience we will omit the arguments of the integrals.

Case $Y_1 = \{y \mid y < b_1(x, s), b_2(x, s)\}$. Then $-U < u_j(x, y, s) \leq \alpha$ so by Condition 2.3, H_1 is Lipschitz on Y_1, thus

$$\begin{aligned}\int_{Y_1} |H_1(u_1) - H_1(u_2)|\, dy &\leq \int_{-\frac{2}{m^i}}^{\frac{2}{m_i}} C_1(U)|u_1 - u_2|\, dy, \\ &\leq C_2(U, m_i)\|u_1 - u_2\|_{C(B(\varepsilon_x^i, m_i)_T)}. \end{aligned} \quad (7.2)$$

Case $Y_2 = \{y \mid b_1(x,s) < y < b_2(x,s)\}$

$$\int_{b_1(x,s)}^{b_2(x,s)} |H_2(u_1) - H_1(u_2)|^p \, dy \le C_3(U)\|b_1 - b_2\|_{C(I(\varepsilon_x^i)_T)}.$$

Arguing similarly to the proof of Lemma (4.15)

$$\|b_1 - b_2\|_{C(I(\varepsilon_x^i)_T)} \le 2C_3(U)m_i\|u_1 - u_2\|_{C(B(\varepsilon_x^i, m_i)_T)}.$$

Case $Y_3 = \{y \mid y > b_1(x,s), b_2(x,s)$ and $\varphi(x,y) \ge \alpha + \frac{1}{m_i^2}\}$. Then by (4.4), $u_j(x,y,s) \ge \alpha + \frac{1}{2m_i^2}$. A similar calculation to the case Y_1 gives

$$\int_{Y_3} |H_2(u_1) - H_2(u_2)| \, dy \le C_4(U, m_i)\|u_1 - u_2\|_{C(B(\varepsilon_x^i, m_i)_T)}.$$

Case $Y_4 = \{y \mid y > b_1(x,s), b_2(x,s)$ and $\varphi(x,y) \le \alpha + \frac{1}{m_i^2}\}$. In this case, $u_{j;y}(x,y,t) > \frac{1}{2m_i}$, so using the mean value theorem

$$\begin{aligned} |u_j(x,y,s) - \alpha| &= u_j(x,y,s) - u_j(x, a_j(x,s), s) \\ &\ge \frac{1}{2m_i}(y - a_j(x,s)) \ge \frac{1}{2m_i}(y - \bar{b}(x)). \end{aligned}$$

Using Condition 2.3, this implies that

$$\int_{Y_4} |H_2(u_1) - H_2(u_2)| \, dy \le \int_{\bar{b}(x)}^{\frac{2}{m_i}} \frac{M|u_1 - u_2|}{2(\frac{1}{2m_i}(y - \bar{b}(x)))^{1-\sigma}} dy. \tag{7.3}$$

$$\begin{aligned} (7.3) &\le C_5(U)\|u_1 - u_2\|_{C(B(\varepsilon_x^i, m_i)_T)} \int_{-\frac{2}{m_i}}^{\frac{2}{m_i}} \left(\frac{2m_i}{y + \frac{2}{m_i}}\right)^{1-\sigma} dy, \\ &\le C_6(U, m_i)\|u_1 - u_2\|_{C(B(\varepsilon_x^i, m_i)_T)}. \end{aligned}$$

Finally, observe that the constants C_1 through C_6 do not depend on (x,s). □

Lemma 7.4. *Let u_1, u_2 be two transverse solutions of* (3.7) - (3.9) *on Q_{T_0}. Then there is a $T > 0$ such that u_1 and u_2 coincide on Q_T.*

