# Incremental Construction of Constrained Delaunay Triangulations

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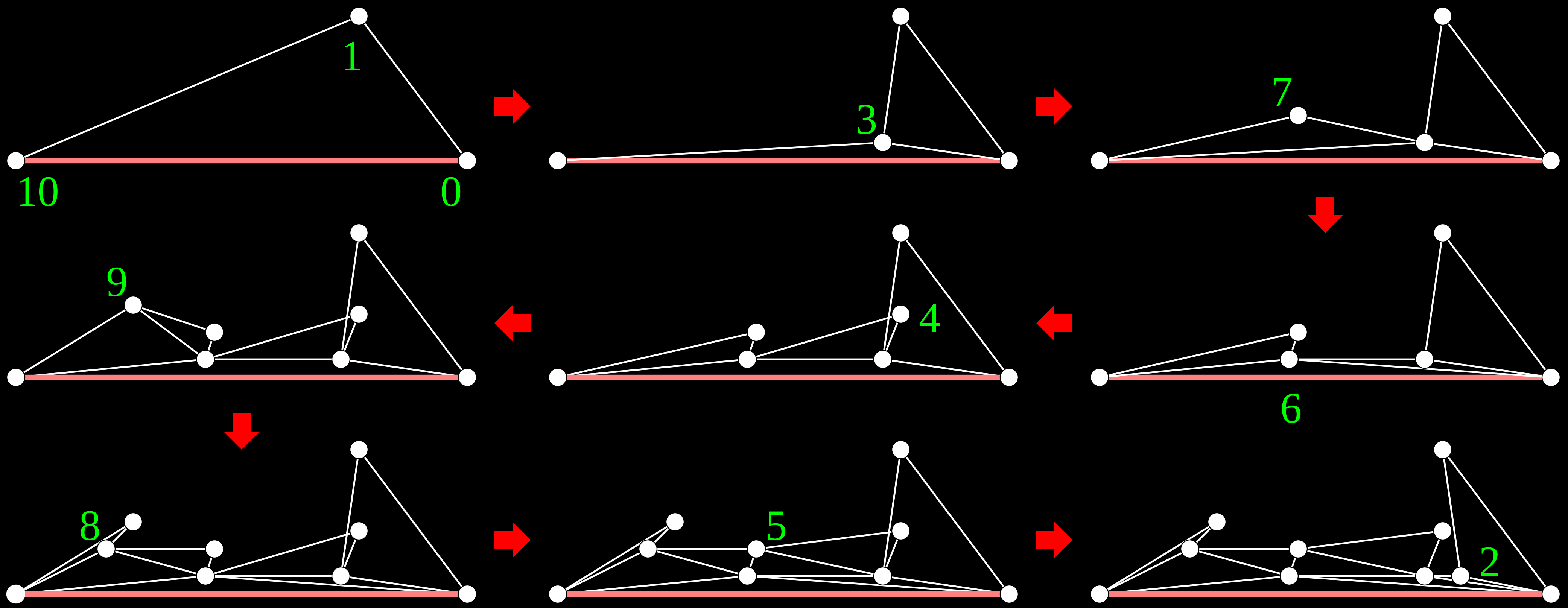
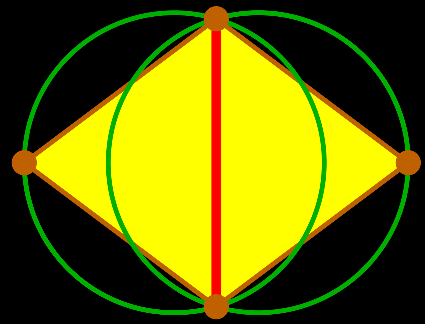
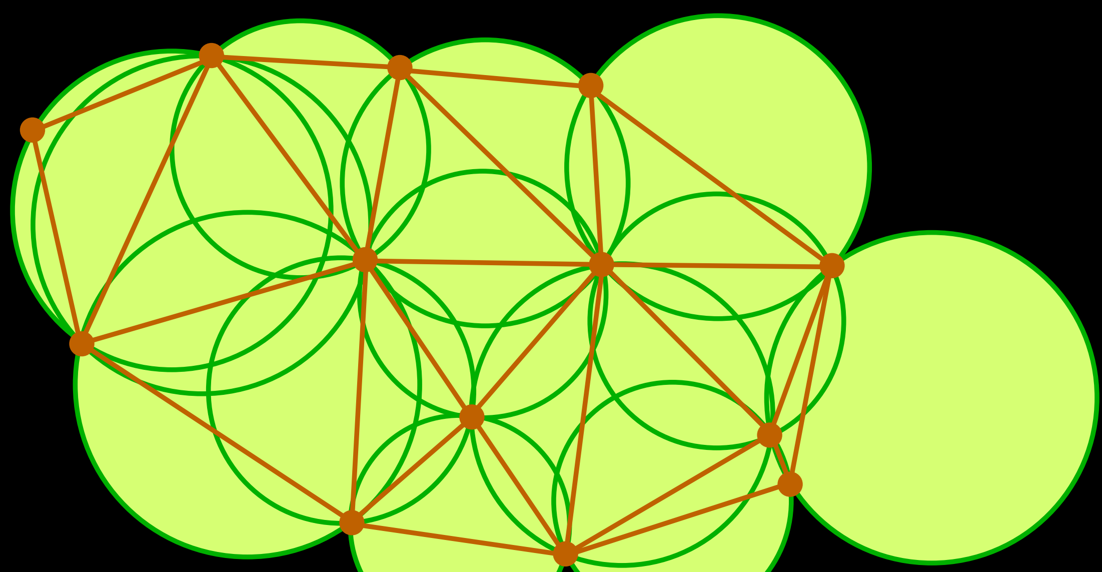


# The Delaunay Triangulation

An edge is *locally Delaunay* if the two triangles sharing it have no vertex in each others' circumcircles.



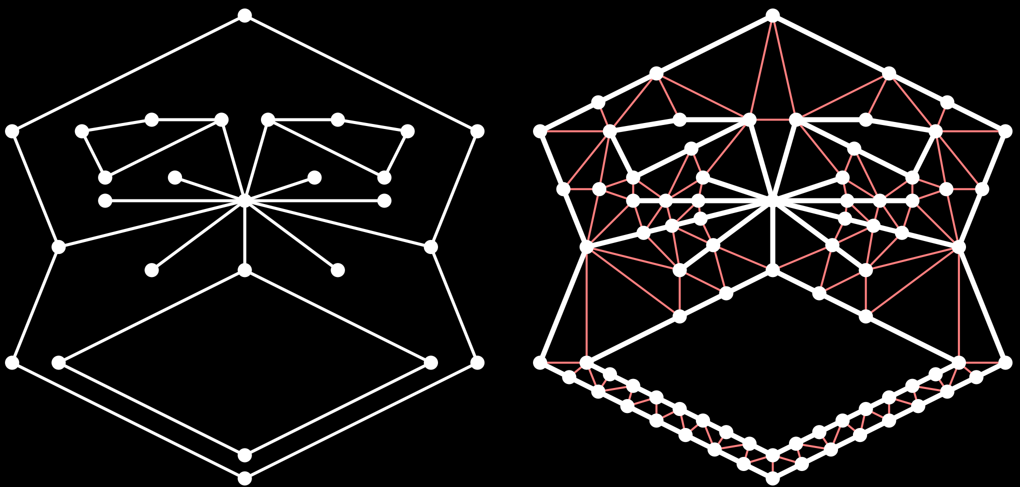
A *Delaunay triangulation* is a triangulation of a point set in which every edge is locally Delaunay.



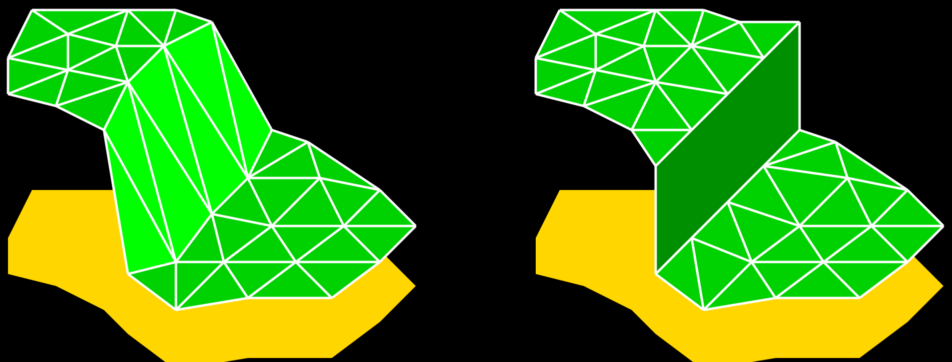
# Constraining Edges

Sometimes we need to force a triangulation to contain specified edges.

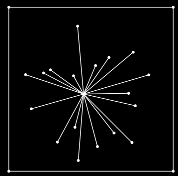
- Nonconvex shapes; internal boundaries

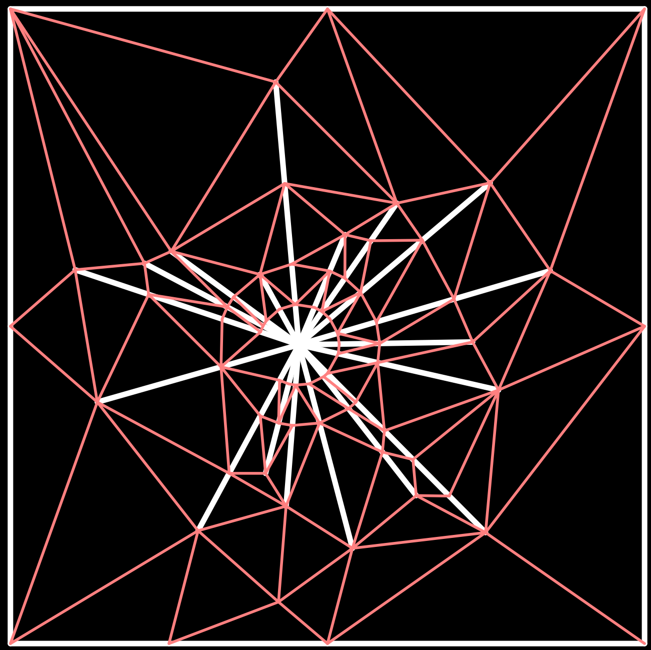


- Discontinuities in interpolated functions



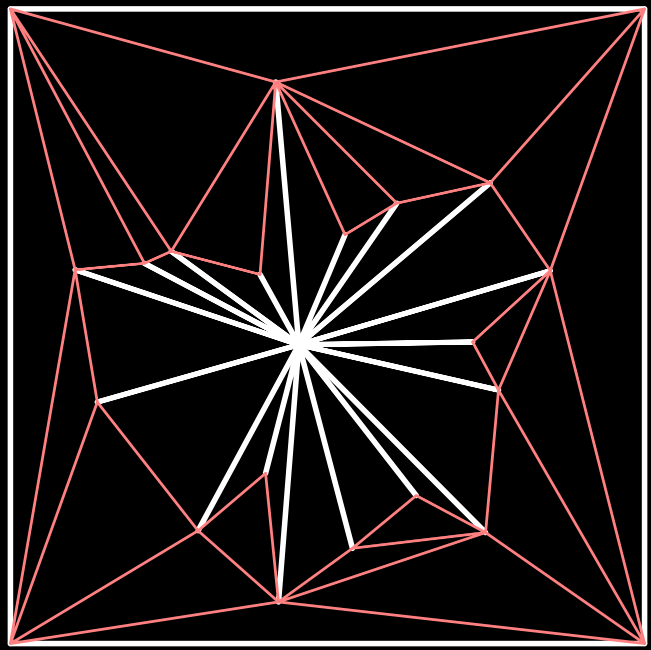
# 2 Ways to Recover Segments





## Conforming Delaunay triangulations

- Edges are all locally Delaunay.
- Worst−case input needs $\Omega(n^2)$ to $O(n^{2.5})$ extra vertices.

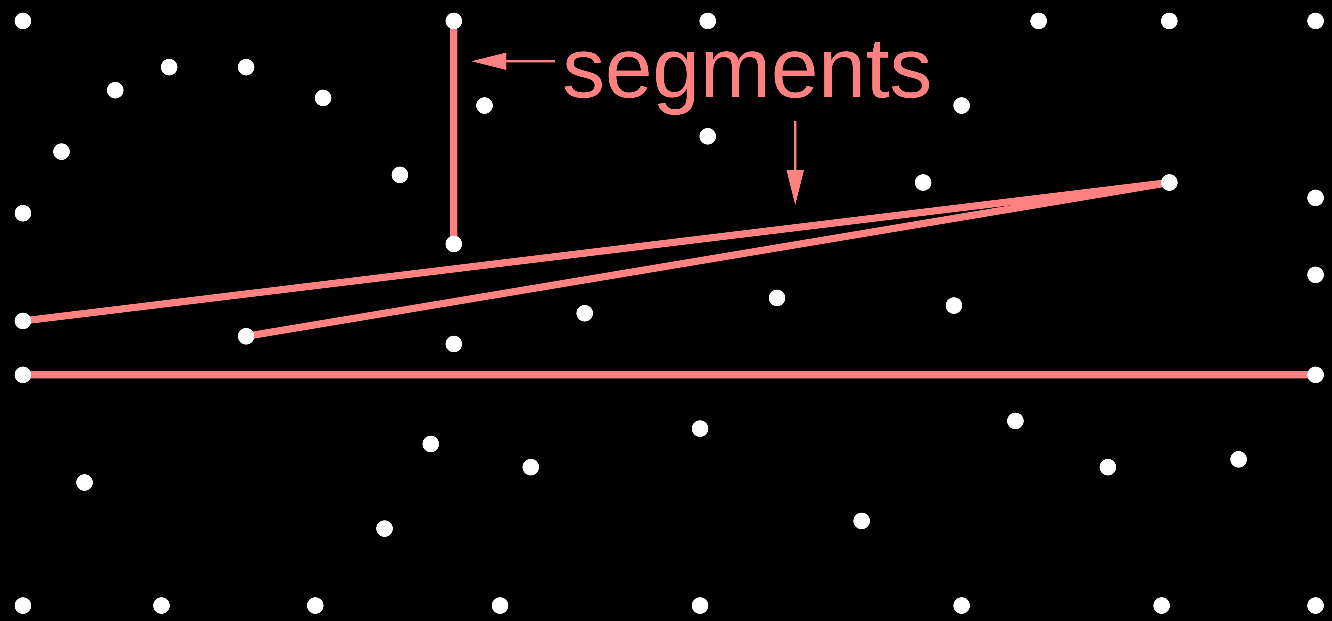


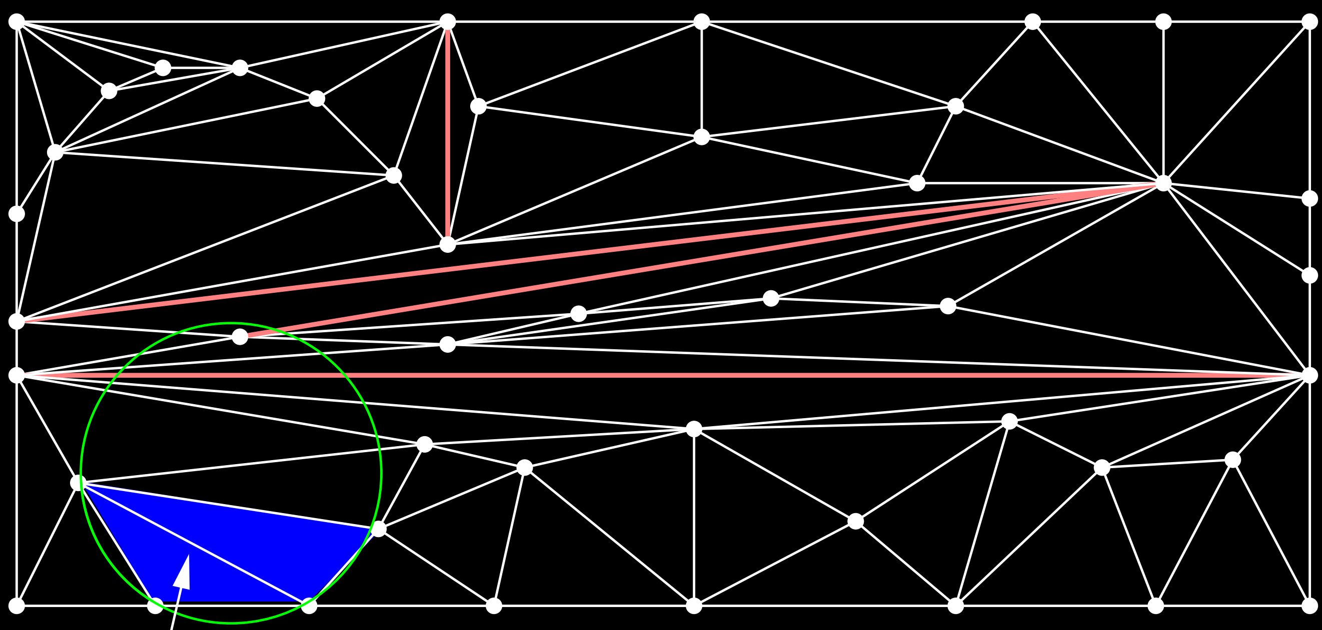
## Constrained Delaunay triangulations (CDTs)

- Edges are locally Delaunay or are domain boundaries.