Proof. Let u_1, u_2 be two transverse solutions of (3.4) - (3.6). Letting $w = u_1 - u_2$ and $h = \mathcal{H}(\eta_0, u_1) - \mathcal{H}(\eta_0, u_2)$ we have

$$w_t - \Delta w = h, \tag{7.4}$$

$$w|_{t=0} = 0, \tag{7.5}$$

$$\left.\frac{\partial w}{\partial \nu}\right|_{Q_T} = 0. \tag{7.6}$$

We will show that there is a decreasing continuous function $F(T)$ such that $F(T) \geq 0$, $F(T) \to 0$ as $T \to 0$ and such that for any $(\kappa, t) \in Q_T$

$$|w(\kappa, t)| \leq F(T)\|w\|_{C(\overline{Q_T})}.$$

Taking $\sup\{|w(\kappa, t)|\}$ gives $\|w\|_{C(\overline{Q_T})} \leq F(T)\|w\|_{C(\overline{Q_T})}$. If $\|w\|_{C(\overline{Q_T})} > 0$ for all $T > 0$, taking $T > 0$ small enough so $F(T) < 1$ one obtains $\|w\|_{C(\overline{Q_T})} < \|w\|_{C(\overline{Q_T})}$; a contradiction. Because $h \in L_\infty(Q_T)$, (7.4) - (7.6) has a solution which can be represented via the Green function (see eg. [4]) in the following way

$$|G(\xi - \kappa, s - t)| \leq \frac{C_1}{t - s} \exp\left(-\frac{\|\xi - \kappa\|^2}{4(t - s)}\right),$$

$$w(\kappa, t) = \int_0^t \int_Q G(\xi - \kappa, s - t) h(\xi, s)\, d\xi ds. \tag{7.7}$$

First note that there is an m such that $(\varphi, \eta_0) \in E_m$. Also observe that u_j $(j = 1, 2)$ and its spatial derivatives are bounded in $C^\gamma(\overline{Q_T})$. Now choose T small enough so that for both u_1 and u_2 and every $t \in [0, T]$, (6.8) holds. By arguing as in the proof of Lemma 6.5, (6.7) holds for both u_1 and u_2. Therefore on $Q_1 \setminus \cup_i A_i$ (resp. $Q_2 \setminus \cup_i A_i$) u_1,u_2 are strictly removed from the threshold β (resp. α) and since $\|u_1\|_{C(\overline{Q_T})}$ and $\|u_2\|_{C(\overline{Q_T})}$ are bounded, Condition 2.3 implies the Lipschitz estimate

$$|h(\xi, s)| = |H_1(u_1(\xi, s)) - H_1(u_2(\xi, s))| \leq C|u_1(\xi, s) - u_2(\xi, s)|,$$

for every $\xi \in Q \setminus \cup_i A_i$ and $s \in (0, T)$ where C is independent of s. An analogous result holds for Q_2, H_2, therefore

$$\begin{aligned}
&\int_0^t \int_{Q \setminus \bigcup_i A_i} \frac{C_1}{t - s} \exp\left(-\frac{\|\xi - \kappa\|^2}{4(t - s)}\right) |h(\xi, s)|\, d\xi ds, \\
\leq &\int_0^t \int_{\mathbb{R}^2} \frac{C_2}{t - s} \exp\left(-\frac{\|\xi - \kappa\|^2}{4(t - s)}\right) d\xi ds\, \|w\|_{C(\overline{Q_T})}, \\
\leq &C_2 \pi t \|w\|_{C(\overline{Q_T})}, \\
\leq &C_2 \pi T \|w\|_{C(\overline{Q_T})}.
\end{aligned}$$

(See Proposition B.1 the estimate on the integral of the Green function). Define $F_1(T) = C_2 \pi T$, then $F_1(T) \to 0$ as $T \to 0$. We have shown that

$$\int_0^t \int_{Q \setminus \bigcup_i A_i} \frac{C}{t - s} \exp\left(-\frac{\|\xi - \kappa\|^2}{4(t - s)}\right) |h(\xi, s)| d\xi ds \leq F_1(T) \|w\|_{C(Q_T)}. \tag{7.8}$$

Recall that ψ_i is defined as a map $\mathbb{R}^2 \to \mathbb{R}^2$. Let $\kappa^i = \psi_i(\kappa) = (\kappa_x^i, \kappa_y^i)$, $h_i(x, y, s) = h(\psi_i^{-1}(x, y), s)$ and consider the following integral over the set A_i.

$$\int_0^t \int_{A_i} \frac{C_1}{t-s} \exp\left(-\frac{\|\xi - \kappa\|^2}{4(t-s)}\right) |h(\xi, s)| \, d\xi \, ds, \tag{7.9}$$

$$= \int_0^t \int_{-\frac{2}{m_i}}^{\frac{2}{m_i}} \int_{-\varepsilon_x^i}^{\varepsilon_x^i} \frac{C_1}{t-s} \exp\left(-\frac{\|(x, y) - (\kappa_x^i, \kappa_y^i)\|^2}{4(t-s)}\right) |h_i(x, y, s)| \, dx \, dy \, ds. \tag{7.10}$$

Note that rotations and translations are volume preserving so there is no extra factor in (7.10) from changing coordinates.

By observing that we have already chosen T small enough for the assumptions of Lemma 7.2 to be valid, we have that for every $(x, s) \in I(\varepsilon_x^i)_T$

$$\int_{-\frac{2}{m_i}}^{\frac{2}{m_i}} h^i(x, y, s) dy \leq C_3 \|w\|_{C(\overline{Q_T})}. \tag{7.11}$$

The constant C_3 is independent of $(x, s) \in I(\varepsilon_x^i)_T$ but depends U and m_i. However since i is indexed over a finite set we can choose the constant C_3 independent of i.

We now distinguish two cases. First, assume that $(\kappa_x^i, \kappa_y^i) \in B(\varepsilon_x^i, m_i)$. Let $x' = x - \kappa_x^i$ and $y' = y - \kappa_y^i$.

$$(7.10) \leq \int_0^t \int_{-\frac{2}{m_i}-\kappa_y^i}^{\frac{2}{m_i}-\kappa_y^i} \int_{-\varepsilon_x^i-\kappa_x^i}^{\varepsilon_x^i-\kappa_x^i} \frac{C_1}{t-s} \exp\left(-\frac{x'^2 + y'^2}{4(t-s)}\right) \cdot$$
$$|h^i(x' + \kappa_x^i, y' + \kappa_y^i, s)| \, dx' \, dy' \, ds.$$

Note that we are using $\cdot$ in a multi-line equation to indicate multiplication. Now observe that

$$\frac{C_1}{t-s} \exp\left(-\frac{x'^2 + y'^2}{4(t-s)}\right) \leq \frac{C_1}{t-s} \exp\left(-\frac{x'^2}{4(t-s)}\right),$$

therefore

$$(7.10) \leq \int_0^t \int_{-\varepsilon_x^i-\kappa_x^i}^{\varepsilon_x^i-\kappa_x^i} \frac{C_1}{t-s} \exp\left(-\frac{x'^2}{4(t-s)}\right) dx' \, ds \cdot$$
$$\int_{-\frac{2}{m_i}-\kappa_y^i}^{\frac{2}{m_i}-\kappa_y^i} |h^i(x' + \kappa_x^i, y' + \kappa_y^i, s)| \, dy'.$$

Applying (7.11) and Proposition B.1 we have

$$
\begin{aligned}
(7.10) &\leq \int_0^t \int_{-\varepsilon_x^i-\kappa_x^i}^{\varepsilon_x^i-\kappa_x^i} \frac{C_1}{t-s} \exp\left(-\frac{x'^2}{4(t-s)}\right) dx'\ ds\, \|w\|_{C(\overline{Q_T})}, \\
&\leq \int_0^t \frac{C_1}{(t-s)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{1}{(t-s)^{\frac{1}{2}}} \exp\left(-\frac{x'^2}{4(t-s)}\right) dx'\, ds\, \|w\|_{C(\overline{Q_T})}, \\
&\leq \int_0^T \frac{C_4}{(t-s)^{\frac{1}{2}}}\, dt\, \|w\|_{C(\overline{Q_T})} \leq C_4 T^{\frac{1}{2}} \|w\|_{C(\overline{Q_T})}, \qquad (7.12)
\end{aligned}
$$