# Goal

Input:
planar straight
line graph
(PSLG)

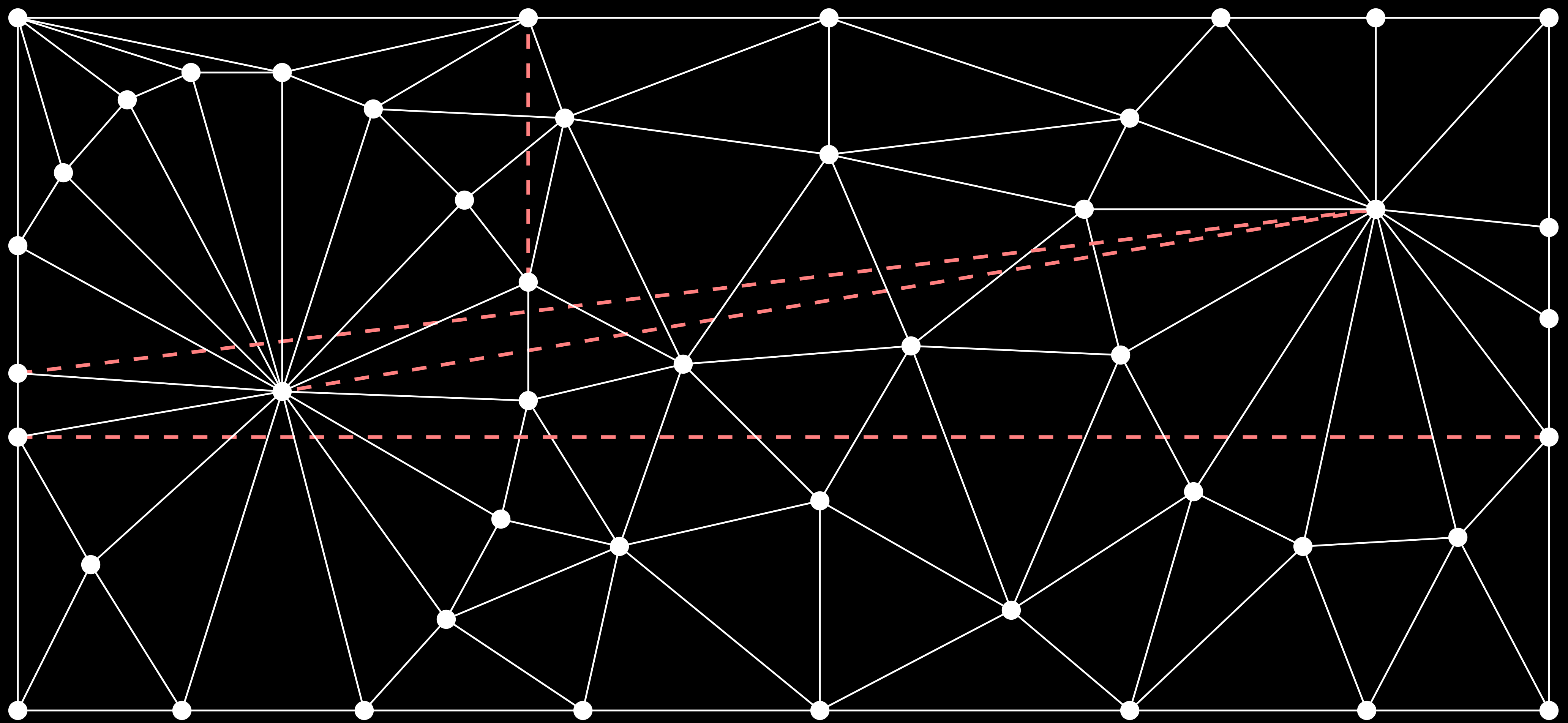


Output:
constrained
Delaunay
triangulation
(CDT)

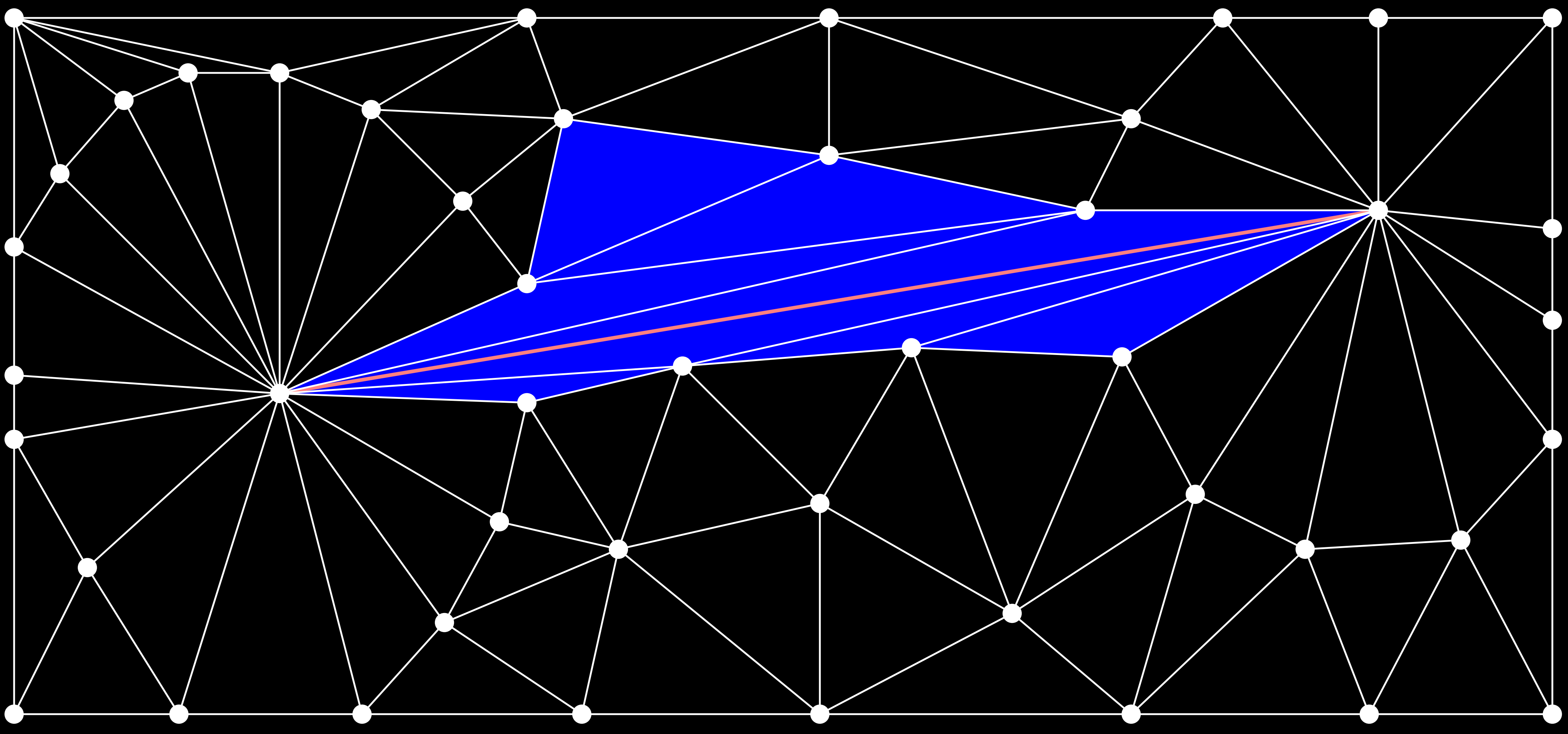
Every edge is locally Delaunay except segments.

# Randomized Incremental CDT Construction



Start with Delaunay triangulation of vertices.

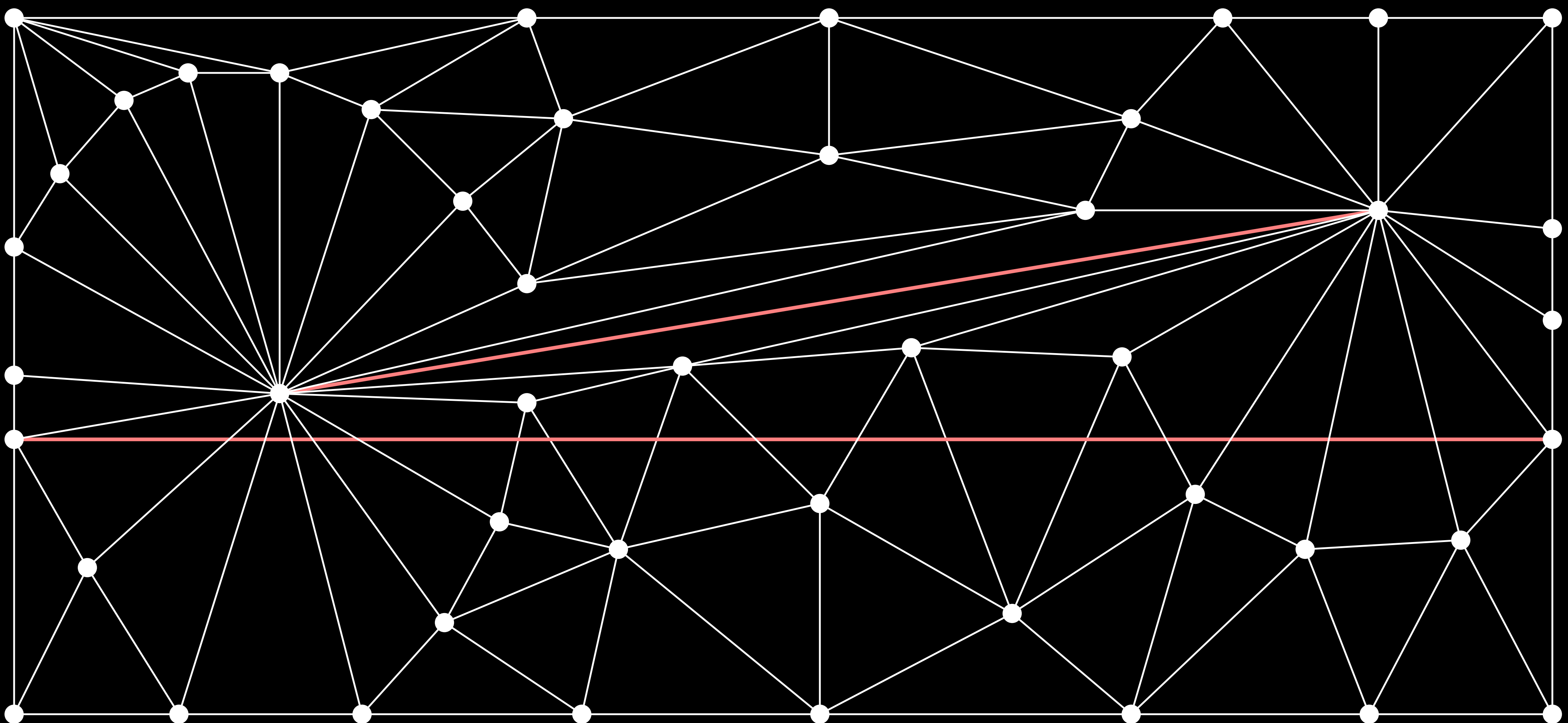
# Randomized Incremental CDT Construction



Start with Delaunay triangulation of vertices.

Do segment location. Insert segment.

# Randomized Incremental CDT Construction



Start with Delaunay triangulation of vertices.

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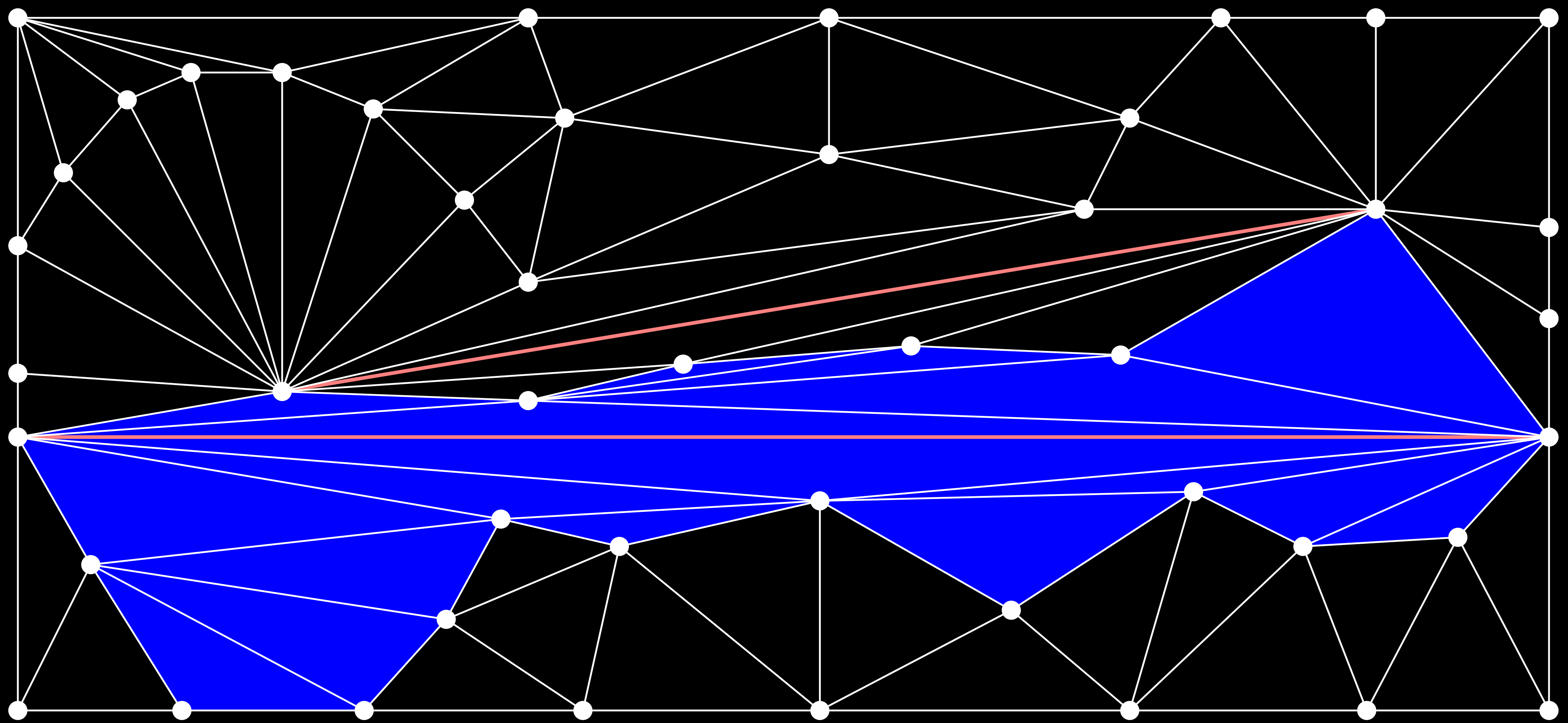

Start with Delaunay triangulation of vertices.