(refer to B.1 in the appendix for the derivation of the inner integral). If $F_2(T) := C_4 T^{\frac{1}{2}}$ then we have shown that

$$(7.9) \leq F_2(T)\|w\|_{C(\overline{Q_T})} \qquad (7.13)$$

Next consider the case where $\kappa^i \notin B(\varepsilon_x^i, m_i)$. Let $K^i = (K_x^i, K_y^i)$ be the point in $B(\varepsilon_x^i, m_i)$ closest to κ^i. Since

$$\|(x,y) - (K_x^i, K_y^i)\|^2 \leq \|(x,y) - (\kappa_x^i, \kappa_y^i)\|^2$$

one has

$$\exp\left(-\frac{\|(x,y) - (\kappa_x^i, \kappa_y^i)\|^2}{4(t-s)}\right) \leq \exp\left(-\frac{\|(x,y) - (K_x, K_y)\|^2}{4(t-s)}\right)$$

It is now possible to proceed as in the previous case with K in place of κ to obtain

$$(7.9) \leq F_2(T)\|w\|_{C(\overline{Q_T})}.$$

Combining (7.7), (7.8) and (7.13)

$$
\begin{aligned}
|w(\kappa,t)| &\leq \int_0^t \int_Q \frac{C}{t-s} \exp\left(-\frac{\|\xi-\kappa\|^2}{4(t-s)}\right) |h(\xi,s)|\, d\xi ds, \qquad (7.14) \\
&\leq \int_0^t \int_{Q\setminus\bigcup_i A_i} \frac{C}{t-s} \exp\left(-\frac{\|\xi-\kappa\|^2}{4(t-s)}\right) |h(\xi,s)|\, d\xi ds, \\
&+ \sum_{i=1}^d \int_0^t \int_{A_i} \frac{C}{t-s} \exp\left(-\frac{\|\xi-\kappa\|^2}{4(t-s)}\right) |h(\xi,s)|\, d\xi ds, \qquad (7.15) \\
&\leq (F_1(T) + d(F_2(T))\|w\|_{C(\overline{Q_T})}. \qquad (7.16)
\end{aligned}
$$

Letting $F(T) = (F_1(T) + dF_2(T))$ gives the result.

□

Remark 7.5. Let us make the observation that the key step in (7.12) can be done in higher dimensions. In effect we are factor out a $(t-s)^{\frac{1}{2}}$ term and then integrate over an area that has one less spatial dimension than the original problem. Indeed, this calculation is the same as integrating a Green function for the heat equation in $\mathbb{R}^{n-1}$, hence we have written the inequality in Appendix B in dimension n.

proof of Theorem 3.19. Consider the set

$$\chi = \{t \in [0, T_0] \mid u_1(\xi, t) = u_2(\xi, t)\, a.e.(\xi, t) \in Q_t\}$$

We know that $0 \in \chi$ so $t_0 = \sup \chi$ is well defined. Assume that $t_0 < T_0$. Because u_1 and u_2 are continuous in Q_{T_0} by Lemma 3.2, we have $u_1(\xi, t_0) = u_2(\xi, t_0)$.

By assumption, $u_j(\xi, t_0)$ $(j = 1, 2)$ is transverse with respect to $\eta(\eta_0, u_j)$. Therefore by Lemma 7.4 there exists a $\tau > 0$ such that any two solutions of (3.4) - (3.6) with initial data $u_j(\cdot, t_0)$, initial configuration $\eta(\eta_0, u_j)$ coincide on the set Q_τ. However by the semigroup property of the hysteresis operator

$$\overline{H}(\eta_0(\xi), u_j(\xi, \cdot))(t_0 + t) = \overline{H}(\eta(\xi, t), u_j(\xi, t_0 + \cdot))(t),$$

we see that $u_1(\xi, t_0 + t)$, $u_2(\xi, t_0 + t)$ with $t \in [0, \tau]$ are solutions of (3.7) - (3.9), therefore they coincide on $[t_0, t_0 + \tau]$, a contradiction. □

Appendix A Uniformly Bounded Solutions

Definition A.1. *Let $u \in C(\overline{Q_T})$ and suppose that $u(\xi, 0) < U_1$ for all $\xi \in \overline{Q}$. We say that $t \in (0, T]$ is a* U_1-attainability moment *of u if there exists a $\xi \in \overline{Q}$ such that $u(\xi, t) = U_1$. We call the set of all U_1-attainability moments the* U_1-attainability set*, denoted by τ.*

Remark A.2. If (ξ_n, t_n) is a sequence of points such that $t_n \to t'$ and $u(\xi_n, t_n) = U_1$, then by taking a convergent subsequence in $\overline{Q_T}$ and noting that u is continuous, it becomes clear that τ is a closed set.