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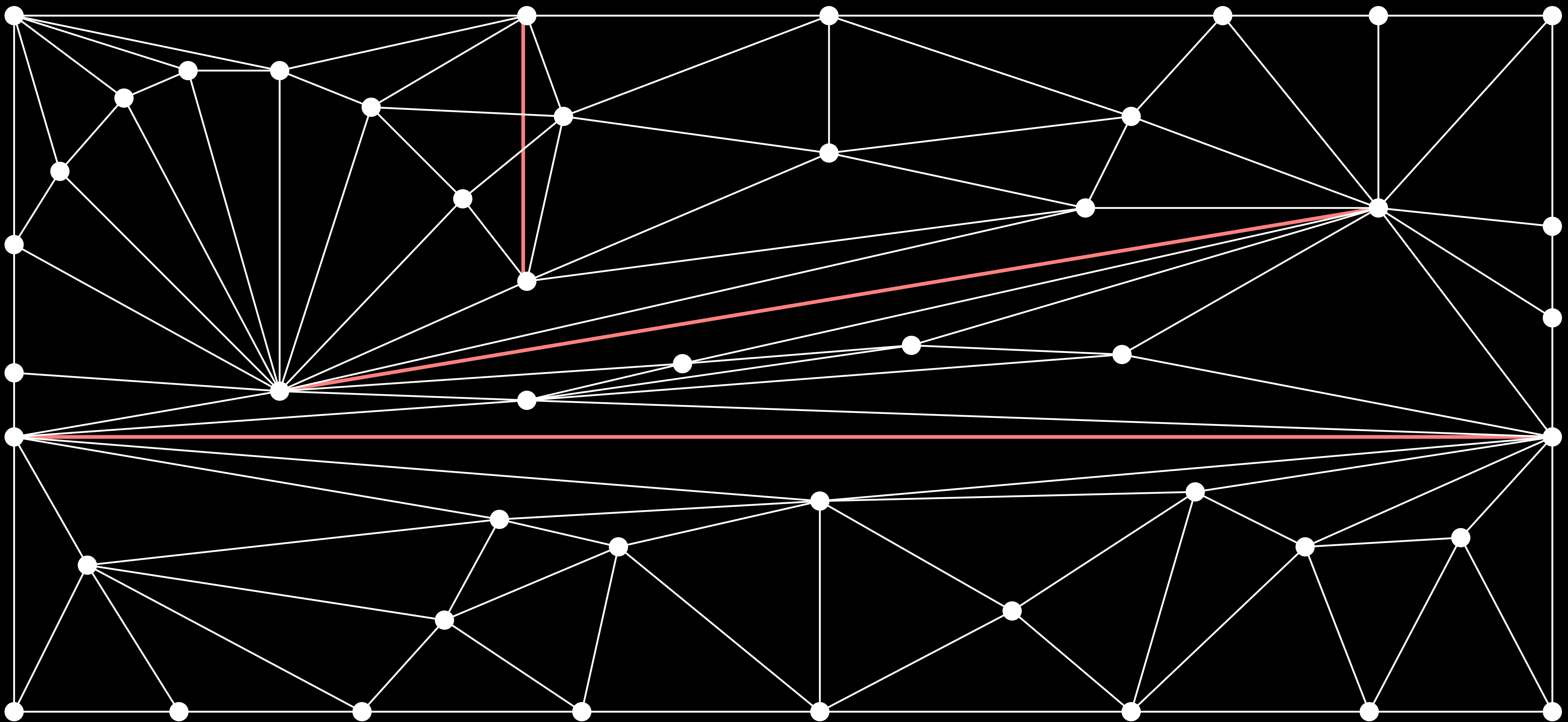
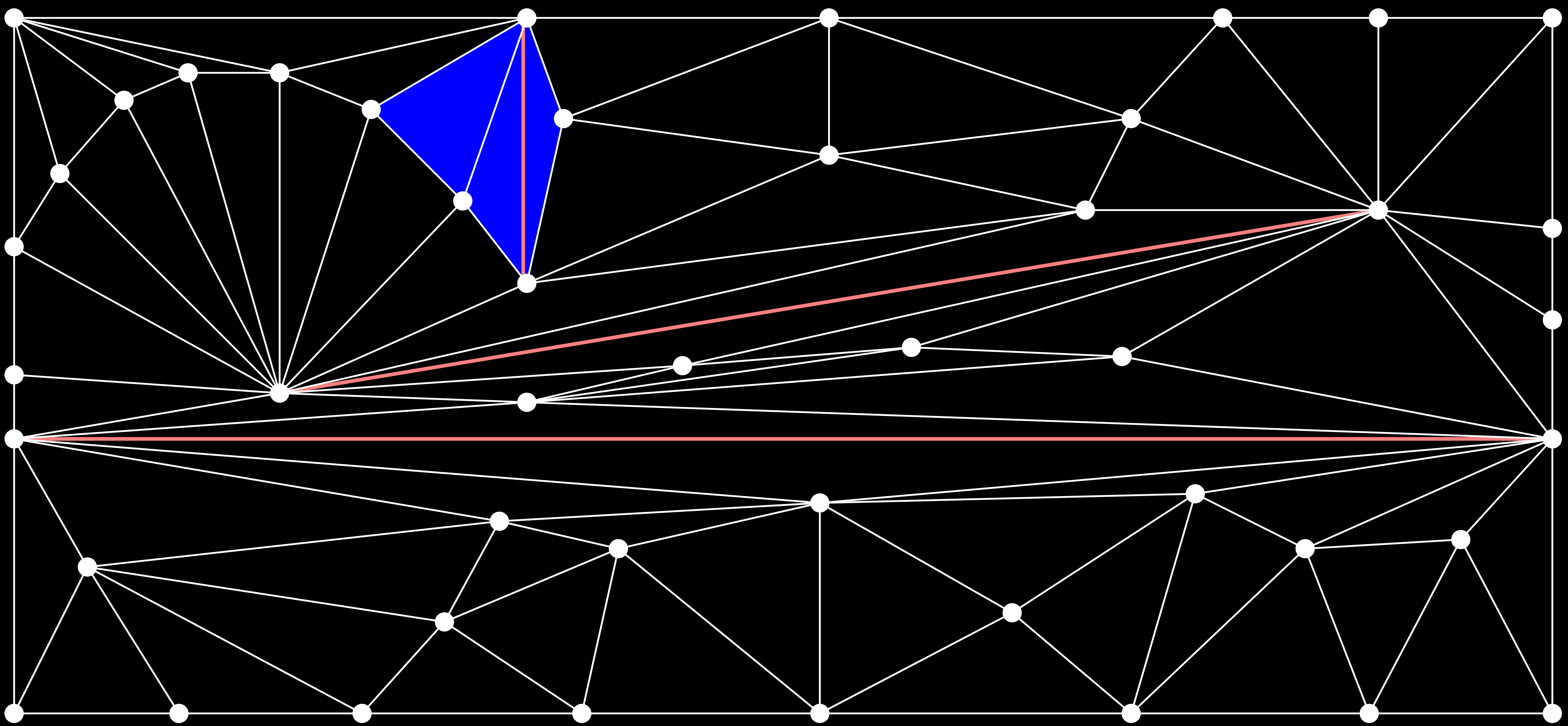


Start with Delaunay triangulation of vertices.

Do segment location. Insert segment.

# Randomized Incremental CDT Construction



Start with Delaunay triangulation of vertices.

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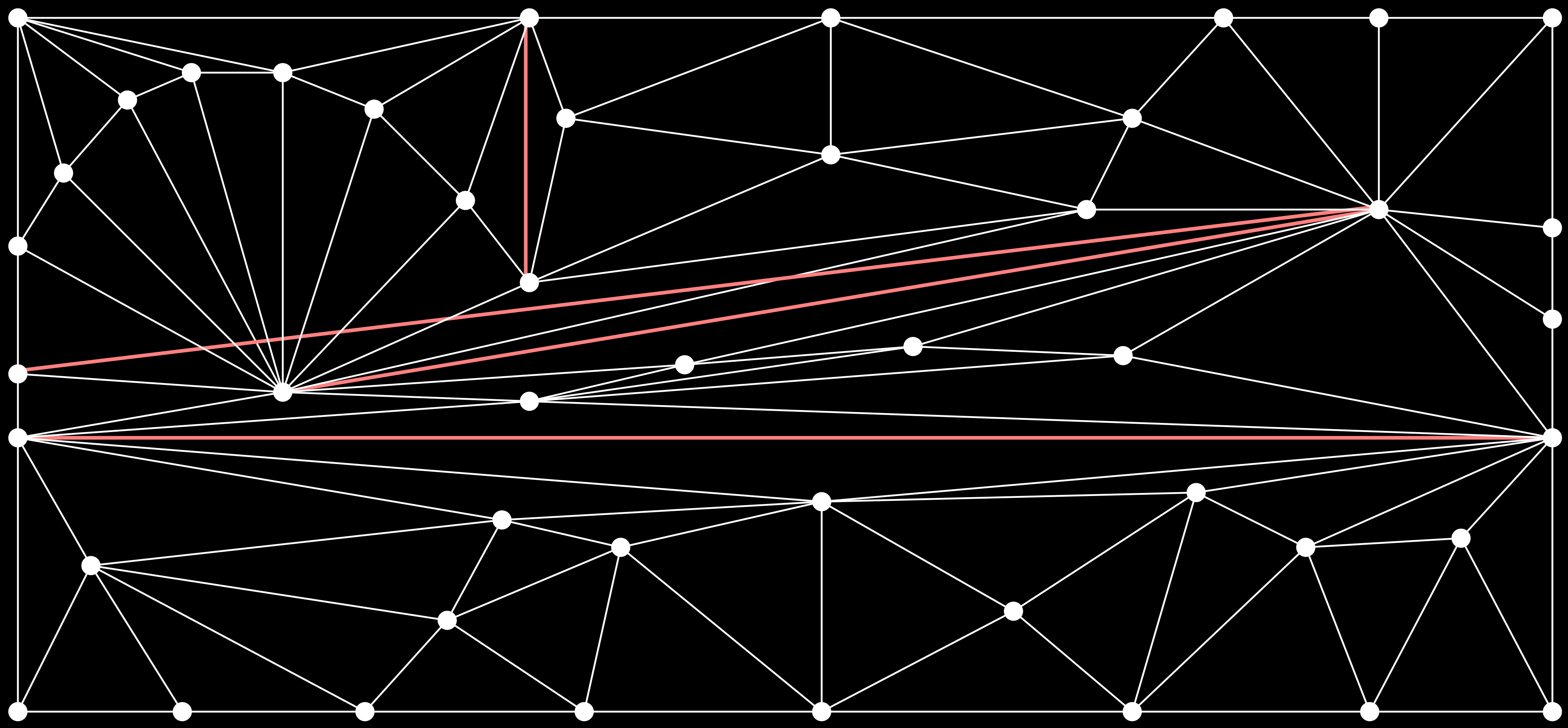

Start with Delaunay triangulation of vertices.

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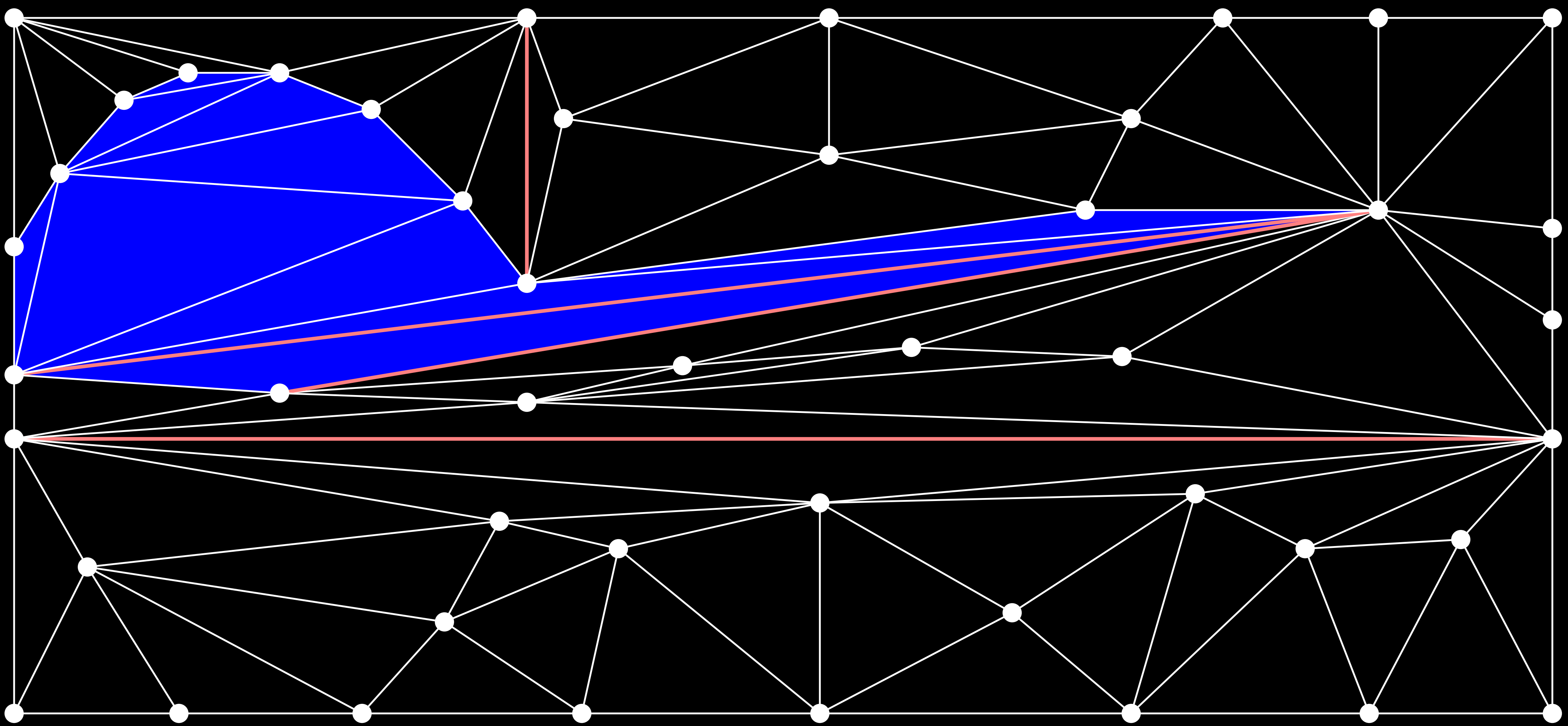
# Randomized Incremental CDT Construction



Start with Delaunay triangulation of vertices.

Do segment location. Insert segment.

# Topics

Inserting a segment in expected linear time.

Randomized incremental CDT construction in expected $\Theta(n \log^2 k)$ time.

# of vertices     # of segments

- Upper bound: Agarwal/Arge/Yi.
- Lower bound: new.

Inserting a polygon into a 3D CDT.

# CDT Construction Algorithms

Divide–and–conquer [Chew 1987, 1989].

$O(n \log n)$ time.

Plane sweep [Seidel 1988].

$O(n \log n)$ time.

Randomized incremental segment insertion.

Expected $O(n \log n + n \log^2 k)$ time.

# of vertices

# of segments

# Why Incremental?

It’s what everyone implements in practice.

(Easiest to implement.)

Leverages the best DT implementations.

The $n \log^2 k$ term is pessimistic.

# Topics

## Inserting a segment in expected linear time.

Randomized incremental CDT construction in expected $\Theta(n \log^2 k)$ time.

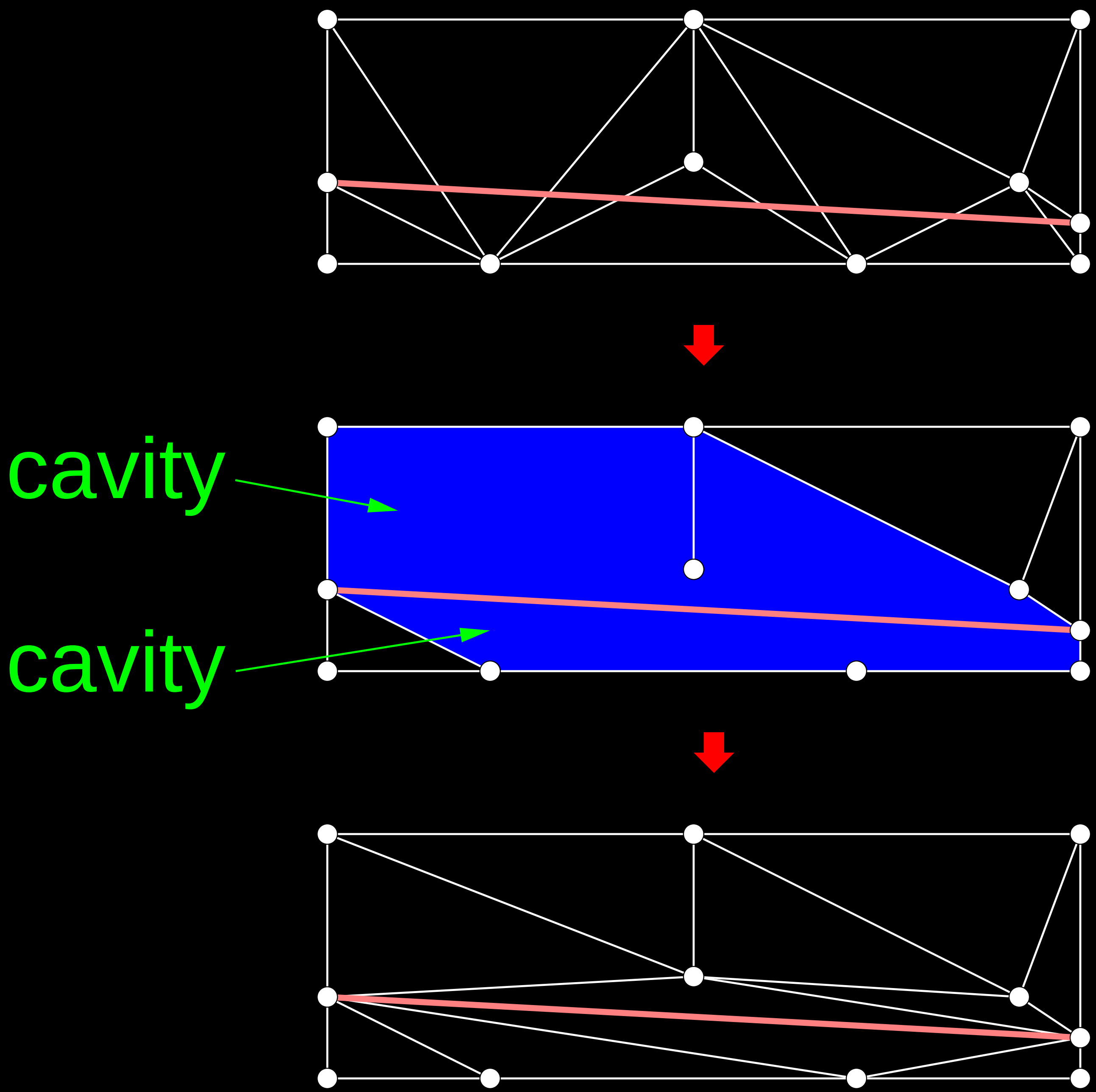
# of vertices — $n$

# of segments — $k$

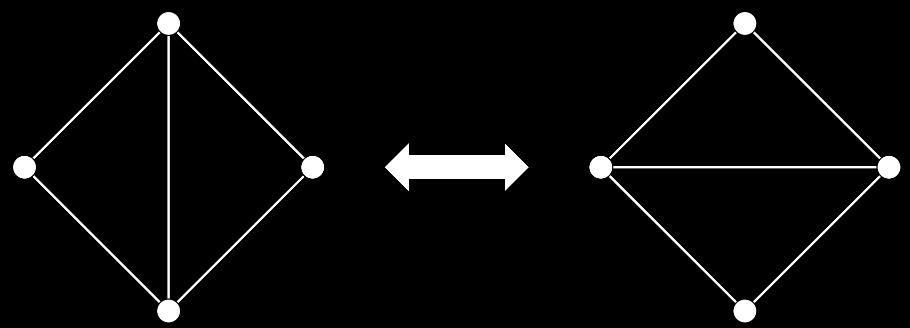
- Upper bound: Agarwal/Arge/Yi.
- Lower bound: new.

Inserting a polygon into a 3D CDT.

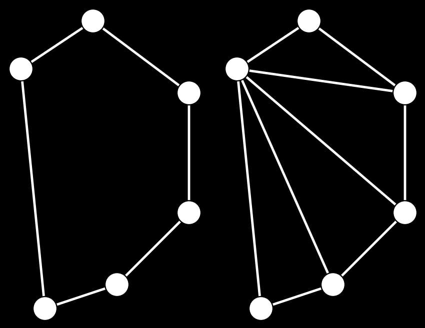
# Segment Insertion



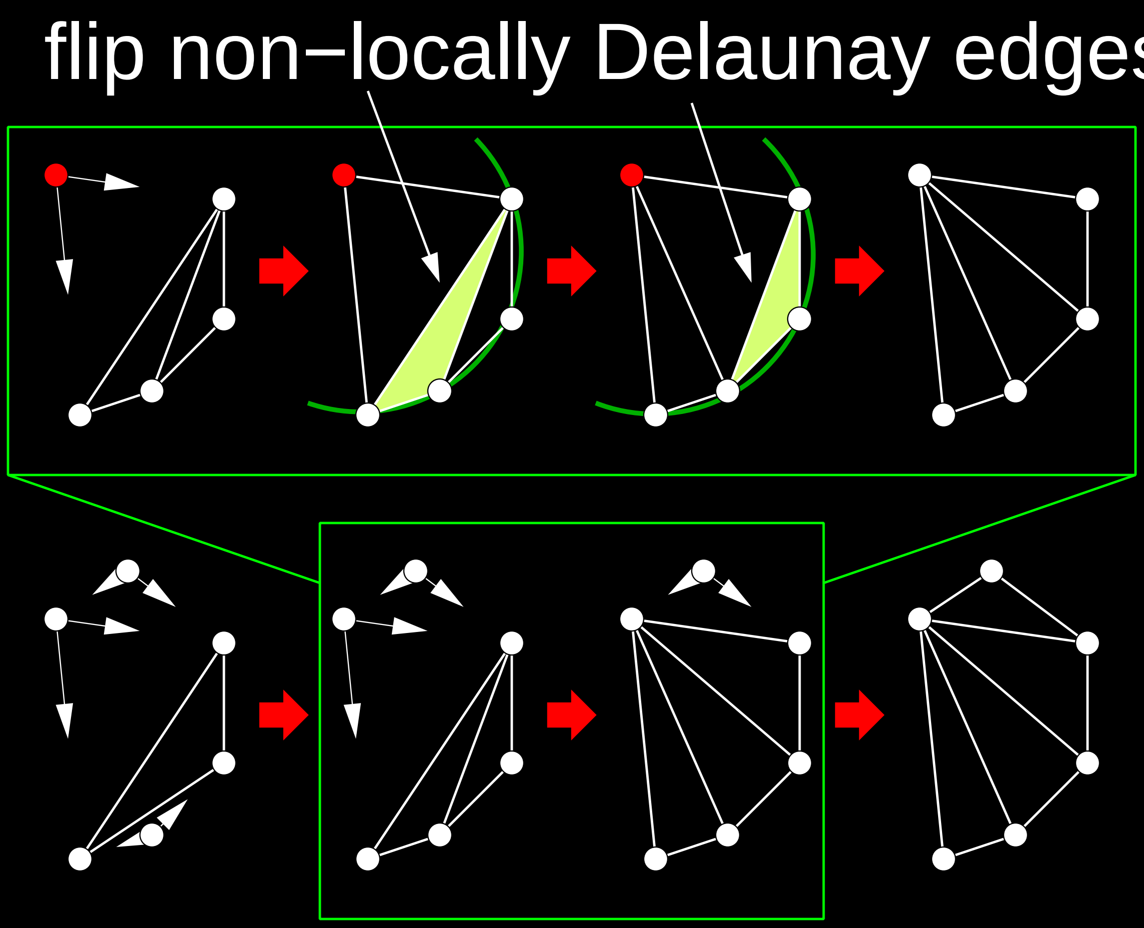
# Edge Flips



# Chew's Algorithm



Computes DT of a convex polygon in expected linear time.



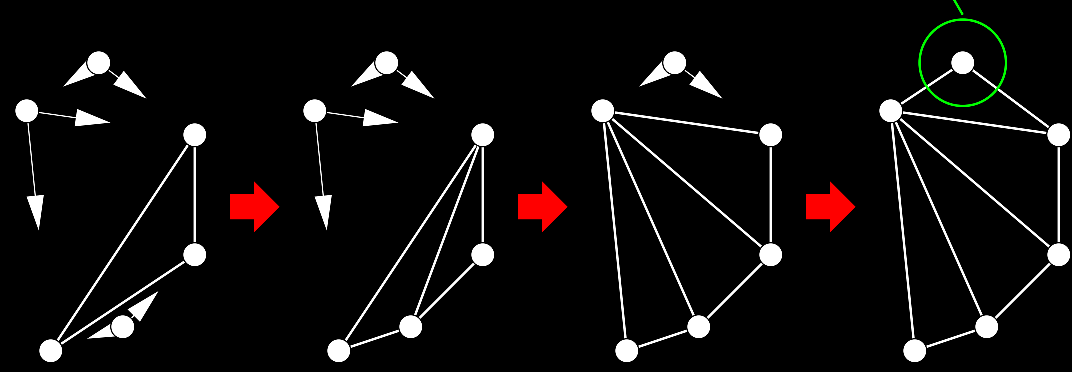

Randomized incremental vertex insertion

# Backward Analysis

Idea: analyze an algorithm as if it were running backward in time.

Pretend we remove a randomly chosen vertex from the final triangulation. Its expected degree is < 4.

Randomized incremental vertex insertion



# Backward Analysis

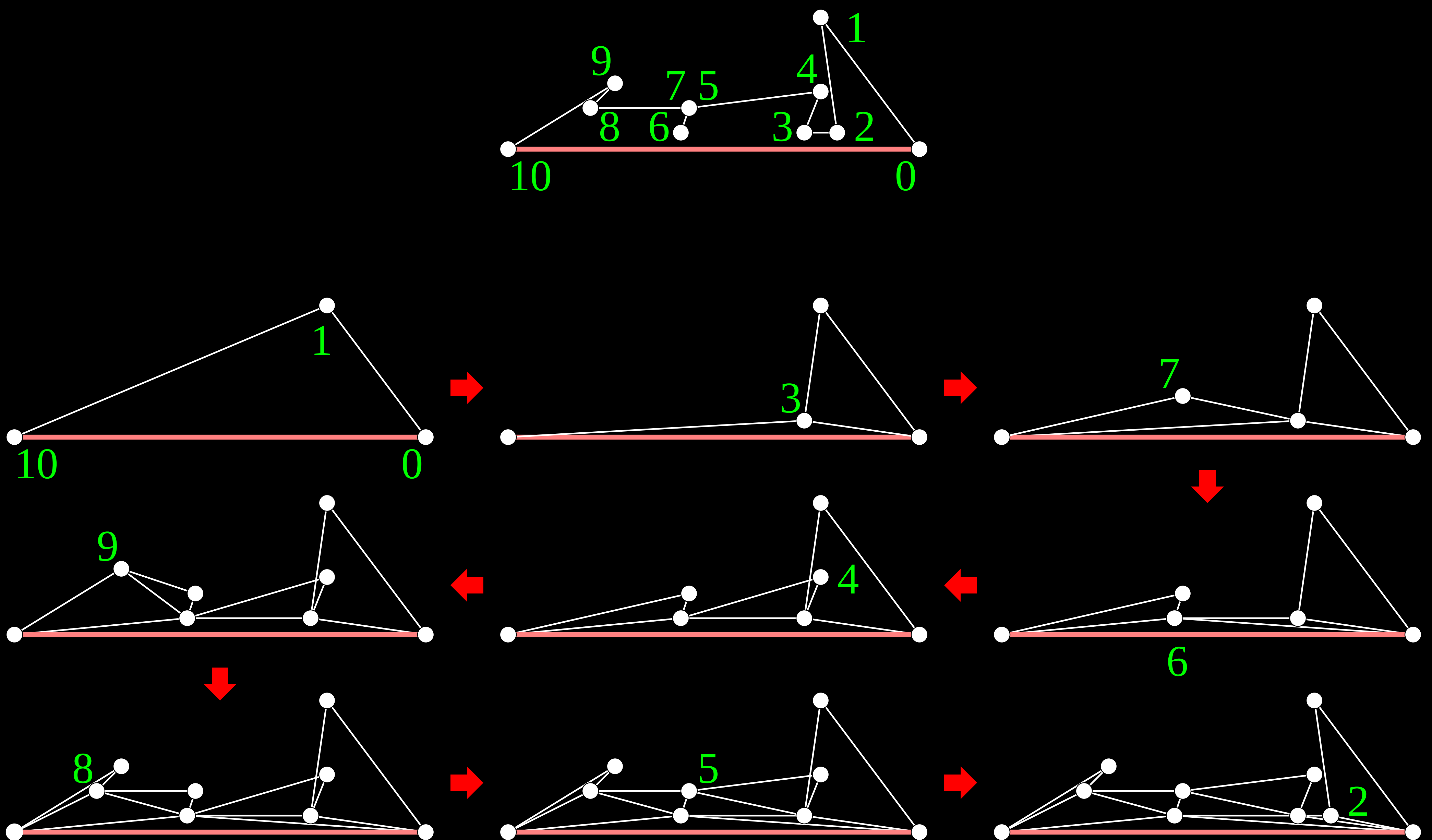
Idea: analyze an algorithm as if
it were running backward in time.