Definition A.3. *We call the minimal element of τ the* first U_1-attainability moment*.*

Lemma A.4. *For any $t \in \tau$ let $X(u, t) = \{\xi \in \overline{Q} \mid u(\xi, t) = U_1\}$. Let $u_n \to u$ in $C(\overline{Q_T})$ and $u(\xi, 0), u_n(\xi, 0) < U_1$, for all $\xi \in \overline{Q}$. Let τ, τ_n denote the U_1-attainability sets of u, u_n respectively, let $\tau, \tau_n \neq \emptyset$ and let t_n be the first U_1-attainability moment of u_n. Suppose that $t_n \to t'$.*

Then $t' \in \tau$ and for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, the set $X(u_n, t_n)$ lies in the ε neighbourhood of $X(u, t')$.

Proof. We will first show that $t' \in \tau$. Take $\xi_n \in X(u_n, t_n)$ and form a sequence $(\xi_n, t_n) \in \overline{Q_T}$. Choose a convergent subsequence, not relabelled, such that $(\xi_n, t_n) \to (\xi', t')$ in $\overline{Q_T}$. Then

$$\begin{aligned} |u(\xi', t') - U_1| \leq\ & |u(\xi', t') - u(\xi_n, t_n)| \\ & + |u(\xi_n, t_n) - u_n(\xi_n, t_n)| + |u_n(\xi_n, t_n) - U_1| \end{aligned} \tag{A.1}$$

Consider the right side of inequality (A.2). Reading from left to right, the first term goes to zero because u is continuous, the second because $u_n \to u$ in $C(\overline{Q_T})$ and the last term is equal to zero because $(\xi_n, t_n) \in X(u_n, t_n)$. Thus $u(\xi', t') = U_1$.

2. It remains to show that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\xi_n \in X(u_n, t_n)$ implies that there is a $\xi' \in X(u, t')$ such that $\|\xi' - \xi_n\| < \varepsilon$. Consider a sequence of points (ξ_n, t_n) with $\xi_n \in X(u_n, t_n)$. Take a convergent subsequence $(\xi_n, t_n) \to (\xi', t')$, then reasoning as above we conclude that $u(\xi', t') = U_1$. If there was an $\varepsilon > 0$ and a sequence $\xi_n \in X(u_n, t_n)$ such that $\inf\{\|\xi_n - \xi\| \mid \xi \in X(u, t')\} > \varepsilon$, then we have just shown that there is necessarily a subsequence converging to something in $X(u, t')$; a contradiction. □

Corollary 5.8. Assume that $u \in W_p^{2,1}(Q_T)$ satisfies (5.6) - (5.8) with $\varphi \in \mathcal{W}$ and $F(\xi, t) = f_0(u, \xi, t) \in L_p(Q_T)$. Further assume that there is some $U > 0$ such that

(1) $f_0(\cdot, \xi, t) : \mathbb{R} \to \mathbb{R}$ is continuous at the points $\pm U$ uniformly with respect to $(\xi, t) \in Q_T$ i.e. for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|U - U'| < \delta$ implies that $|f_0(U, \xi, t) - f_0(U', \xi, t)| < \varepsilon$, where δ can be chosen independently of (ξ, t),
(2) $f_0(U, \xi, t) < 0$, $f_0(-U, \xi, t) > 0$ for all $(\xi, t) \in Q_T$,
(3) $\|\varphi\|_{C(\overline{Q})} < U$.

Then $\|u\|_{C(\overline{Q_T})} < U$.