Pretend we remove a randomly chosen
vertex from the final triangulation.
Its expected degree is < 4.

Expected time to insert one vertex = constant.

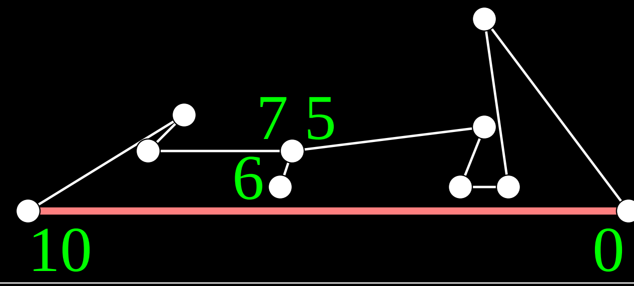
Expected time to compute Del tri = linear.

# Retriangulating a Segment Cavity



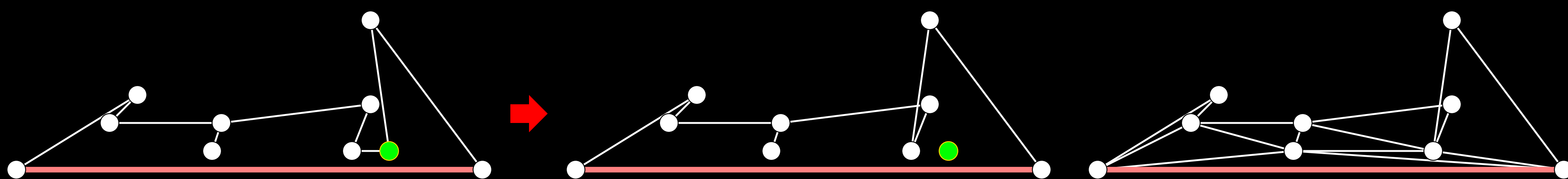
# How Our Algorithm Differs from Chew's

Polygons with dangling edges.

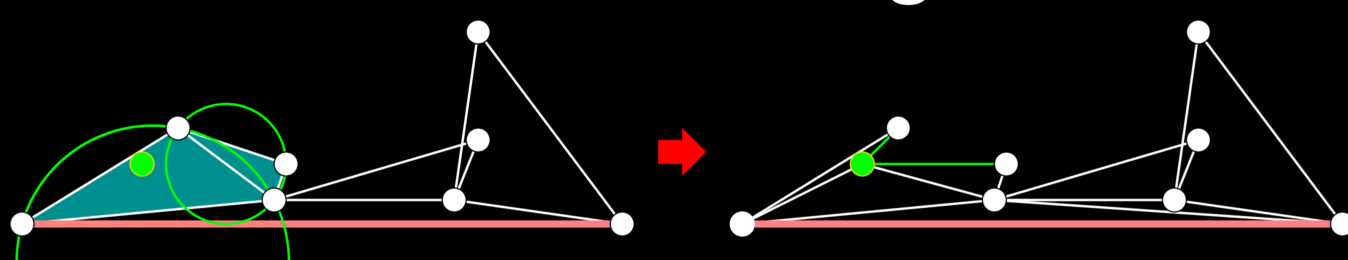



Always insert segment endpoints first.

An intermediate polygon can self−intersect.



CDT triangles are deleted for 2 reasons:
new vertex in circumcircle; new edges cross.



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Inserting a polygon into a 3D CDT.

# Coupon Collecting

15 14 13 12 11 10 9 8 7 6 5 4 3 2 1

Shuffle numbered coupons in random order.

6 12 7 9 14 3 5 15 11 1 10 8 13 2 4

How often do we collect a coupon preceded by every larger coupon?

Expected $\Theta(\log k)$ times.

# Coupon Collecting

5 5 5 5 5 4 4 4 4 3 3 3 2 2 1

Shuffle numbered coupons in random order.

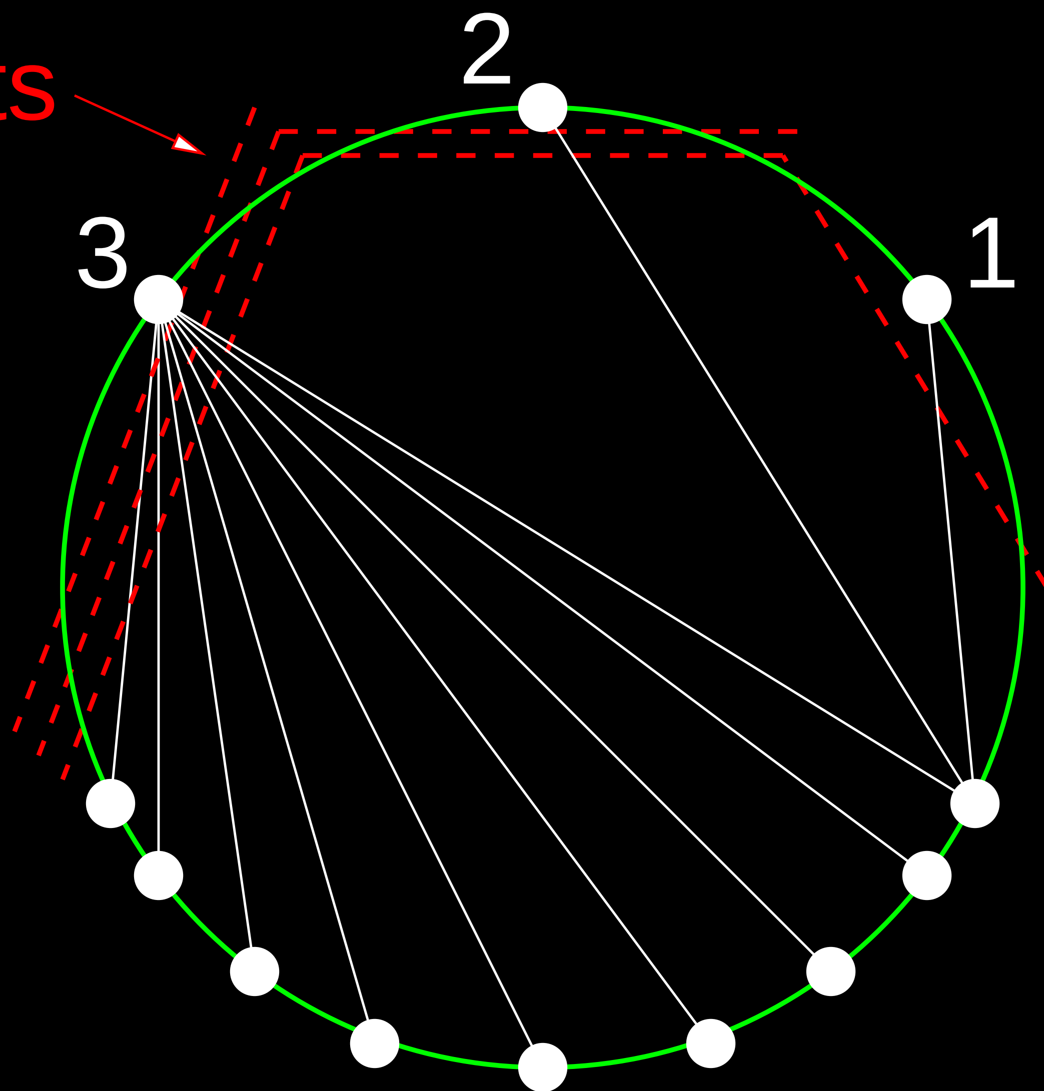
2 5 4 4 5 3 3 5 1 2 3 4 5 5 4

How often is the first collection of a coupon preceded by (one of) every larger coupon?

Expected $\Theta(\log^2 k)$ times.

# Lower Bound Example

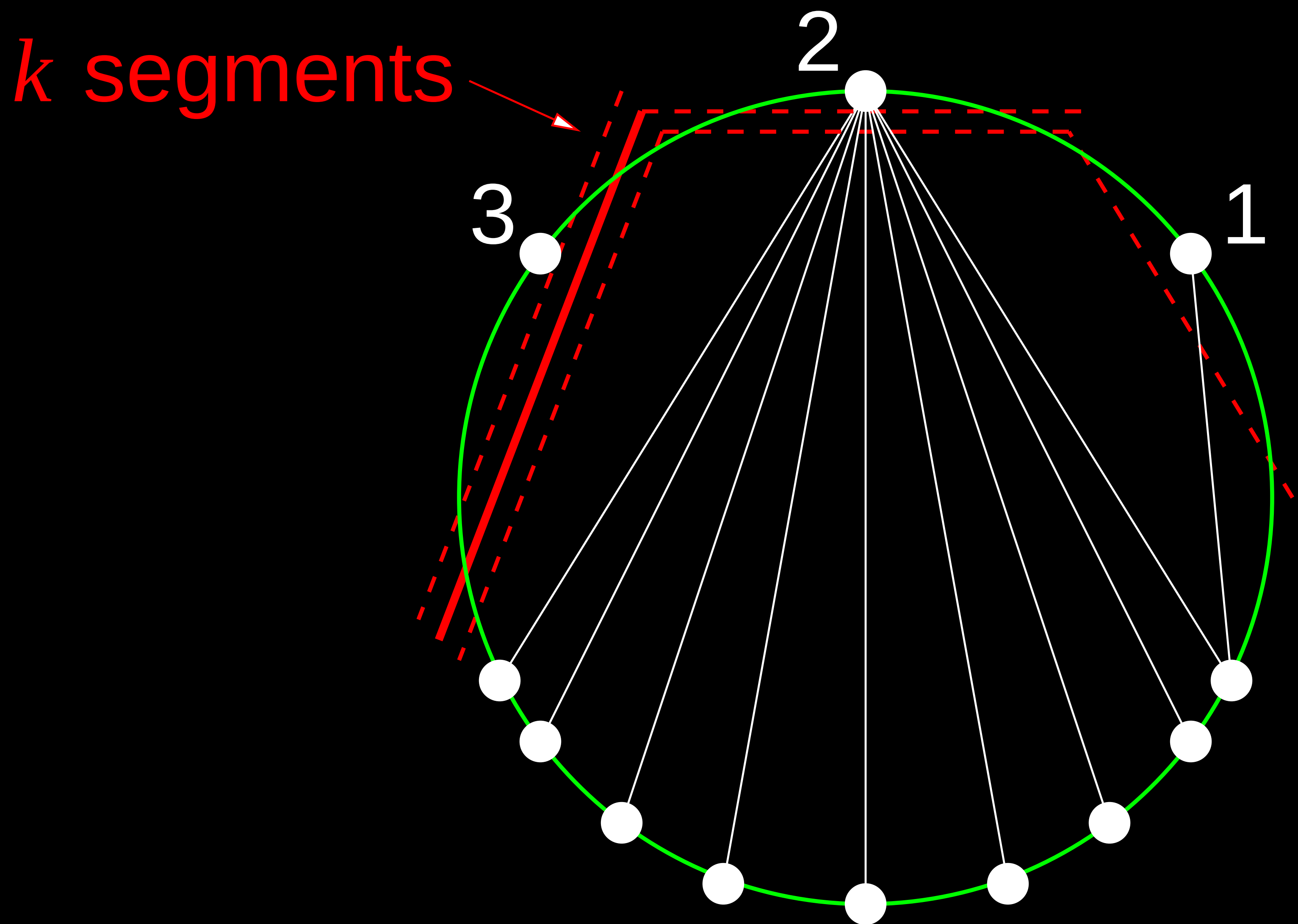
$\Theta(\sqrt{k})$ pulling vertices




$\Theta(n)$ pushing vertices

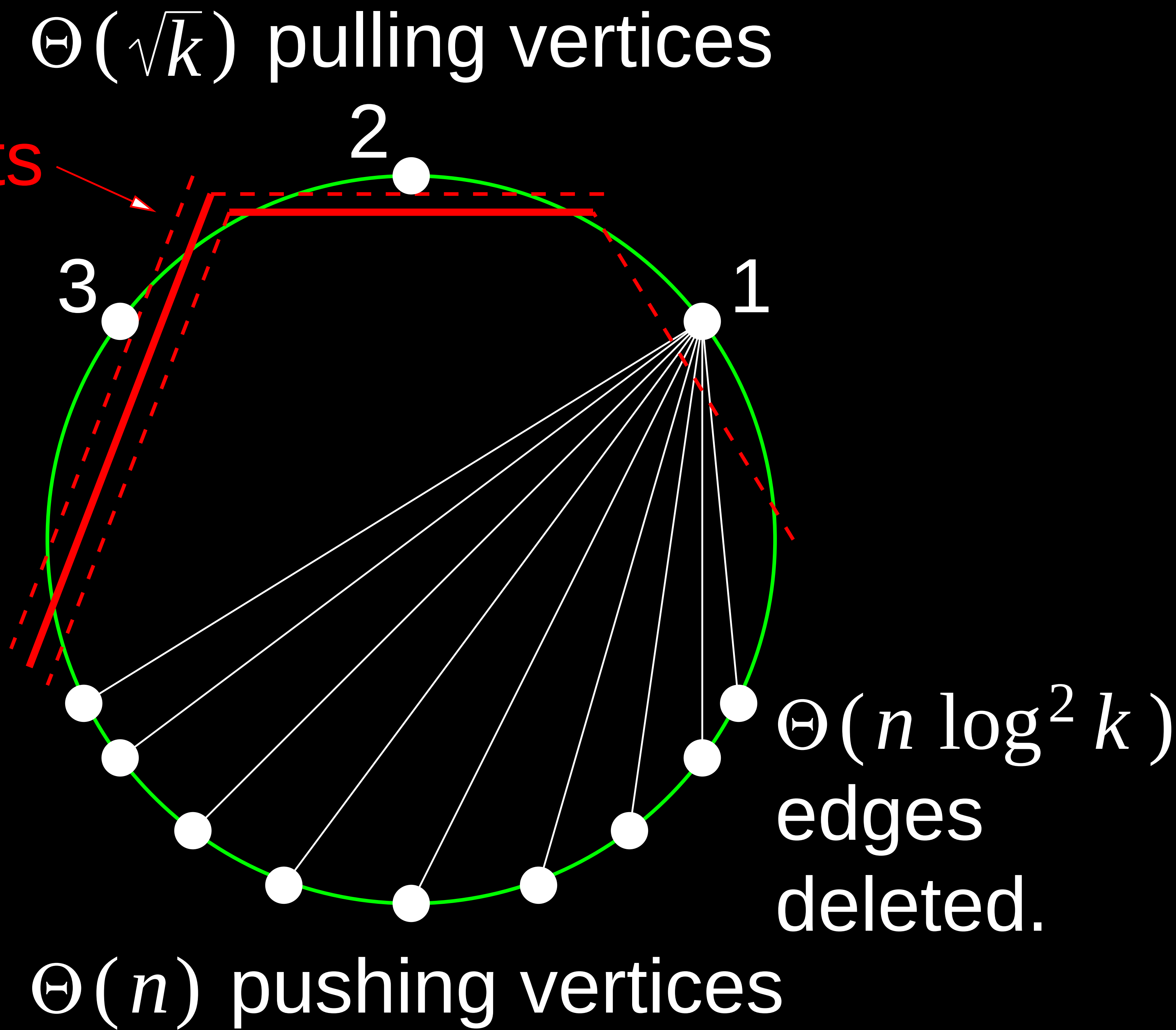
# Lower Bound Example

$\Theta(\sqrt{k})$ pulling vertices




$\Theta(n)$ pushing vertices

# Lower Bound Example



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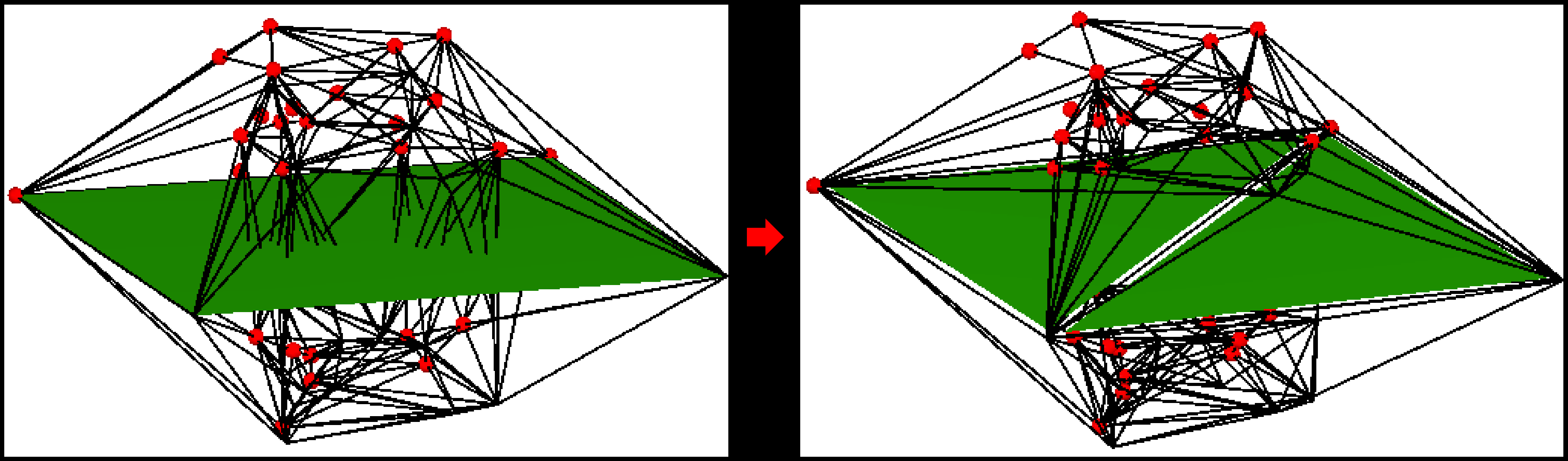
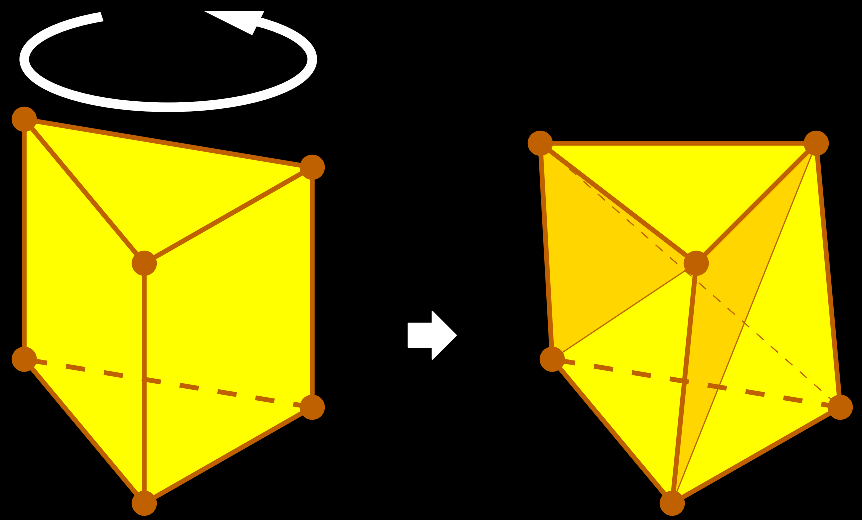
Inserting a polygon into a 3D CDT.

# Polygon Insertion

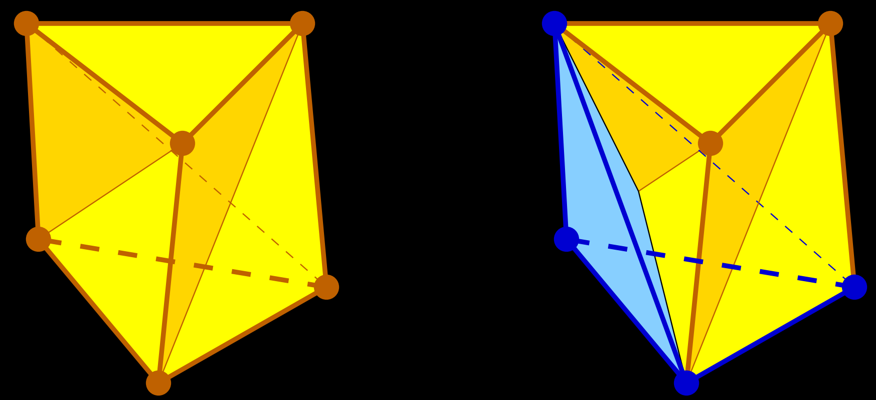


# Some Polyhedra Have No Tetrahedralization



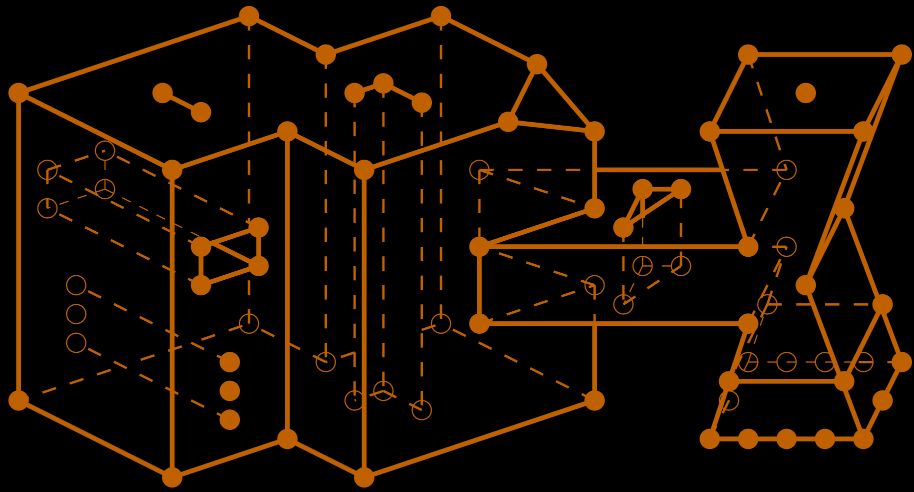
Schönhardt’s polyhedron

# Some Polyhedra Have No Tetrahedralization



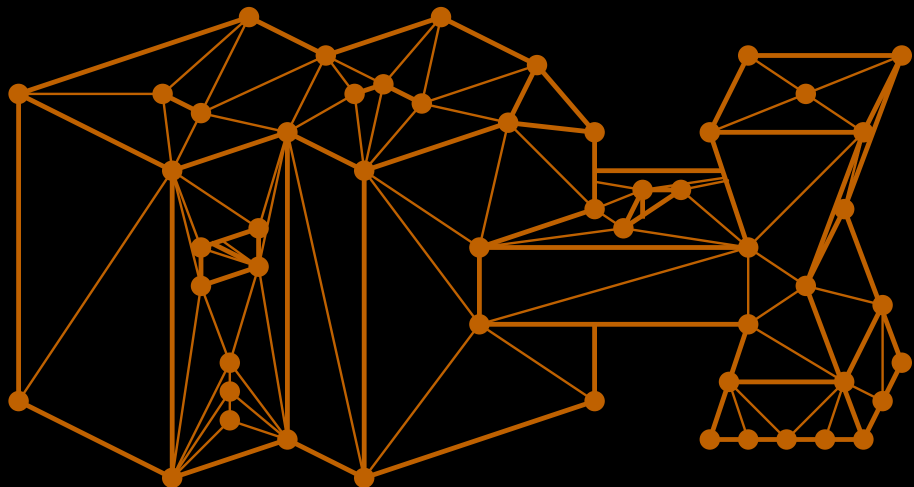
Any four vertices of Schönhardt’s polyhedron yield a tetrahedron that sticks out a bit.

# Input: A Piecewise Linear Complex



PLC

A set of vertices, segments, and polygons.

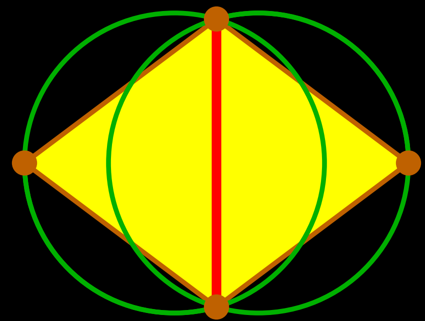


CDT

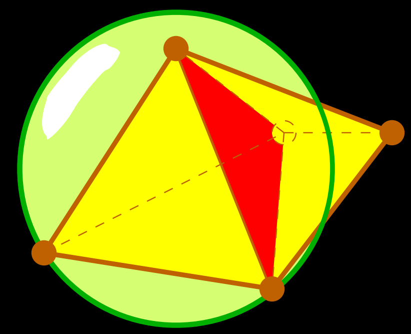
Each polygon appears as a union of triangular faces.

# The Delaunay Triangulation

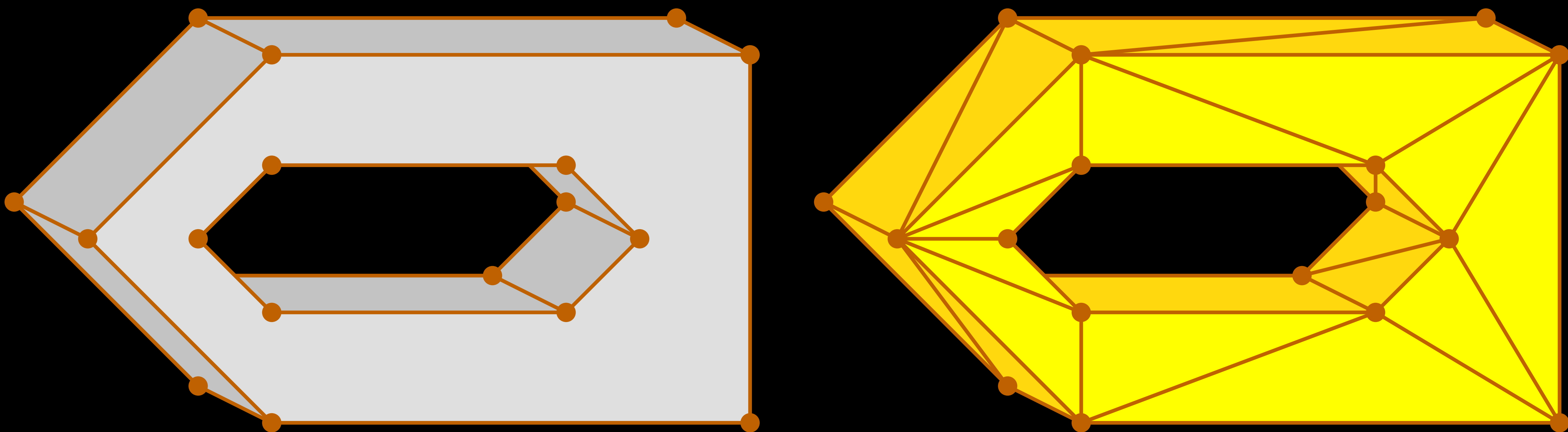
An edge is *locally Delaunay* if the two triangles sharing it have no vertex in each others' circumcircles.



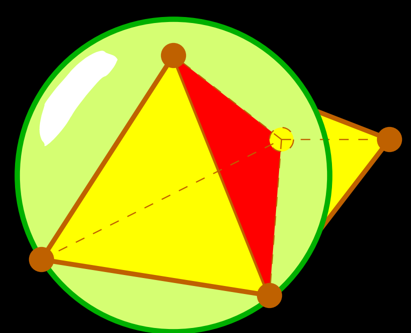
A triangular face is *locally Delaunay* if the two tetrahedra sharing it have no vertex in each others' circumspheres.



# Constrained Delaunay Triangulations (CDTs)

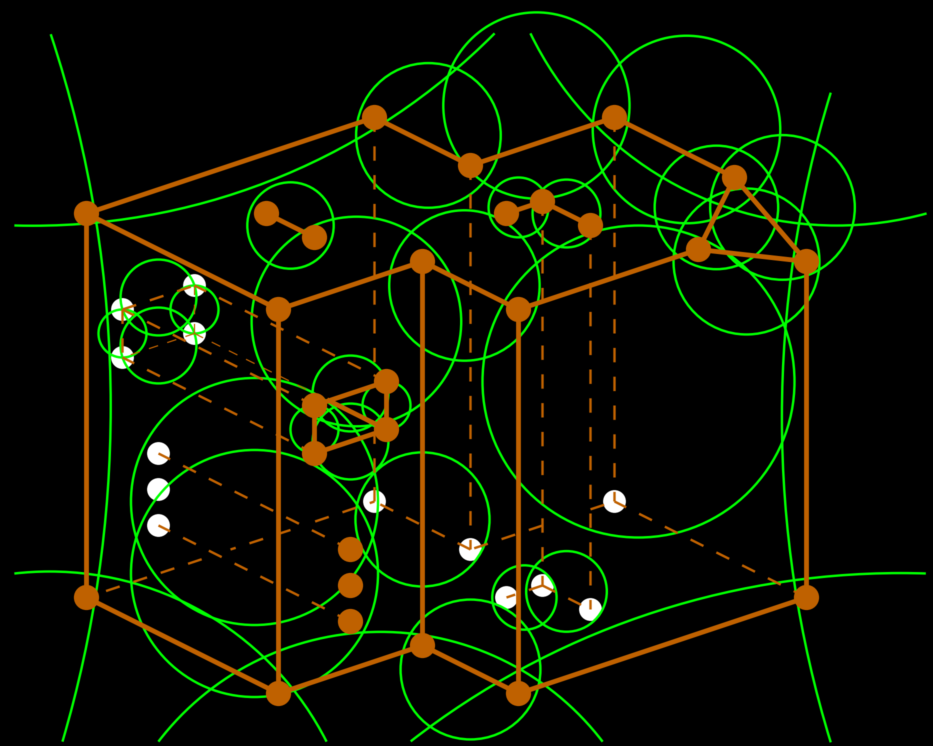


A tetrahedralization of a polyhedron/PLC is a CDT if every triangular face not included in an input polygon is locally Delaunay.