Proof. Choose $U_1 < U$ sufficiently close to U such that

$$\|\varphi\|_{C(\overline{Q_T})} < U_1,$$

and using the continuity of f_0 in the first argument further assume that

$$f(U_1, \xi, t) < 0 \text{ and } f(-U_1, \xi, t) > 0. \tag{A.2}$$

We will show that $\max_{(\xi,t)\in\overline{Q_T}} u(\xi, t) \leq U_1$ (the statement $\min_{(\xi,t)\in\overline{Q_T}} u(\xi, t) \geq -U_1$ is proved similarly).

Let $\varphi_n \in C^\infty(\overline{Q})$ and $F_n \in C^\infty(\overline{Q_T})$ be sequences of functions such that $\varphi_n \to \varphi$ in $\mathcal{W}$ and $F_n \to F$ in $L_p(Q_T)$. The functions F_n can be constructed by mollifiers (c.f. appendix C, Theorem 6 of [3]).

By Theorem 5.3 in chapter 4 of [11], for each n the problem

$$\begin{aligned} u_t &= \Delta u + F_n(\xi, t), \\ u|_{t=0} &= \varphi_n, \\ \frac{\partial u}{\partial \nu}\Big|_{\partial Q_T} &= 0, \end{aligned} \tag{A.3}$$

has a unique classical solution which we will denote by u_n. By Theorem 5.1 $u_n \to u$ in $W_p^{2,1}(Q_T)$ and thus by Lemma 3.2 also in $C(\overline{Q_T})$. Therefore it suffices to show $\max_{(\xi,t)\in\overline{Q_T}} u_n(\xi, t) < U_1$.

To this end, we begin by taking n sufficiently large and then relabel the index in such a way that $\varphi_n(\xi) < U_1$ for all $\xi \in \overline{Q}$ and $n \in \mathbb{N}$. Let τ_n, τ' be the U_1-attainability sets of u_n, u respectively.

If $\tau' = \emptyset$ or if there is a subsequence $\tau_n = \emptyset$ (i.e. $u_n(\xi, t) < U_1$ $\forall(\xi, t) \in \overline{Q_T}$), then the lemma is proved. Therefore assume that $\tau', \tau_n \neq \emptyset$. Let t_n, be the first U_1-attainability moment of u_n. Choose a converging subsequence, not relabelled, with limit $t_n \to t'$ and note by Lemma A.4 that $t' \in \tau'$.

Because of (A.2), there is an $L > 0$ such that

$$f_0(u(\xi', t'), \xi, t) = f_0(U_1, \xi, t) < -L \ \ \forall(\xi', t') \in X(u, t') \text{ and } (\xi, t) \in \overline{Q_T}.$$

Let Y_ε be the intersection of the ε-neighbourhood of the set $X(u, t') \times \{t'\}$ with $\overline{Q_T}$. Since u is continuous and f_0 is continuous in the first argument at the point U (uniformly in (ξ, t)), we can choose ε small enough so that for all $(\xi, t) \in Y_\varepsilon$, $u(\xi, t)$ is sufficiently close to U_1 to imply that

$$F(\xi, t) = f_0(u(\xi, t), \xi, t) \leq -\frac{L}{2}.$$

If the F_n were constructed via mollifiers, then for sufficiently large n

$$F_n(\xi, t) \leq -\frac{L}{2} \ \ \forall (\xi, t) \in Y_{\frac{\varepsilon}{2}}. \tag{A.4}$$

By Lemma A.4, $X(u_n, t_n) \times \{t_n\}$ can be chosen within distance $\frac{\varepsilon}{2}$ of $X(u, t') \times \{t'\}$ for sufficiently large n i.e. $X(u_n, t_n) \times \{t_n\} \subset Y_{\frac{\varepsilon}{2}}$. Hence

$$F_n(\xi, t_n) \leq -\frac{L}{2} \ \forall \xi \in X(u_n, t_n). \tag{A.5}$$

However since $\varphi_n(\xi) < U_1$ and t_n was was the first U_1-attainability moment we must have for $j = 1, 2$

$$\frac{\partial^2 u_n}{\partial \xi_j^2} < 0 \ \ \forall \xi \in X(u_n, t_n), \tag{A.6}$$

and hence by (A.4) and (A.6)

$$\frac{\partial u_n}{\partial t} < 0 \ \ \forall \xi \in X(u_n, t_n).$$

Therefore t_n is not the first U_1 attainability moment of u_n. This contradiction proves that $\tau_n = \emptyset$, which in turn proves the Lemma. $\square$