# **The CDT Theorem** (makes 3D CDTs useful)

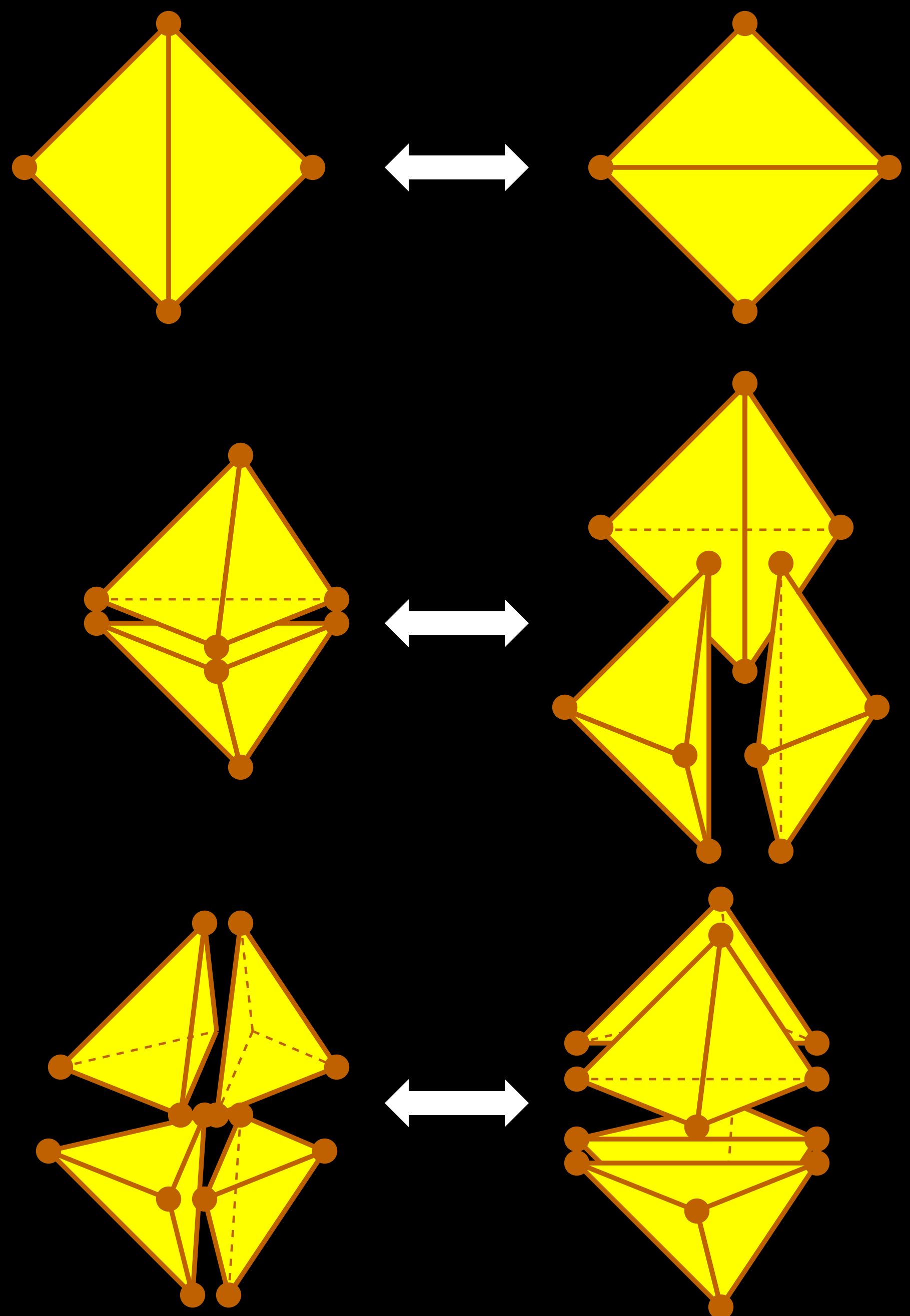
Say that a PLC is *edge−protected* if every segment has an enclosing ball that contains no vertex but the segment's endpoints.



Theorem:

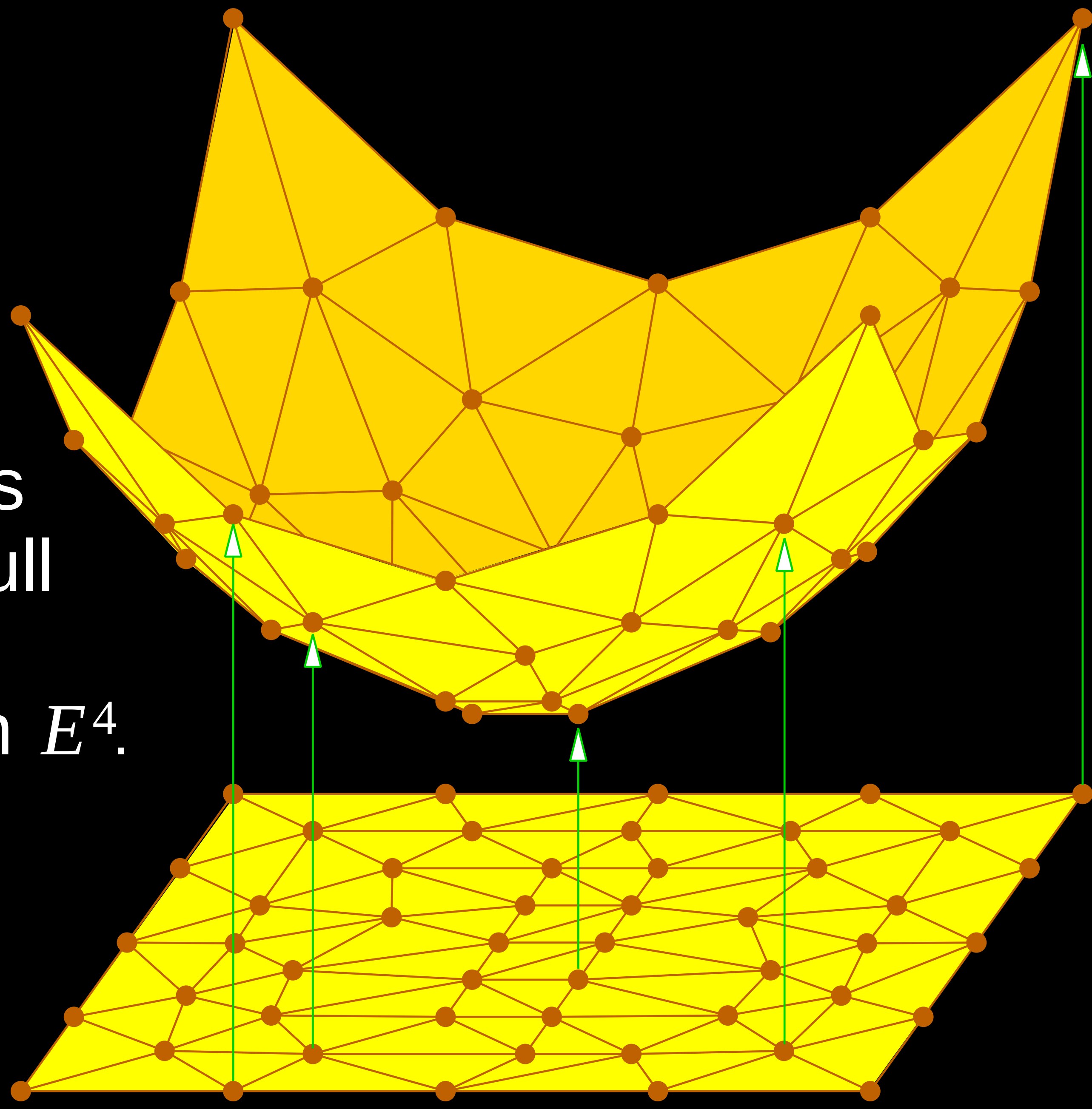
Every edge−protected PLC has a CDT.

# Bistellar Flips

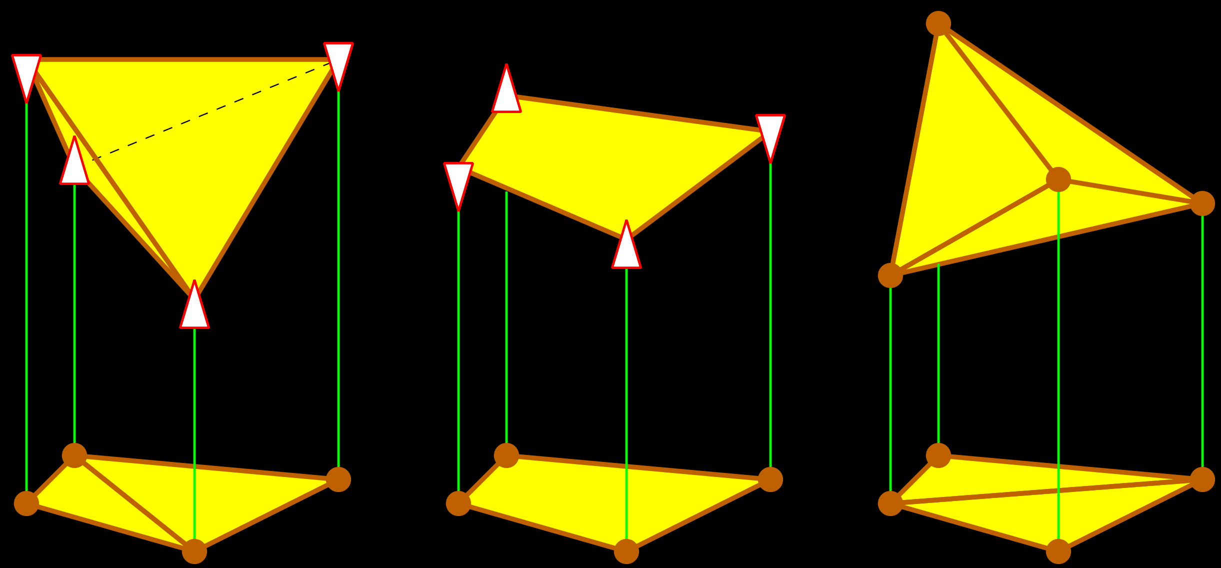


# Seidel's Parabolic Lifting Map

The 3D DT matches the lower convex hull of the vertices lifted onto a paraboloid in $E^4$.

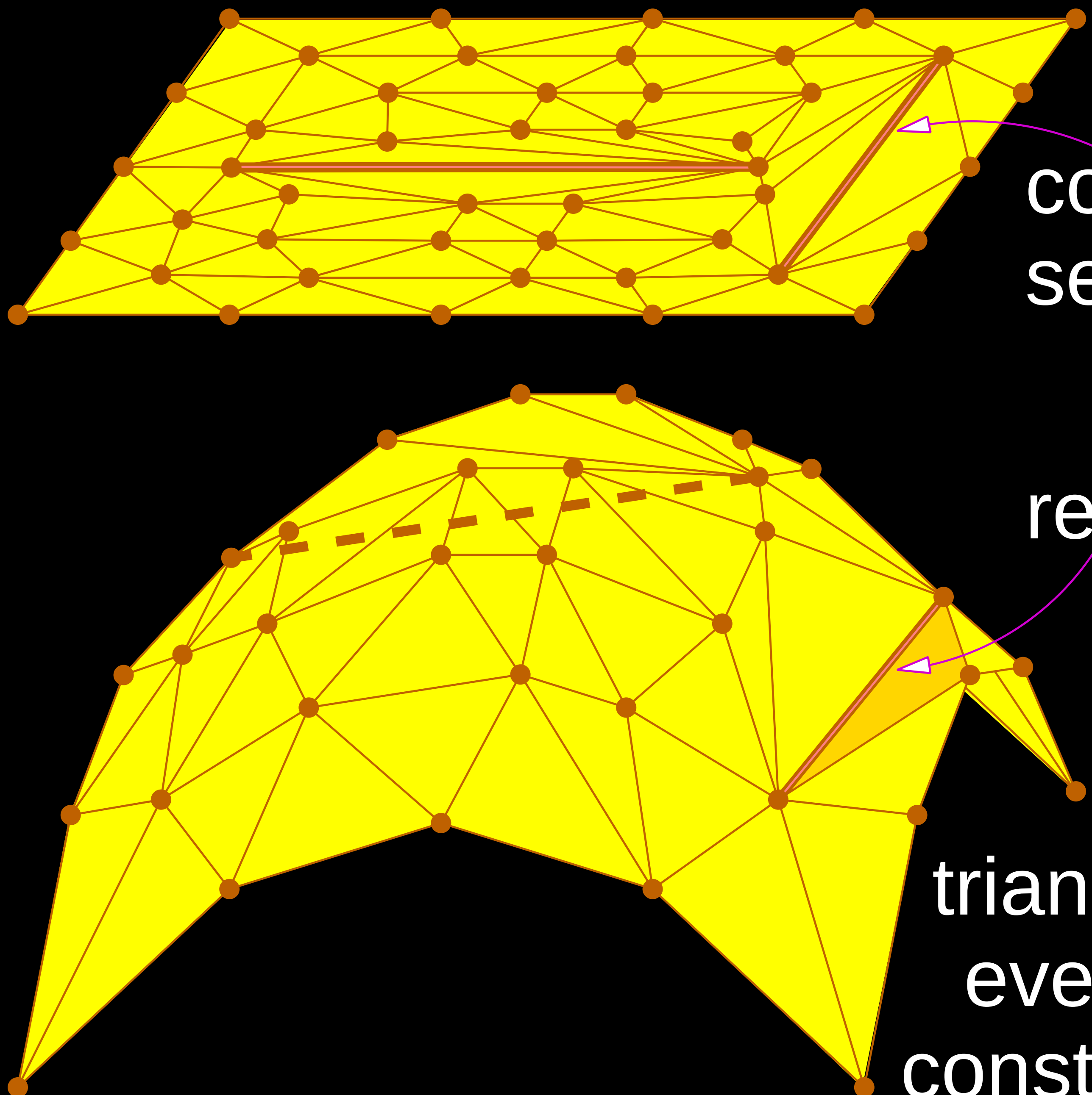


# Convex Hull of Moving Points



As the lifted vertices move vertically,
use flips to maintain the lower convex hull.

# A Lifted CDT

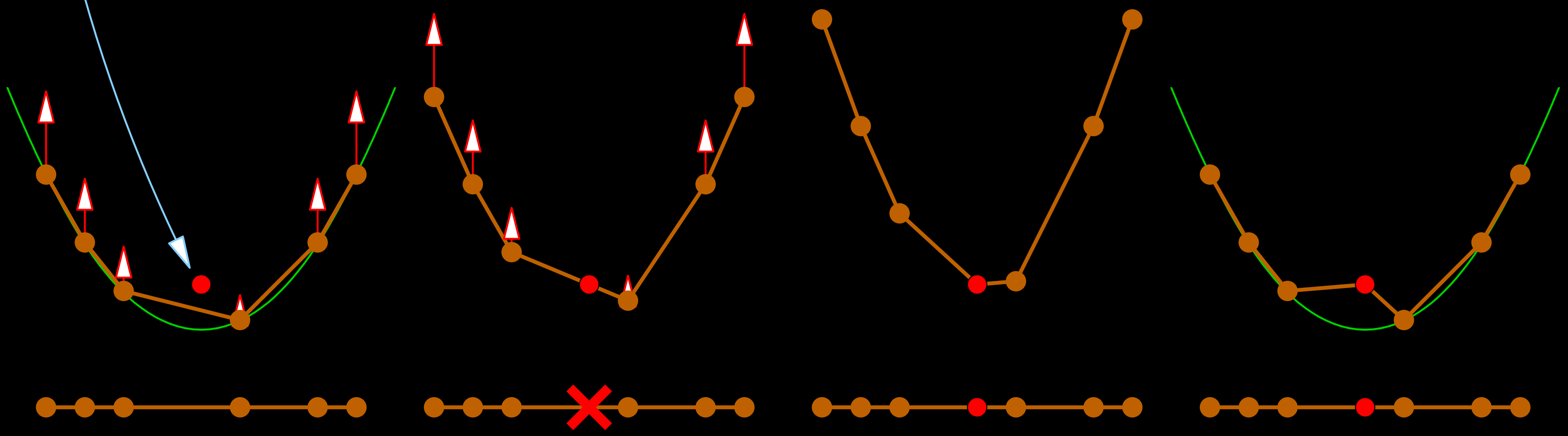



Lifted triangulation is convex everywhere except at constraining segments.