Appendix B Green Function Inequalities

Let G be the Green function for the heat equations for a bounded domain $Q \subset \mathbb{R}^n$ with smooth boundary. By [4], G satisfies the estimate

$$|G(\xi - \kappa, t - s)| \leq \frac{C_1}{(t - s)^{\frac{n}{2}}} \exp\left(-\frac{\|\xi - \kappa\|^2}{4(t - s)}\right).$$

Proposition B.1. *There is a constant $C(n)$ such that*

$$\int_{\mathbb{R}^n} |G(\xi - \kappa, t - s)| d\xi ds \leq C(n).$$

Proof. Consider

$$\int_{\mathbb{R}^n} \frac{C_1}{(t-s)^{\frac{n}{2}}} \exp\left(-\frac{\|\xi-\kappa\|^2}{4(t-s)}\right) d\kappa ds.$$

Let $\tau = t-s$ and $x' = \xi - \kappa$. One obtains

$$\int_{\mathbb{R}_n} \frac{C_1}{\tau^{\frac{n}{2}}} \exp\left(-\frac{\|x'\|^2}{4\tau}\right) dx'd\tau.$$

Convert to polar coordinates in $\mathbb{R}^n$ with radial direction denoted r.

$$\underbrace{\int_0^{2\pi} \cdots \int_0^{2\pi}}_{n-1} \int_0^{\infty} \frac{C_1}{\tau^{\frac{n}{2}}} \exp\left(-\frac{r^2}{4\tau}\right) r^{n-1} dr d\theta^{n-1}.$$

Letting $z' = \frac{r^2}{4\tau}$. Then $dr = \frac{\tau}{2r} dz'$

$$\begin{aligned}
&= \frac{C_1(2\pi)^{n-1}}{2} \int_0^{\infty} \frac{r^{n-2}}{\tau^{\frac{n-2}{2}}} \exp(-z')\, dz', \\
&= \frac{C_1(2\pi)^{n-1}}{2} \int_0^{\infty} z'^{\frac{n-2}{2}} \exp(-z')\, dz', \\
&= \frac{C_1(2\pi)^{n-1}}{2} \Gamma\left(\frac{n}{2}\right) := C(n).
\end{aligned}$$

□

Appendix C Mild solutions of semilinear parabolic problems

Consider the following operator A_0,

$$D(A_0) = \{u \in C^2(\overline{Q}) \mid \left.\frac{\partial u}{\partial \nu}\right|_{\partial Q} = 0\},$$

$$A_0(u) = \Delta u.$$

By Lemma 1 pg.15 of [13], for $1 < p < \infty$, A_0 has a closure in $L_p(Q)$ (i.e. the closure of the graph of A_0 in $L_p(Q)$ corresponds to the graph of an operator on $L_p(Q)$) and generates an analytic semigroup $S_p(t)$ on $L_p(Q)$.

Moreover for $1 < p \leq q < \infty$

$$D(A_q) \subset D(A_p),\ A_p u = A_q u \ \ \forall u \in D(A_q),$$

$$S_p(t)u = S_q(t)u, \ \ t \in [0, t) \ u \in L_q(Q) \subset L_p(Q). \tag{C.1}$$

By Lemma 2 pg.19 of [13]

$$\sup\{\|S_p(t)u\|_{L_\infty(Q)} \mid t \in [0, T]\} \leq C(T), \|u\|_{L^\infty(Q)}$$

(For the operator Δ one can take $C(T) = 1$, however this may not be the case for an elliptic operator with lower order terms).

Therefore, for $u \in L_\infty(Q)$ one can define $P(t)u := S_p(t)u$ which is well defined by (C.1). Since $S_p(t_1 + t_2) = S_p(t_1)S_p(t_2)$, $P(t_1 + t_2) = P(t_1)P(t_2)$ i.e. P is a semigroup on $L_\infty(Q)$.

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