# “Polygon” Insertion in 1D by Flips

Let’s insert this “polygon”!

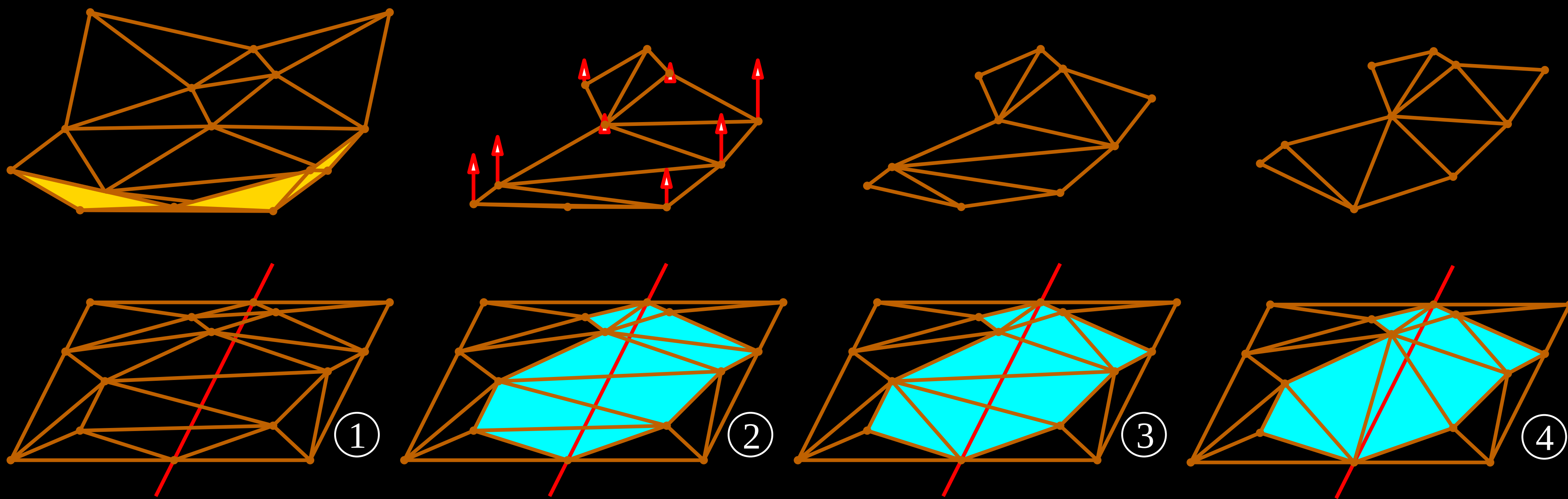
Lifted vertices rise at a velocity proportional to distance from “polygon.”



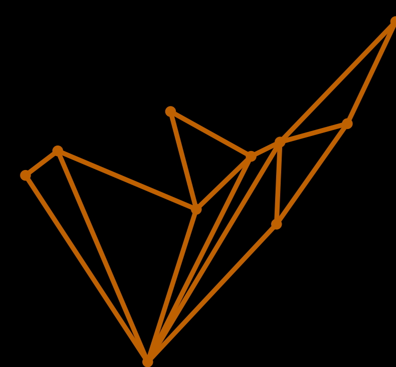
1D triangulation

CDT

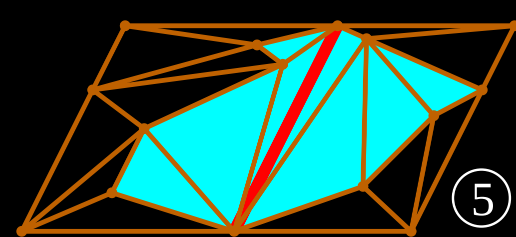
# “Polygon” Insertion in 2D by Flips




Lift vertices at speed proportional to distance from new segment.



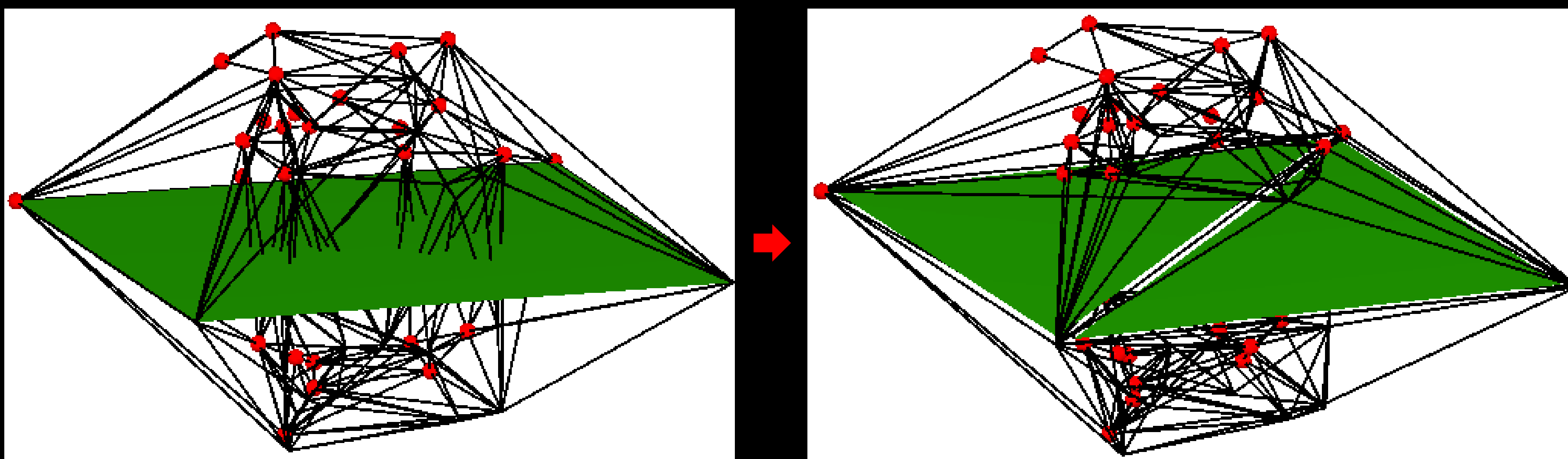
Maintain lower convex hull with flips.



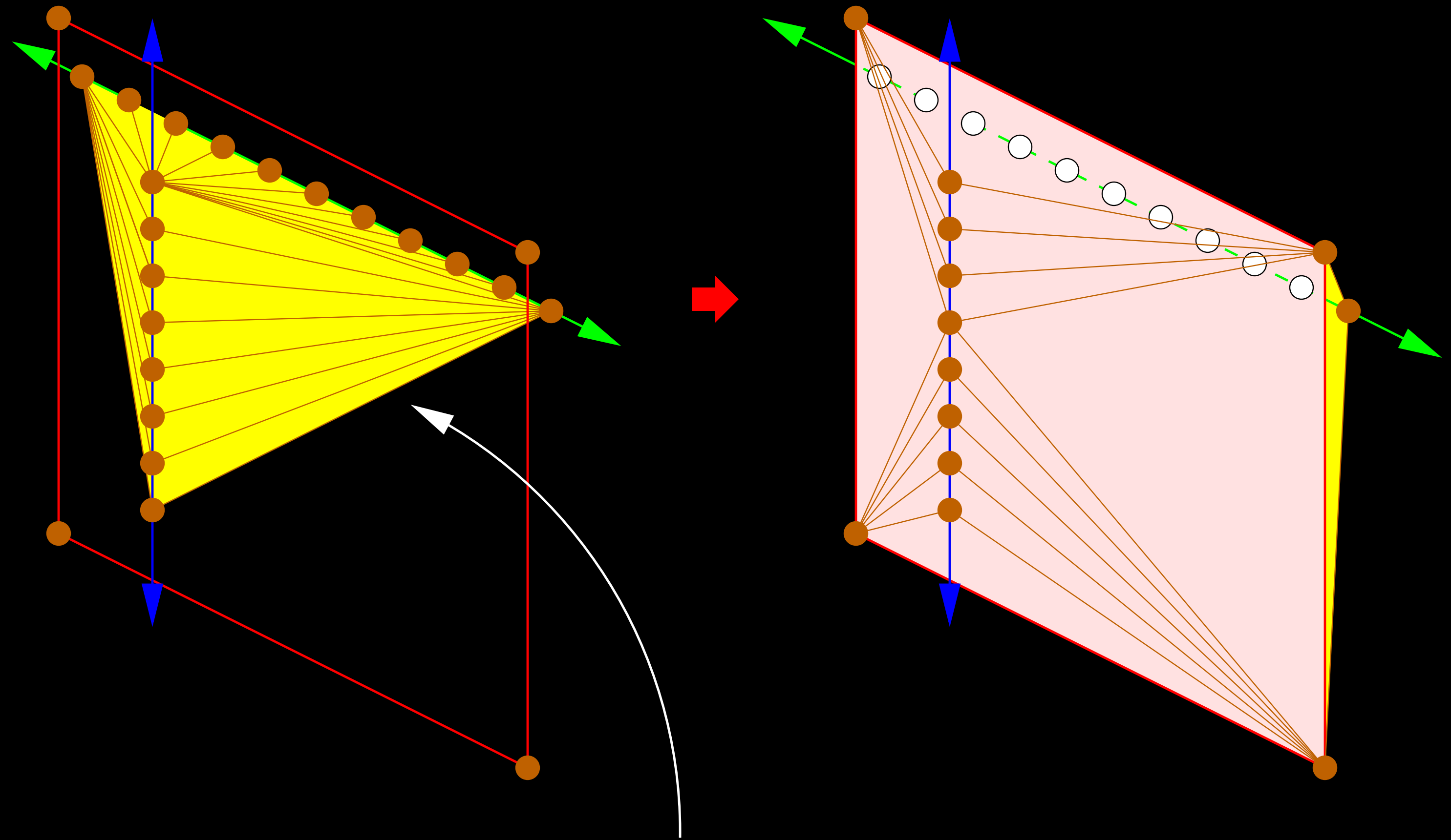
# Running Time for Polygon Insertion

Worst–case time to insert one polygon:

$$O(n^2 \log n)$$



# Running Time: Lower Bound



$\Theta(n^2)$ tetrahedra cut

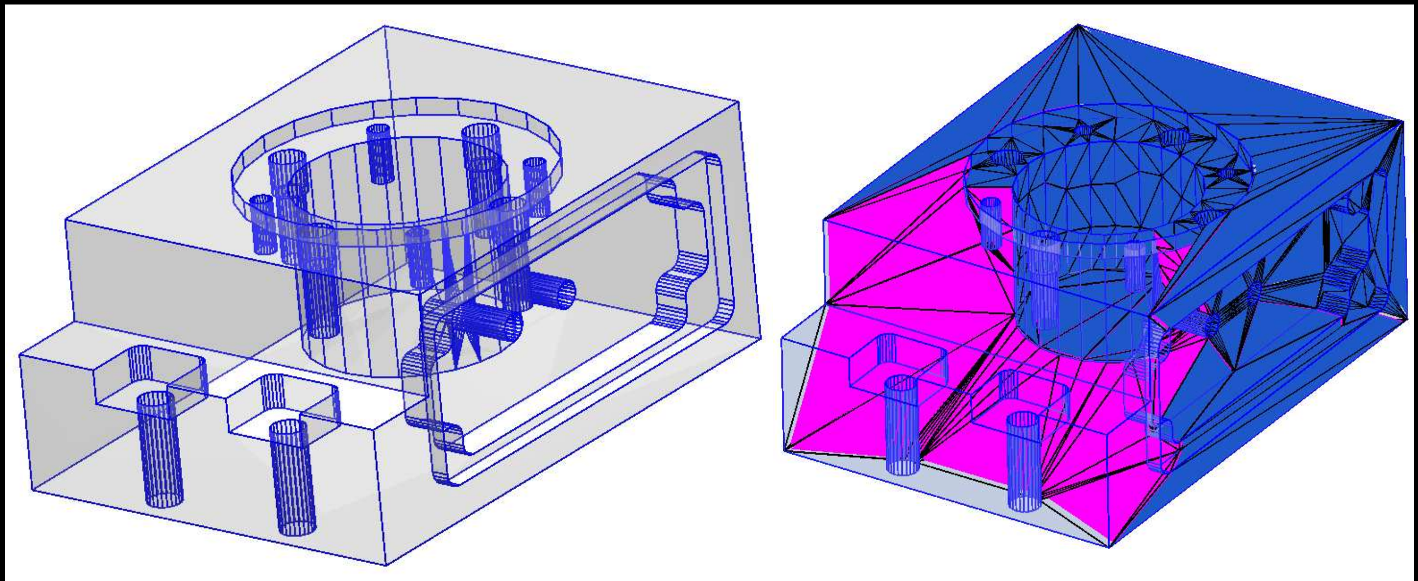
# Running Time for Polygon Insertion

Worst−case time to insert one polygon:

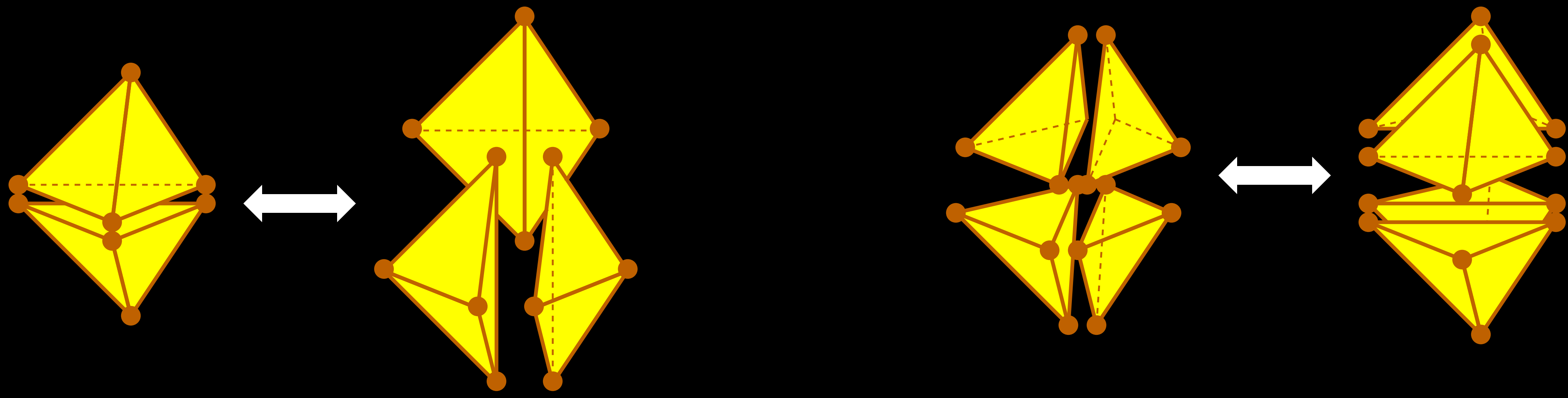
$O(n^2 \log n)$

Worst−case time to insert *any number* of polygons and construct a CDT:

$O(n^2 \log n)$



# Running Time:  Upper Bound



Every bistellar flip either creates or deletes an edge.

Once deleted, an edge never reappears.

# Open Problems (Theoretical)

Does any incremental insertion algorithm run in $\Theta(n \log n)$ time? E.g. biased random?

Upper bound does not depend on Delaunay property! Other optimal triangulations?

Chapman & Hall/CRC
Computer & Information Science Series

# Delaunay Mesh Generation

Siu-Wing Cheng
Tamal Krishna Dey
Jonathan Richard Shewchuk

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK