Combinatorial rigidity with (forced) symmetry

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joint work with
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Frameworks, rigidity, flexibility

- A *framework* is a *graph* $G = (V,E)$ and an assignment of a *length* $\ell(ij)$ to each edge ij

- A *realization* $G(\mathbf{p})$ is a mapping $\mathbf{p} : V \to \mathbf{R}^d$ such that

$$|\mathbf{p}(i) - \mathbf{p}(j)| = \ell(ij)$$

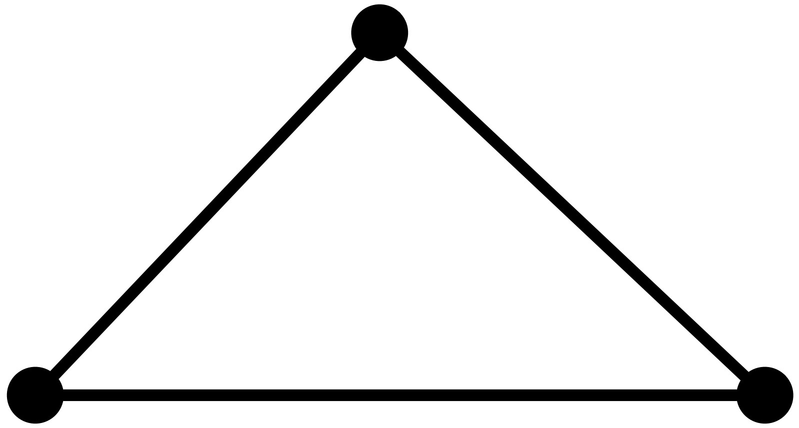
- A *realization* is *rigid* if all continuous motions are Euclidean isometries

Rigidity, flexibility

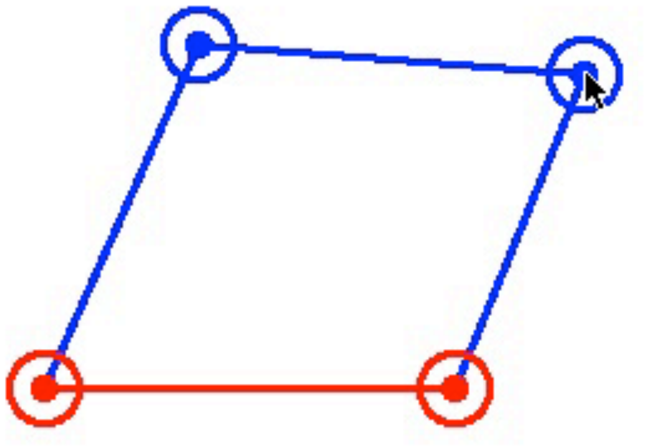
Rigid

Flexible

Rigidity, flexibility



Rigid



Flexible

Degrees of freedom

- Take the coordinates of the n points $\mathbf{p}$ as variables
  - and subtract off dim Euc(d) for "trivial motions"

Total d.o.f: $$dn - \dim \mathrm{Euc(d)}$$

- The edges of G index equations in these variables

For rigidity $$\#\mathrm{E}(G) \geq dn - \dim \mathrm{Euc(d)}$$

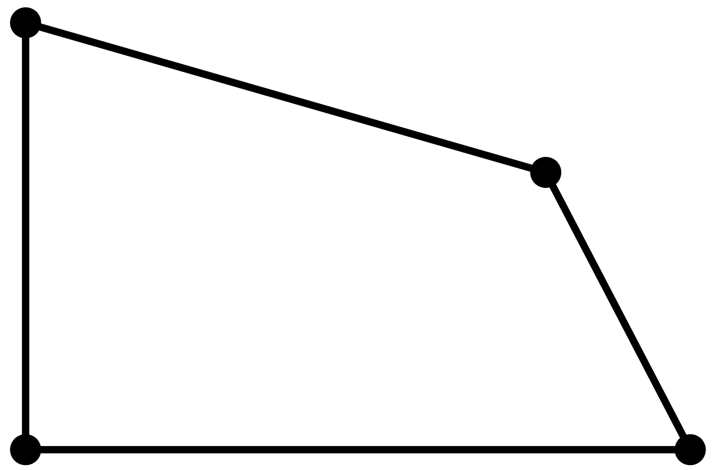
Laman's Theorem

- The "combinatorial rigidity" problem is

  Which graphs are graphs of rigid frameworks?

- **Theorem:** For $d = 2$, G generically rigid "$m \leq 2n - 3$" for all subgraphs and "#E(G) = 2 #V(G) – 3".
  - Generic: except a proper algebraic subset of choices of $\mathbf{p}$
- Can test efficiently.
- *Not* sufficient in higher dimensions.

Genericity



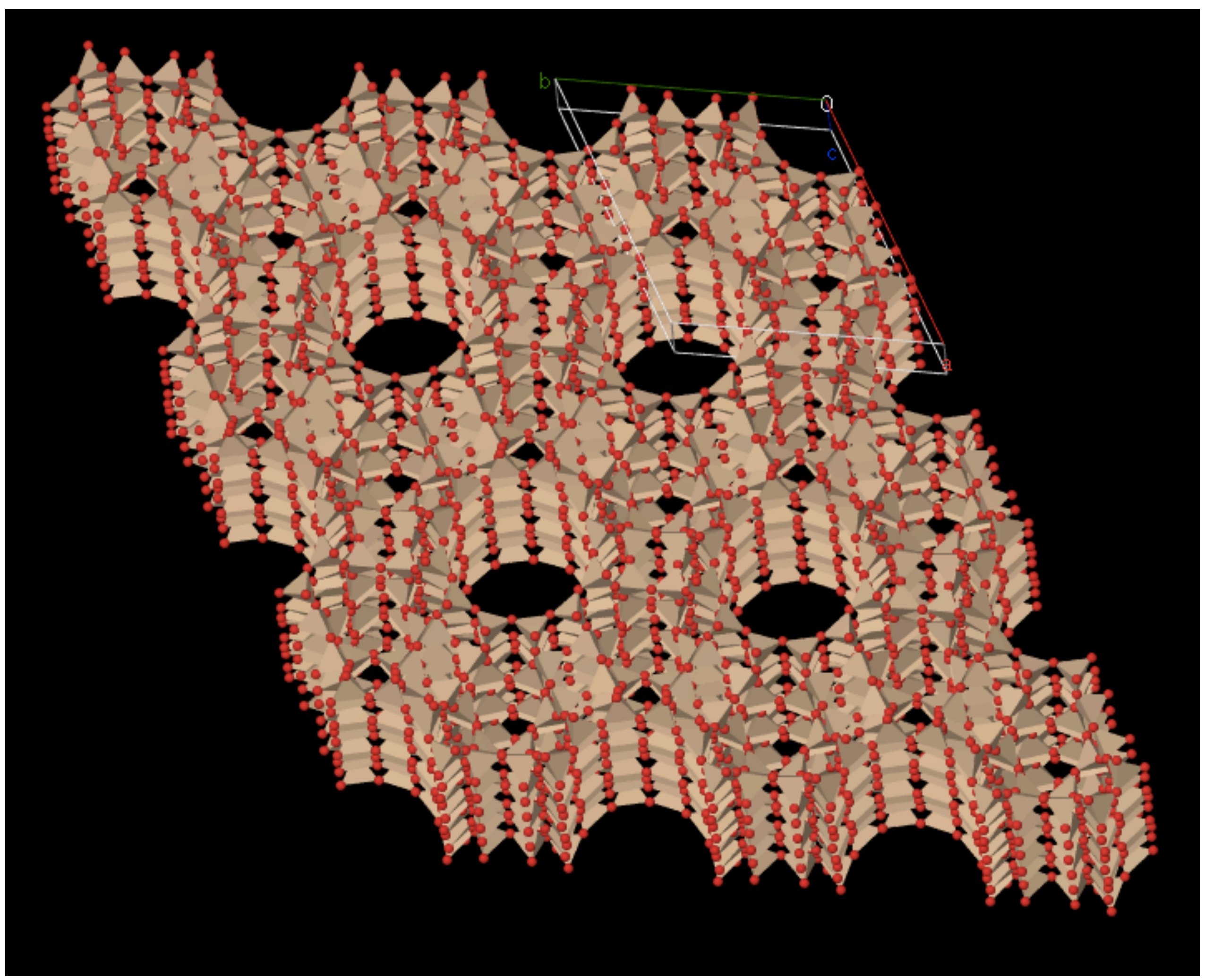
Generic

Non-generic

Intermezzo: motivations

- Frameworks go back to the time of Maxwell
- Applications in: polyhedral geometry, structural biology, robotics, **crystallography**, cond. mat., computer-aided design
- *Combinatorial* methods need inputs that are "generic enough" and treatable via "finite information"

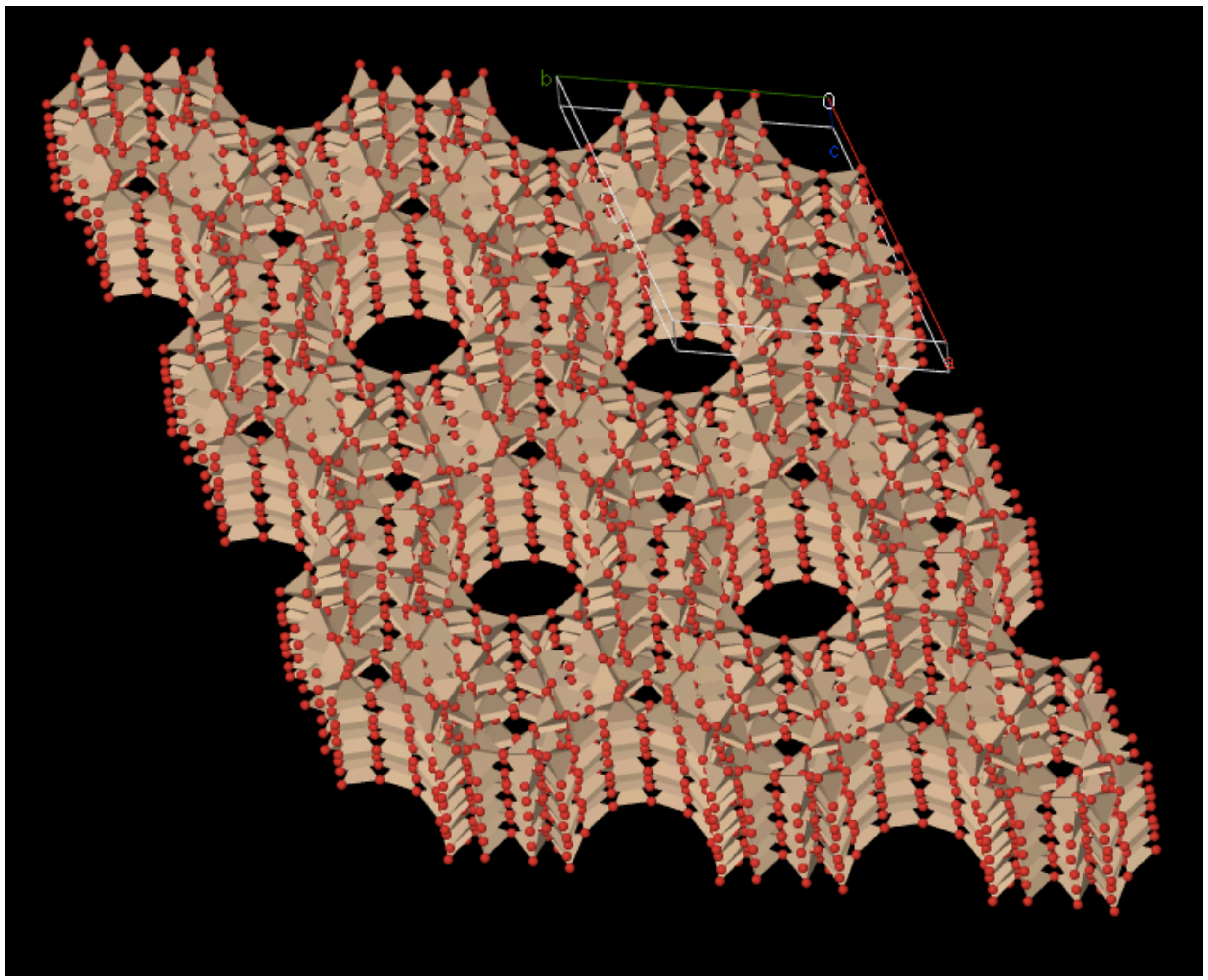
Zeolites



[Rivin-Treacy-Randall, Hypothetical Zeolite Database]

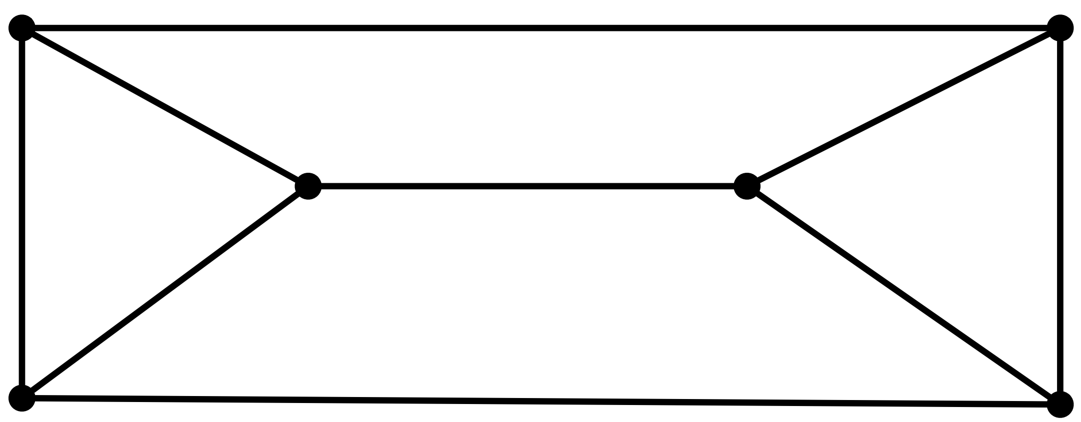
- Class of aluminosilicates with broad industrial applications
- Geometrically, crystals of corner-sharing tetrahedra
- Useful ones are *flexible*
- Underlying graphs are *regular*
- Database of potential structures as *crystallograpic* frameworks

Combinatorial analysis?

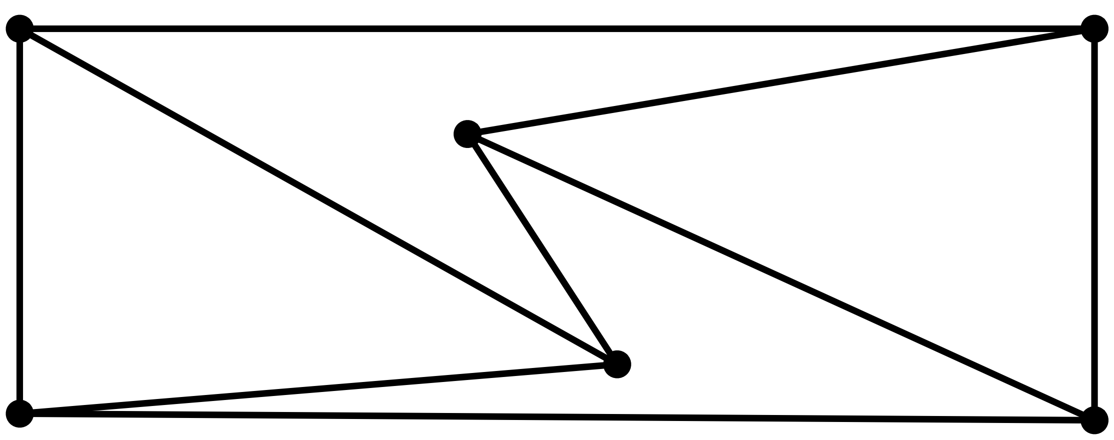


- Would like to be able to combinatorially test rigidity / flexibility of potential zeolite
- Underlying graph is *infinite* (so what would an algorithm look like?)
- Geometry in crystallography is always *special* (so not generic enough?)

Symmetry



Infinitesimally
flexible



Rigid

Infinite frameworks

- For G with a countable vertex set, the solutions to the length equations are an *inverse limit* (of varieties).
- **Thm (Owen-Power):** Configuration spaces of infinite frameworks can be very wild (e.g., homeo to space-filling curves)

Periodic frameworks

- A *periodic framework* (G, ℓ, Γ) is an infinite framework with

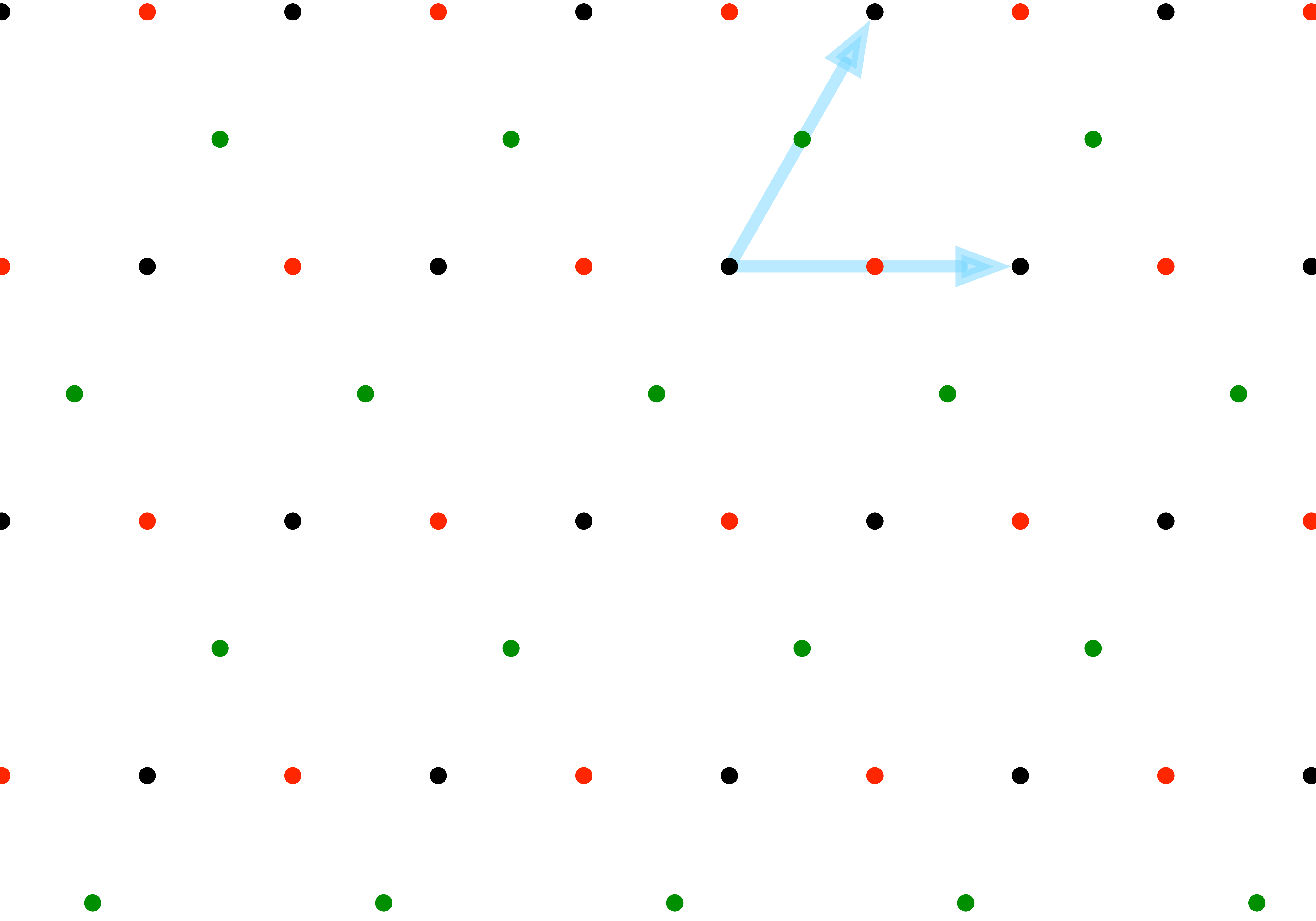
$$\Gamma < \mathrm{Aut}(G)$$

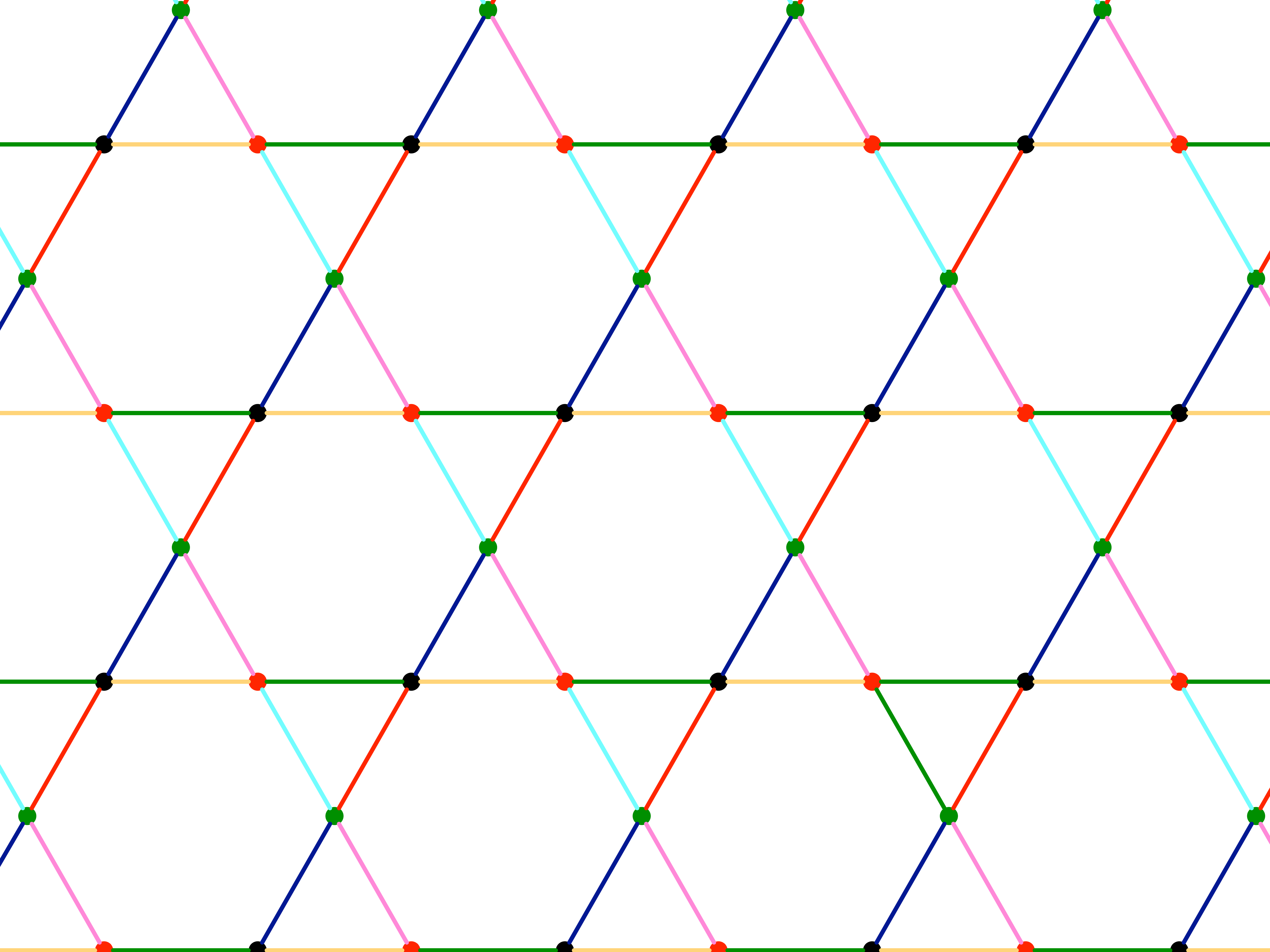
Γ free abelian,
rank d

$$\ell(\gamma(ij)) = \ell(ij)$$

- A *realization* $G(\mathbf{p},\Lambda)$ is a realization periodic with respect to a *lattice of translations* Λ, which realizes Γ

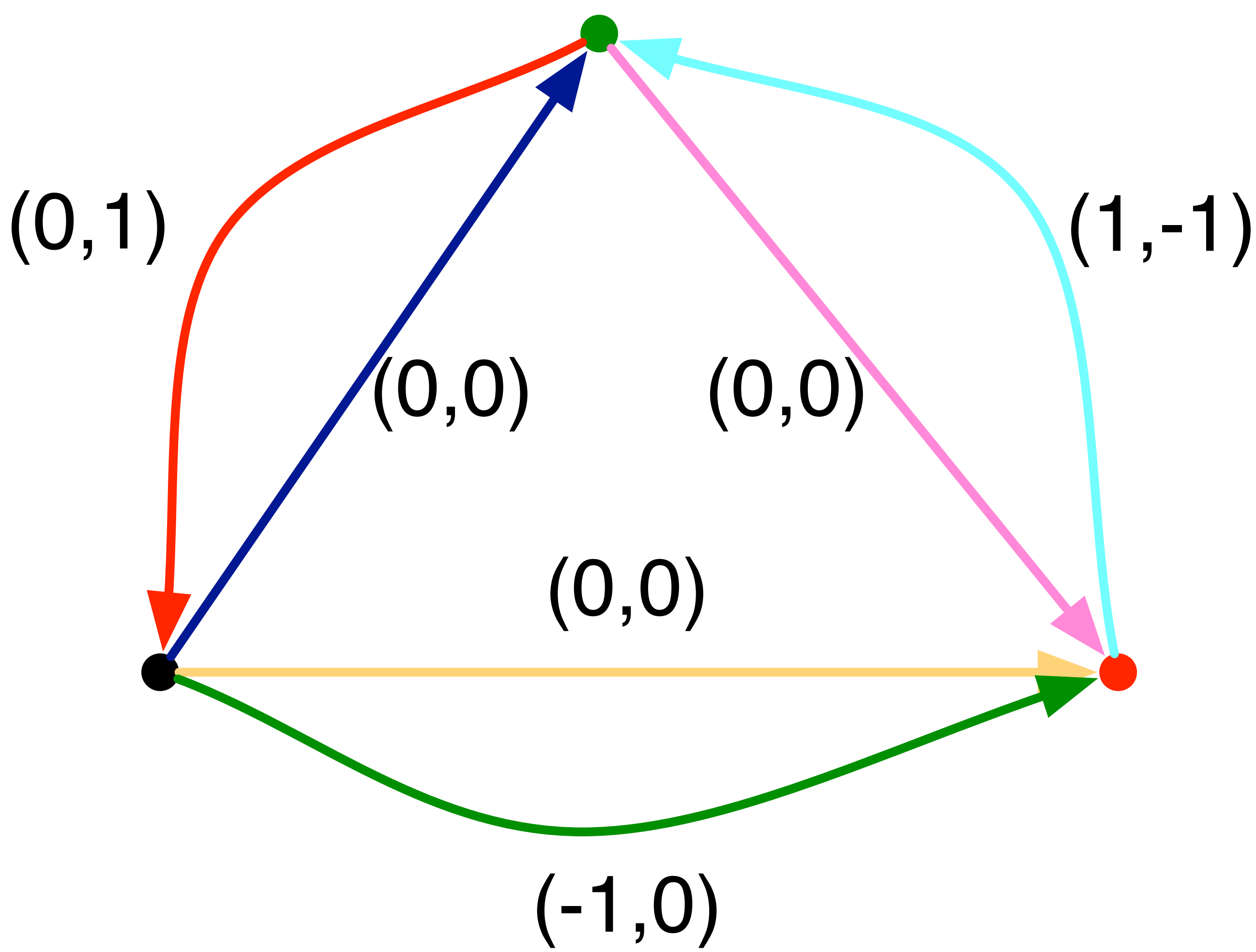
- Motions *preserve the* Γ*-symmetry*



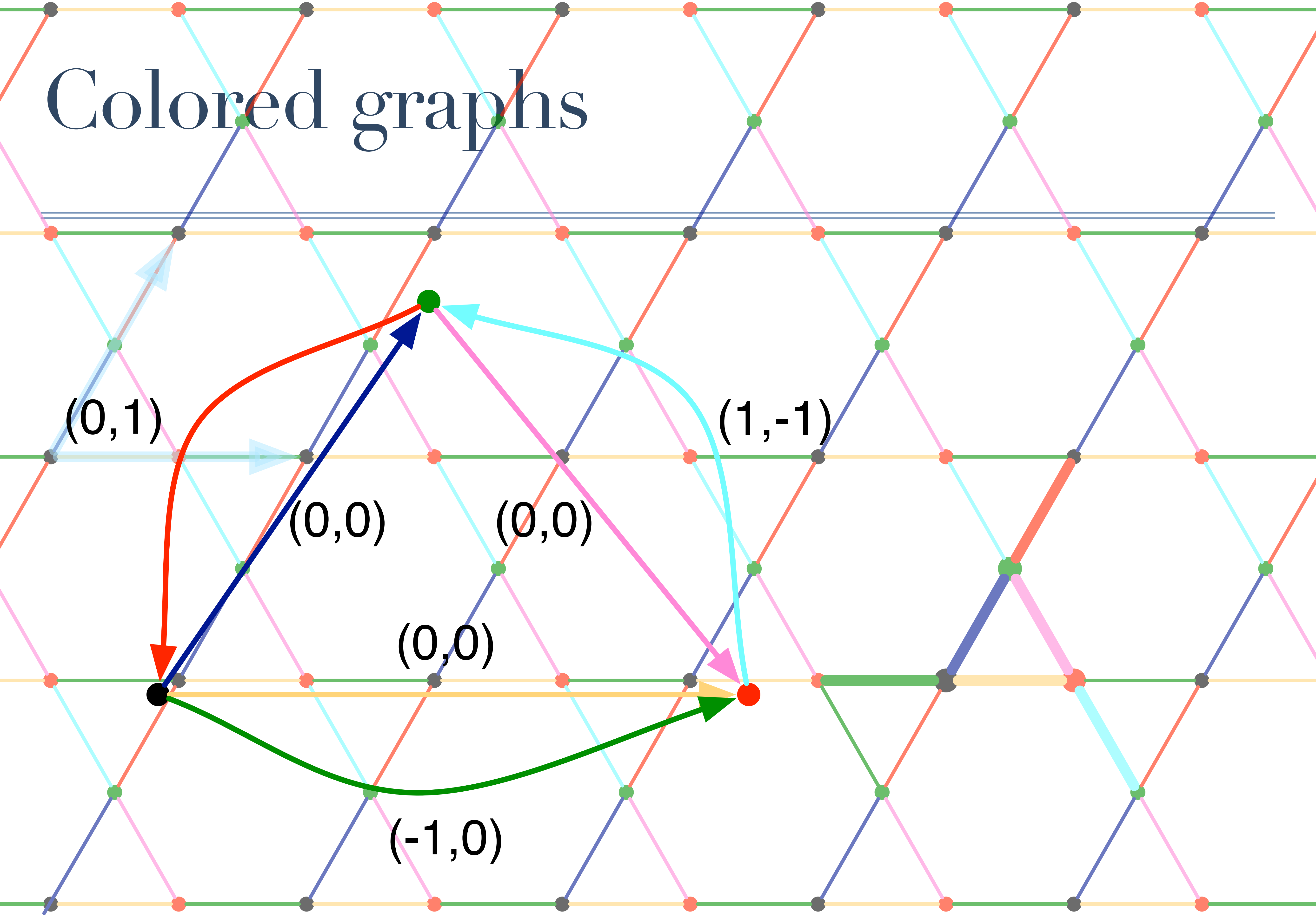


Colored graphs

- Finite directed graph
- Edges "colored" by elements of Γ
- Equivalent to periodic graphs

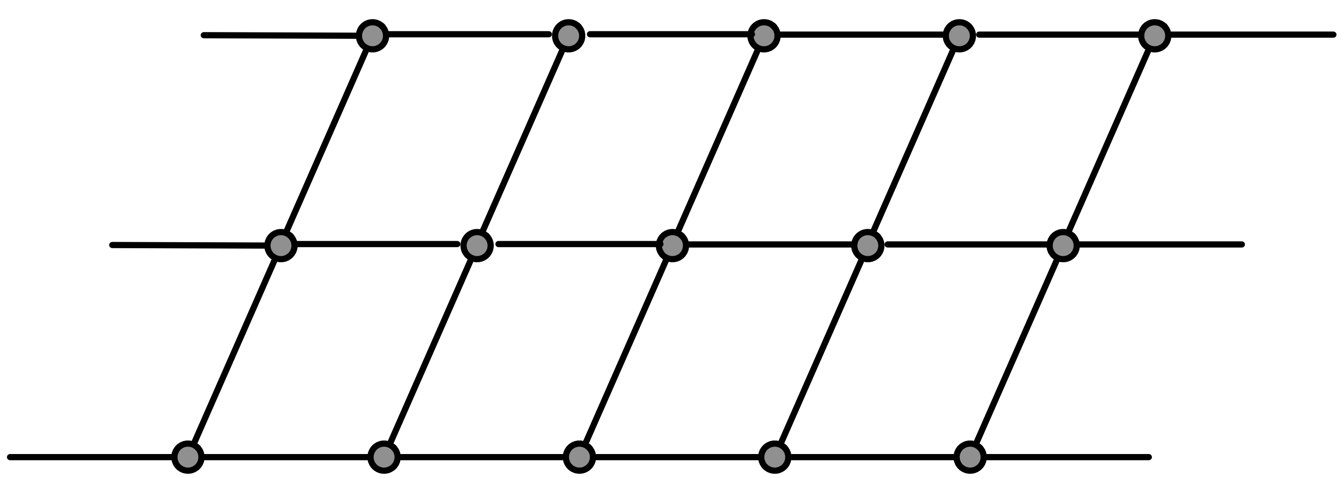


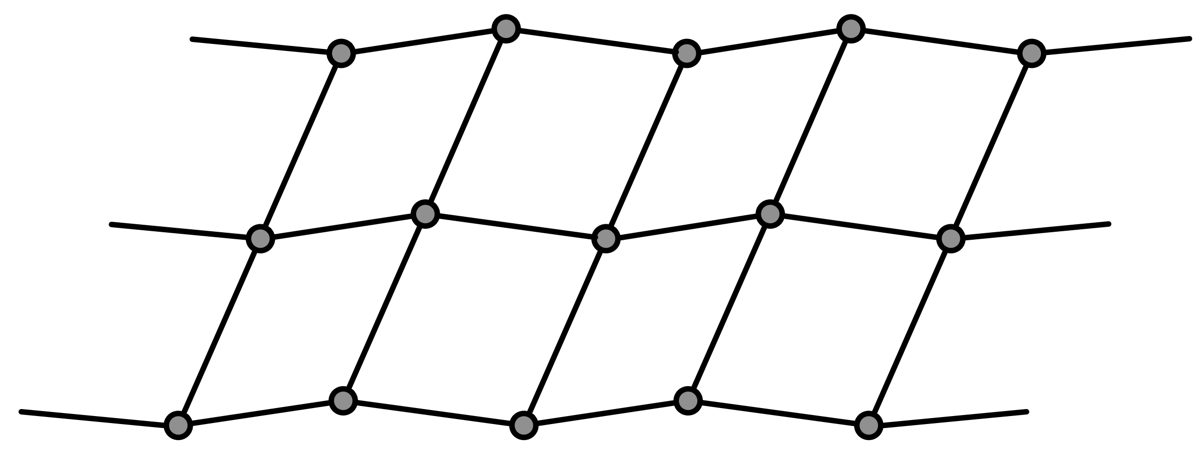
Colored graphs



Symmetry forcing

- Motions *preserve the* Γ*-symmetry!*
- An **essential** feature of the model
- Not allowed:





Algebraic setup

- **Theorem (Borcea-Streinu '10):** The configuration space of a periodic framework is homeomorphic to a finite (real) algebraic variety.

Edges ij $\quad |\mathbf{p}(j) + \Lambda(\gamma(\mathrm{ij})) - \mathbf{p}(i)|^2 = \ell(ij)$

- Λ not regarded as fixed
- Periodic rigidity / flexibility are generic properties

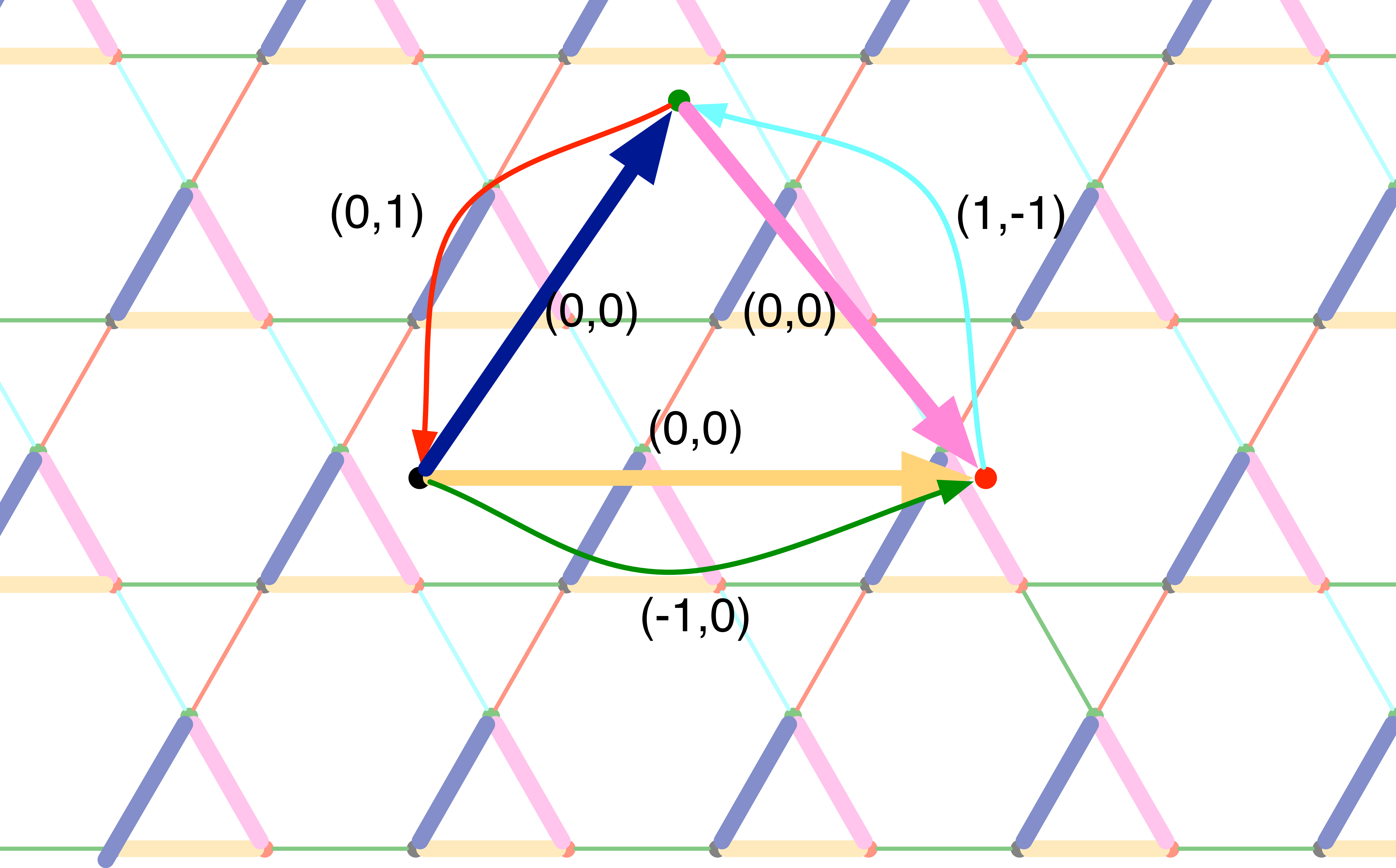
Counting d.o.f.s

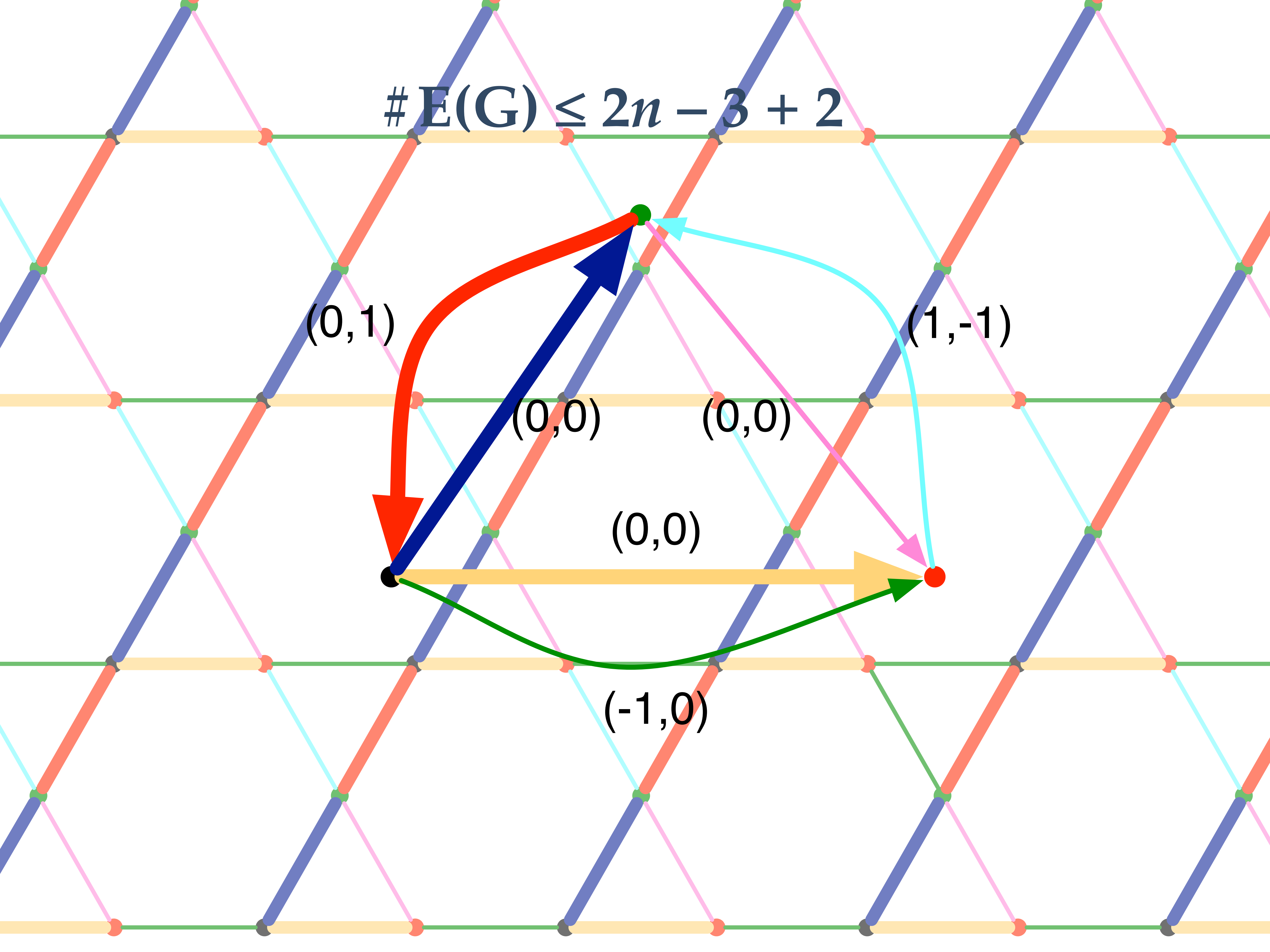
- Variables are *coordinates of the* $\mathbf{p}(i)$ and *entries of a matrix representing* Λ

Rigidity $$\#E(G) \geq dn + d^2 - \dim \mathrm{Euc}(d)$$

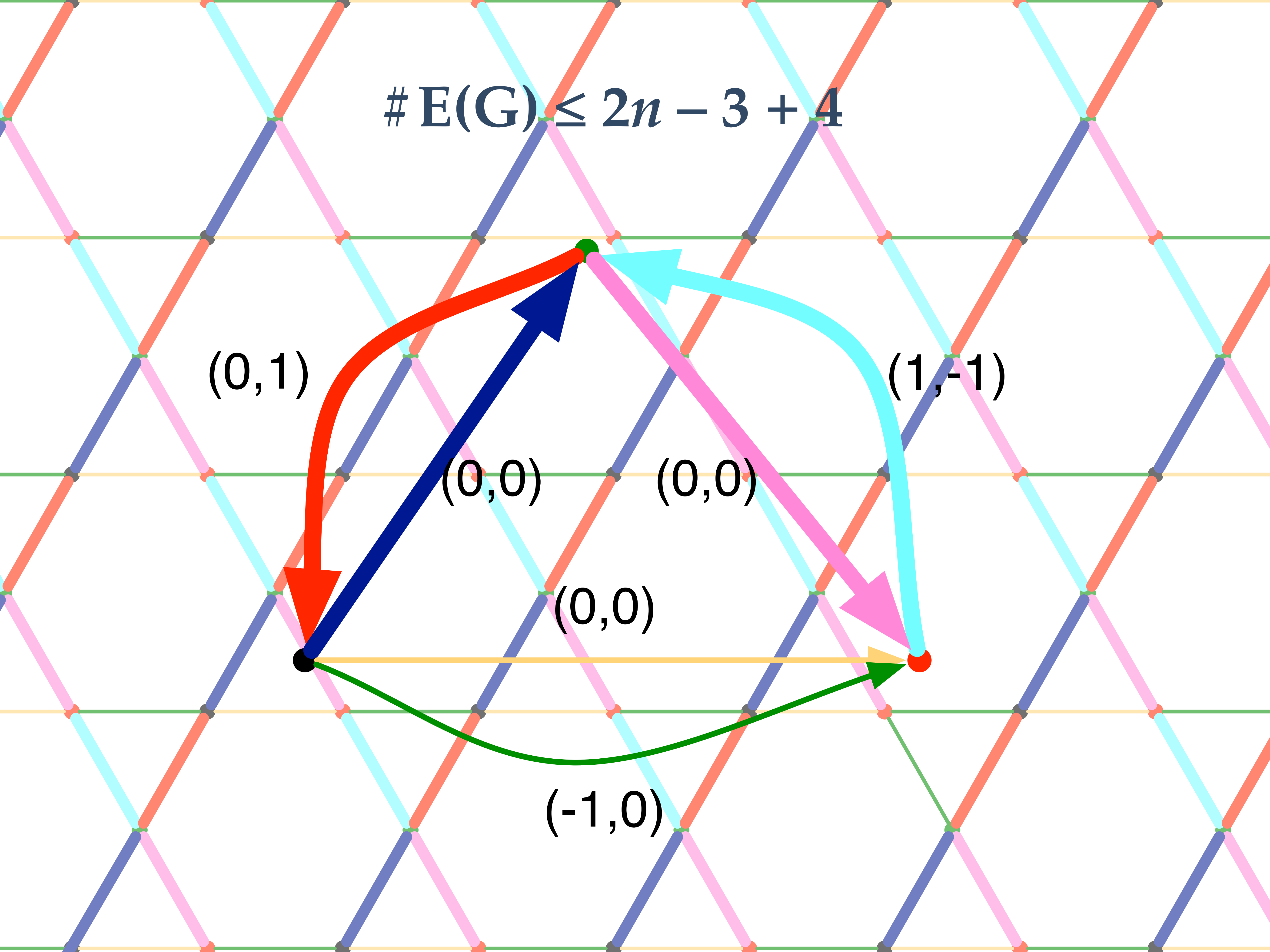
- Now subgraphs are more complicated...

$\# E(G) \leq 2n - 3$




$\#E(G) \leq 2n - 3 + 2$
(0,1)
(0,0)
(0,0)
(1,-1)
(0,0)
(-1,0)

$\# E(G) \leq 2n - 3 + 4$



Counting for periodic frameworks

- For a colored graph, there is a natural map

$$H_1(G,\mathbf{Z}) \rightarrow \Gamma$$

- The *rank* of a colored graph is the rank of this image
- Heuristic for 2d

$$\#E(G) \leq 2(n + \mathrm{rank}(G)) - 3 - 2(c - 1)$$

Laman-like theorem

- **Theorem (Malestein-T '10/13):** For $d = 2$, a colored graph is generically rigid iff, for all subgraphs

$$m \leq 2(n + \mathrm{rank}(G)) - 3 - 2(c - 1)$$

and $2n + 1$ edges.

- **Cor:** 4-regular graphs are ≥ 1 degree of freedom

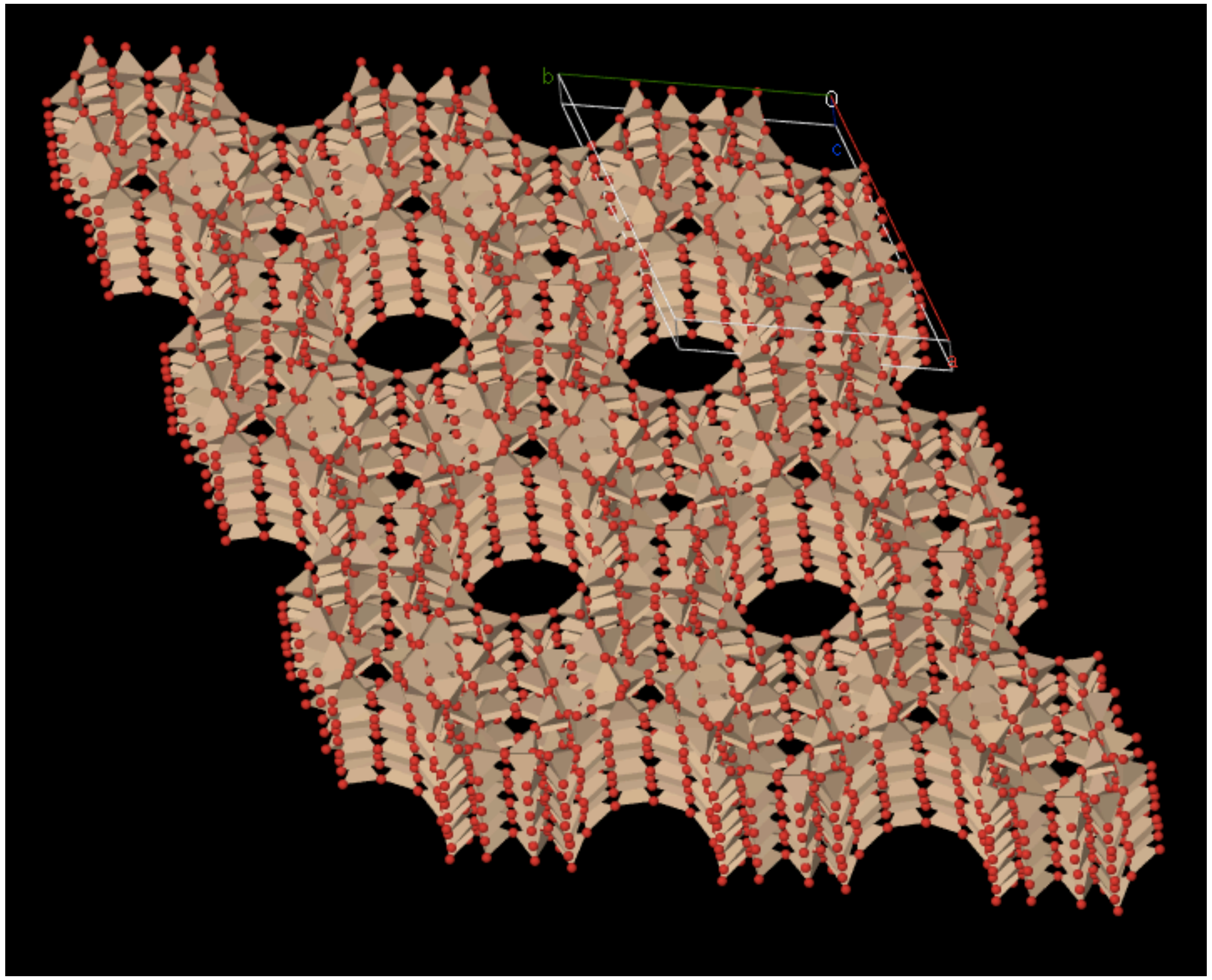
History

- Similar models in engineering and physics for some time
- Whitely ’88 “uncolored” result for fixed-lattice; flexible Borcea-Streinu ’10
- Other counting heuristics from engineering (e.g., Guest-Kangawi ’98)

What happened next

- **Theorem (Malestein-T ′11/13):** Combinatorial characterizations for symmetry groups generated by translations and rotations or a reflection in 2d.
- **Theorem (Jordán-Kasinitzsky-Tanigawa ′13):** Similar statement for all odd-order dihedral groups.

Zeolites again



[Rivin-Treacy-Randall, Hypothetical Zeolite Database]

- Aside from being stuck in 2D, did we answer the question?
- Nobody "told" the zeolite *which* lattice of periods its motion should come from
- What can we say about motions with respect to *any* possible sublattice?

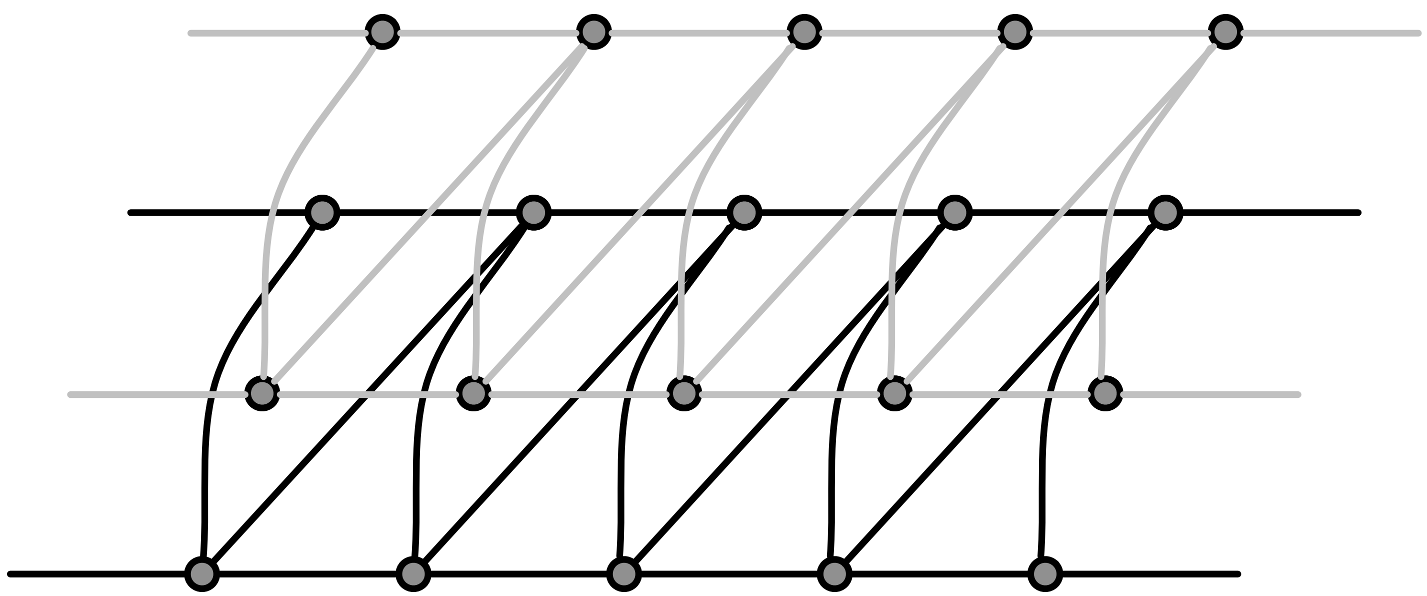
Sub-lattices

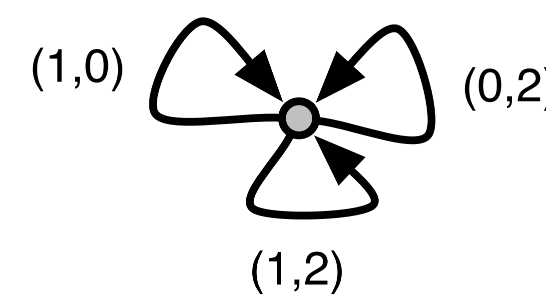
2011

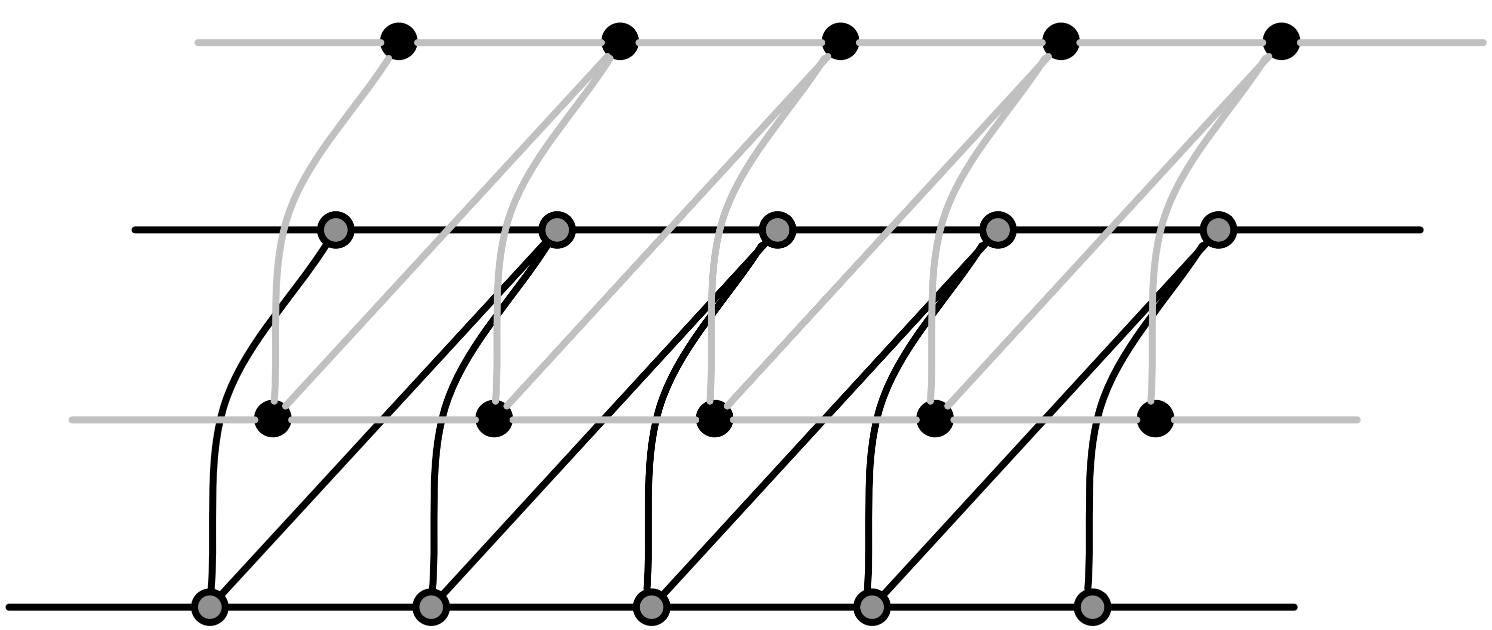
Conjecture 8.2.21. *If a framework* $(\langle G, m\rangle, p)$ *is infinitesimally rigid on the flexible torus, then it is infinitesimally rigid as an incidentally periodic (infinite) framework* $(\widetilde{G}, \widetilde{p})$.

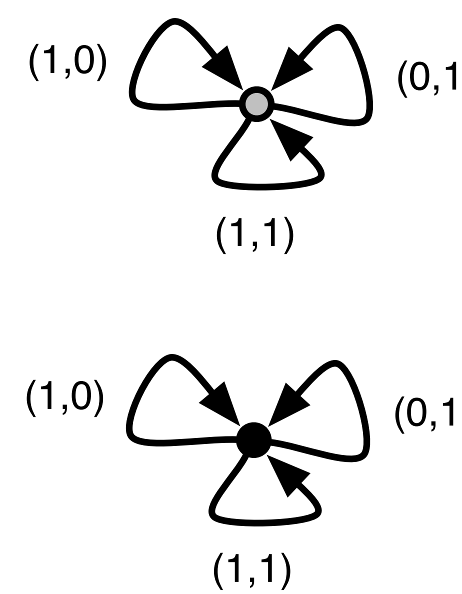
Maybe it doesn't matter...

Sub-lattices

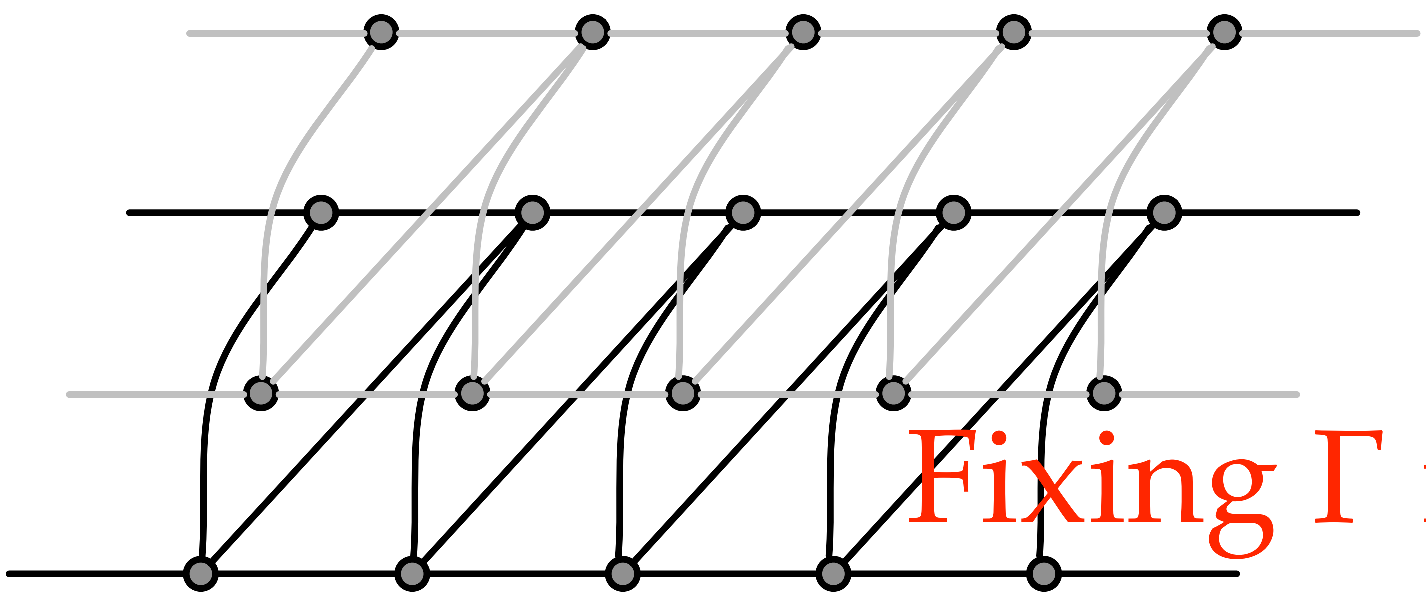


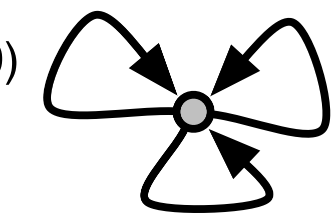




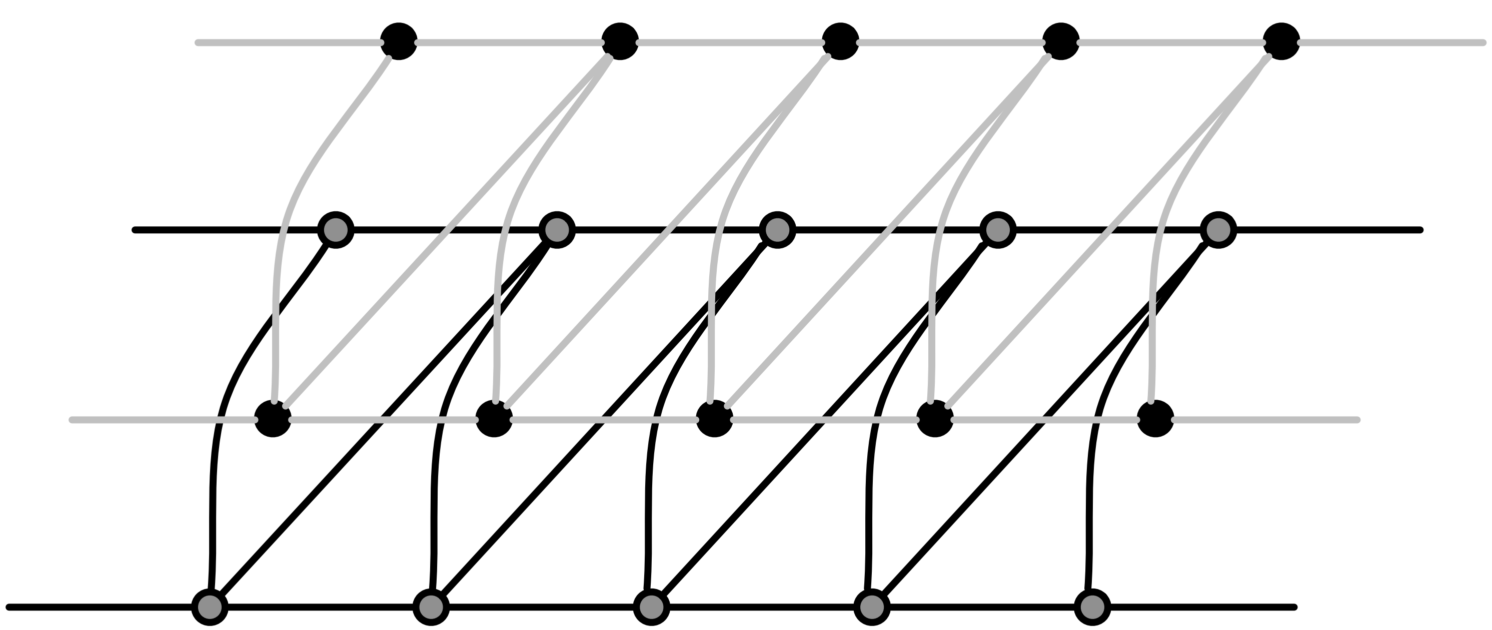


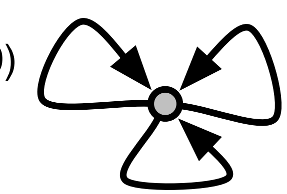
Sub-lattices

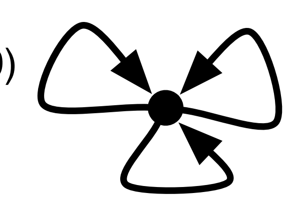




Fixing Γ is a
non-trivial constraint



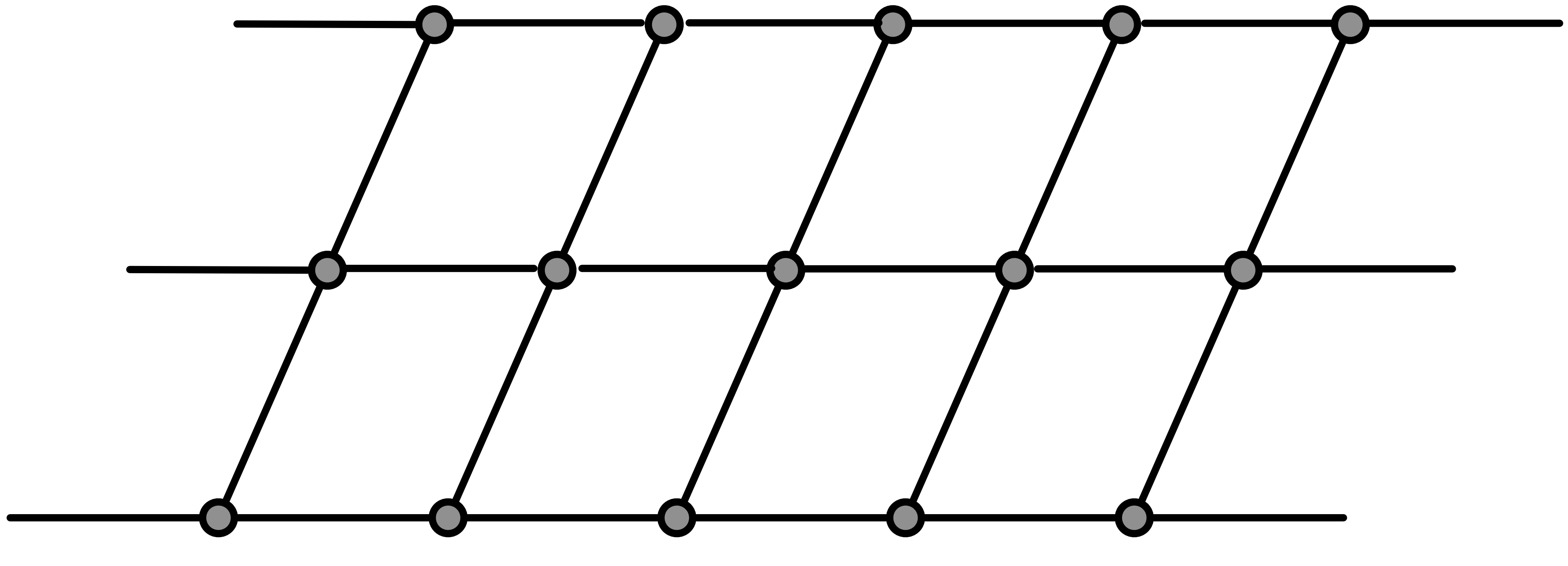


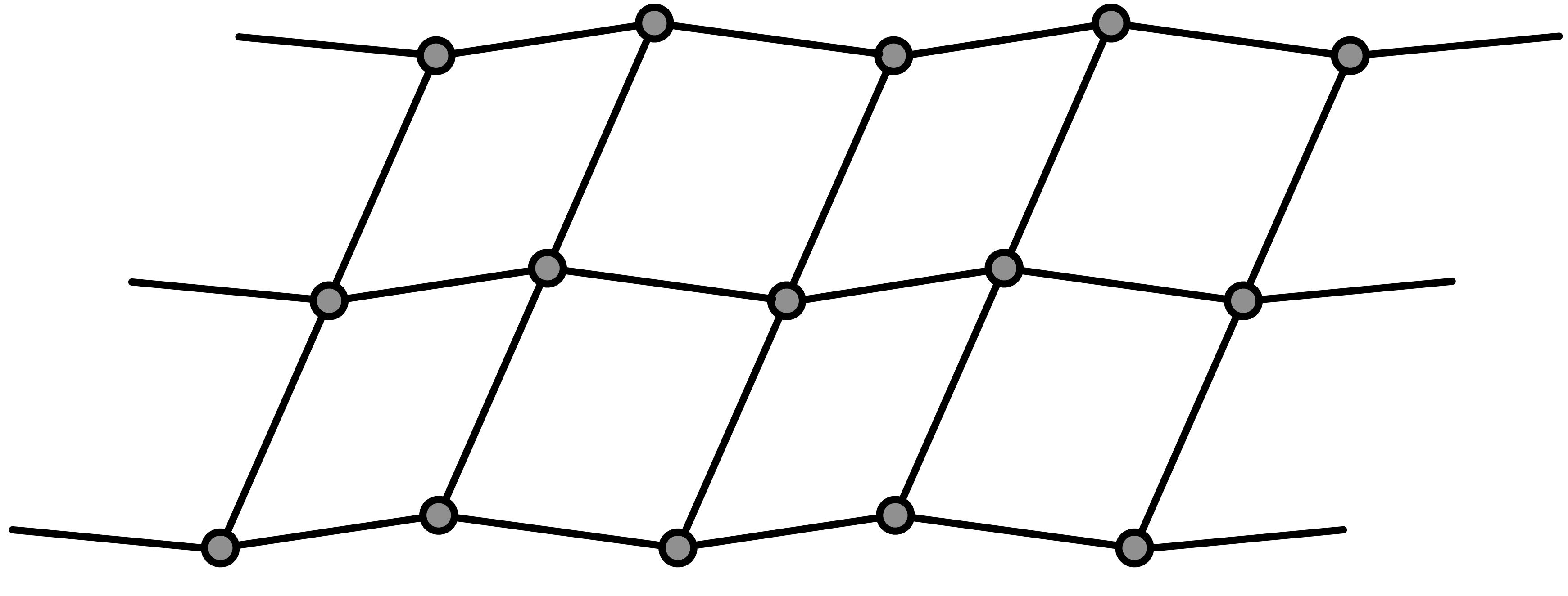


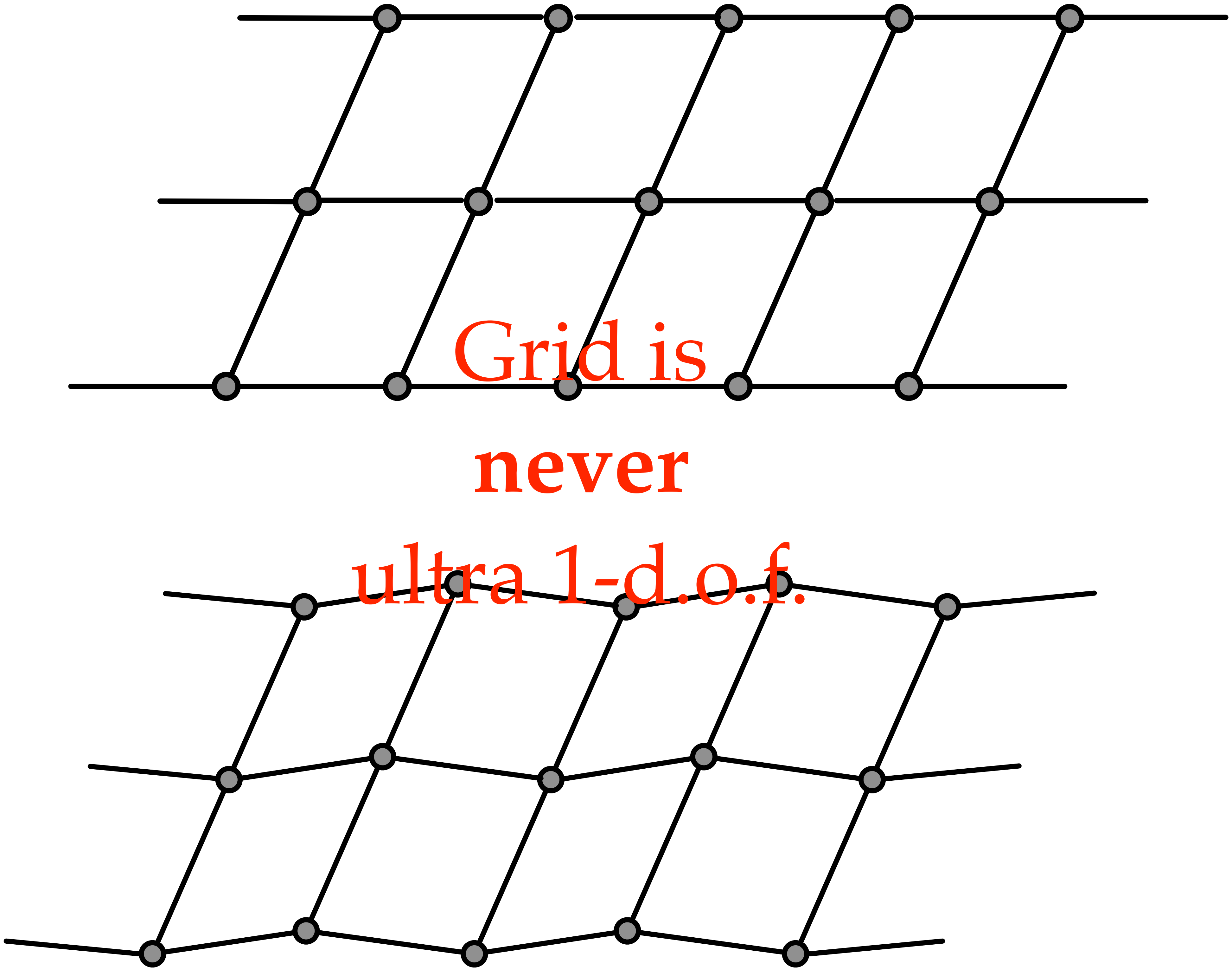
Ultrarigidity

- A periodic framework $G(\mathbf{p}, \Lambda)$, is *ultrarigid* if it is rigid, and remains so *if the periodicity constraint* is relaxed to *any finite index* $\Gamma' < \Gamma$

- A periodic framework is *"ultra 1-d.o.f."* if it remains 1-d.o.f.
  - Ultra 1-d.o.f. is interesting in 2D, since 4-regular colored graphs (like 2D-zeolites) can be.

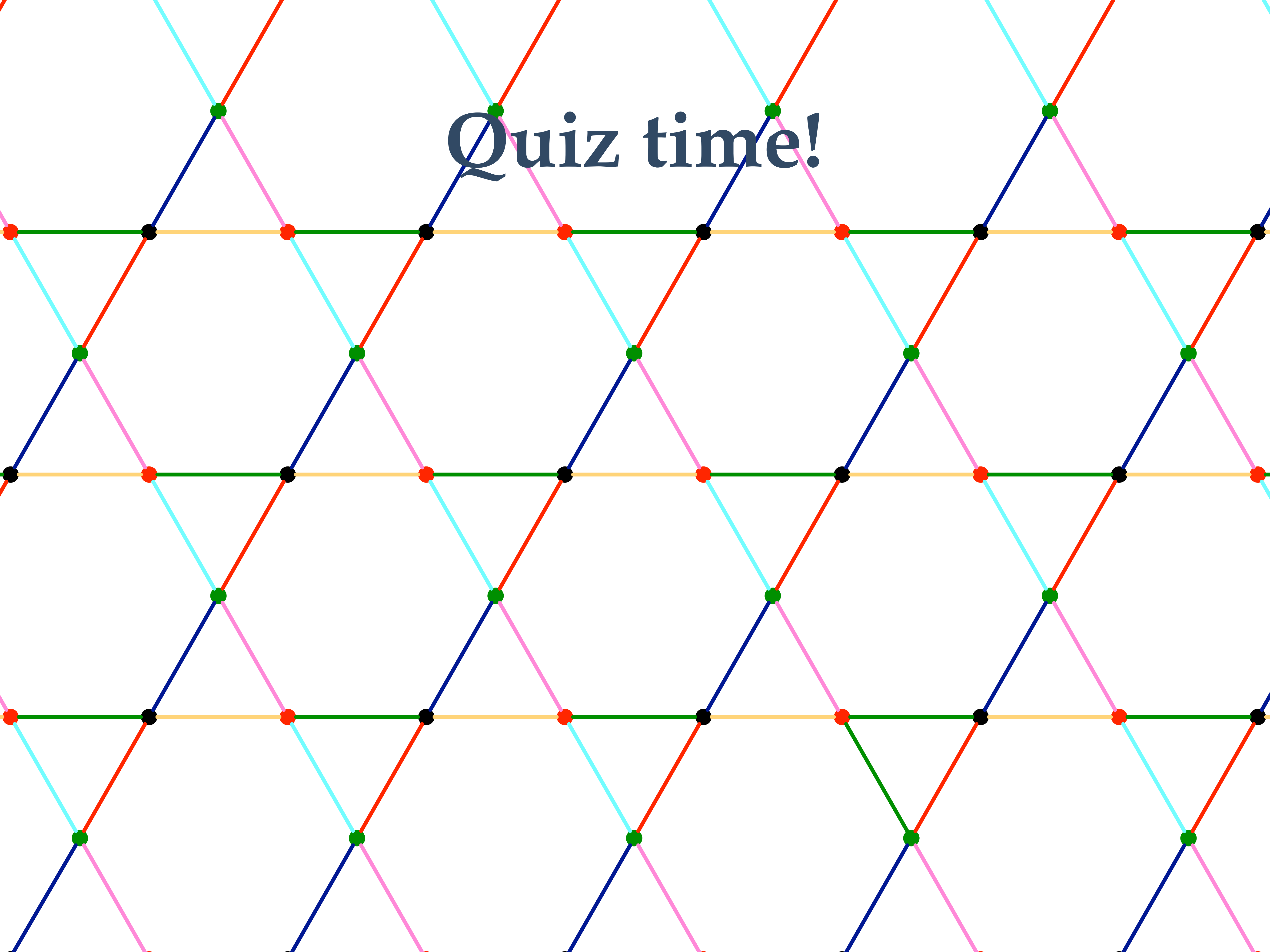


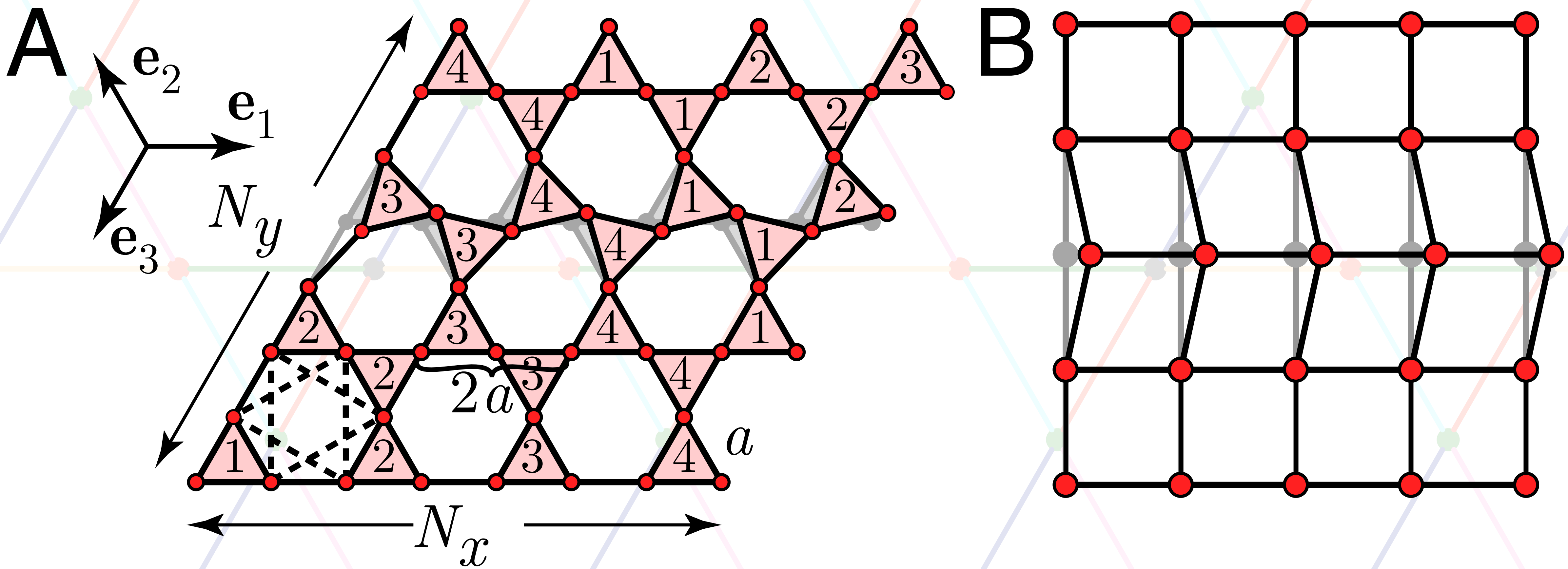




Grid is **never** ultra 1-d.o.f.

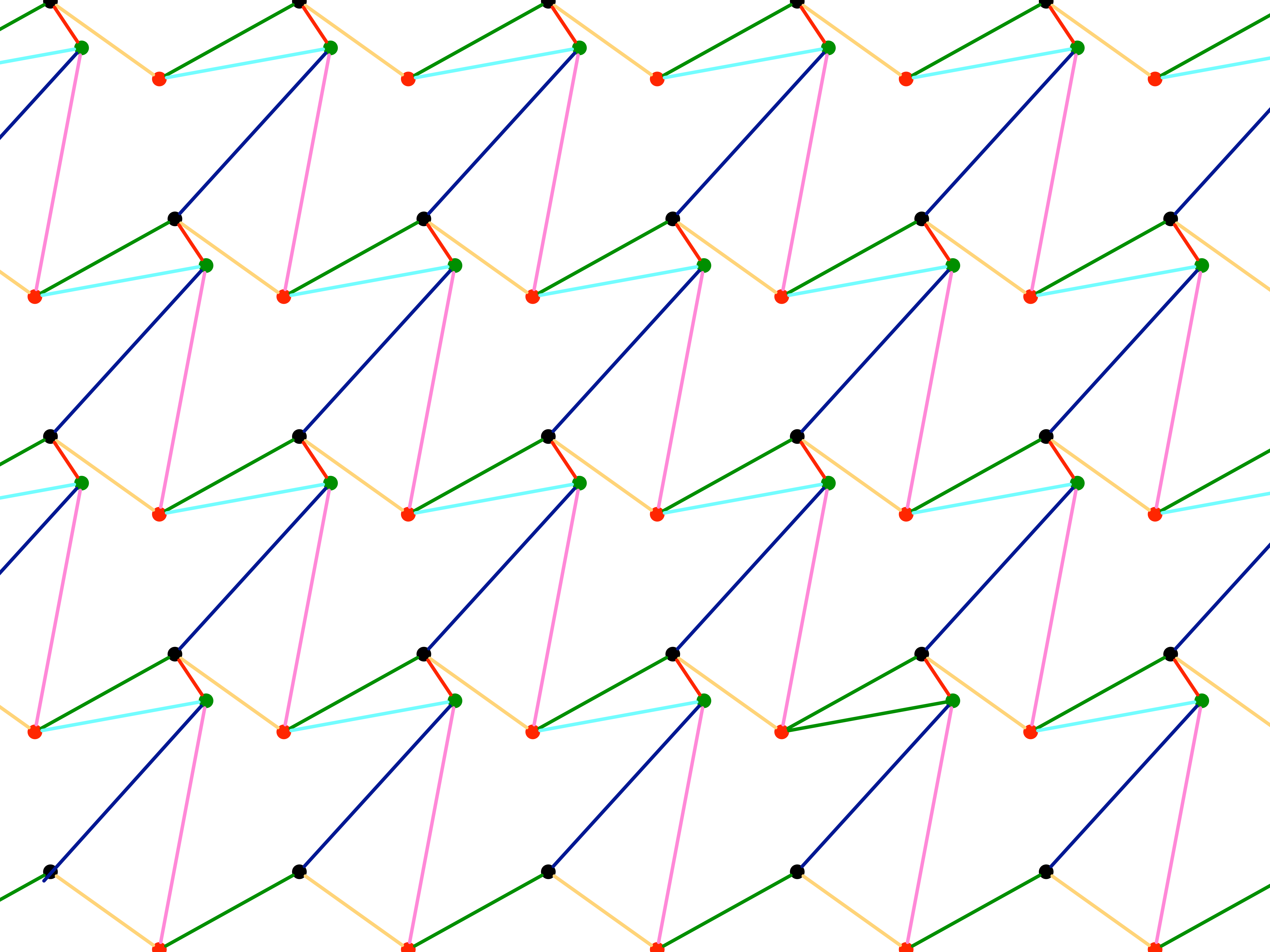
Quiz time!



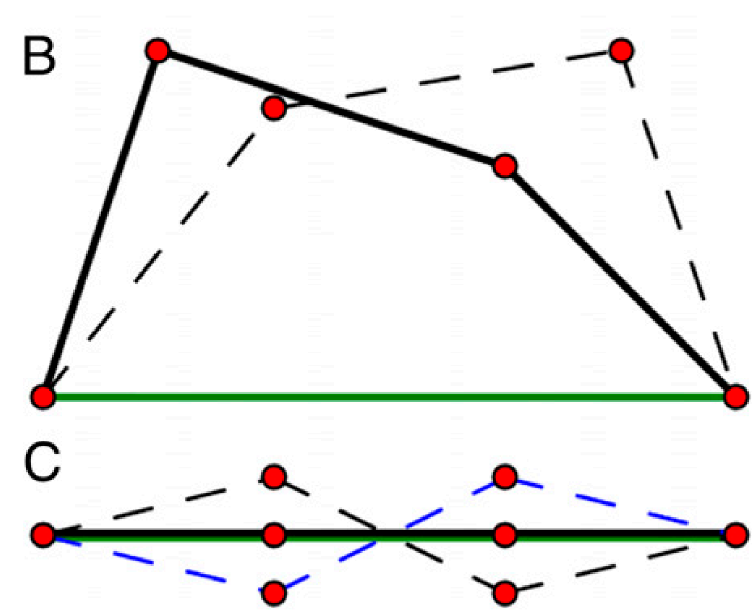


(infinitesimal motions)

Sun et al. ’**12** (PNAS)



(3). However, the character of these unusual phonons is not purely dictated by network topology; rather, any smooth deformation (i.e., a gentle twist) matters.



Vitelli '12 (PNAS)

Characterizing ultrarigidity

- Could start with a rigid periodic framework G($\mathbf{p}$, Λ)
- For each possible finite-index sub-lattice Λ′, lift to a framework G′($\mathbf{p'}$, Λ′)
- Check the rank of the rigidity matrix
- G′($\mathbf{p'}$, Λ′) is *not generic*… can't apply M-T theorem
- This is *not a finite process*

Algebraic characterization

- **Theorem (Malestein-T '13+):** An infinitesimally rigid periodic framework G(p, Λ) is ultrarigid if and only if the system

$$\mathbf{d}(\mathrm{ij}) := \mathrm{p}(j) + \Lambda(\gamma(\mathrm{ij})) - \mathrm{p}(i)$$

$$<-\mathbf{d}(\mathrm{ij}), \mathbf{v}(i)> + <\boldsymbol{\omega}^{\gamma(ij)}\mathbf{d}(\mathrm{ij}), \mathbf{v}(j)> = 0$$

is full rank for all d-tuples of primitive roots of unity $\boldsymbol{\omega}$.

Decidability

- The theorem says that to prove ultrarigidity, it is enough to show that:
  - a *finite* collection of polynomials (minors)
  - has no *torsion points* (solutions in roots of unity) except for all 1's
- Counting/computing torsion points and torsion cosets is well-studied
  - **Fact:** There are algorithms that compute the maximal torsion cosets for varieties defined over number fields (Bombieri-Zannier, others)
  - **Cor:** Ultrarigidity is decidable

An algorithm

- **Theorem (Malestein-T '13+):** There is an effective constant N depending only on #V(G) and the max. 1-norm of the $\gamma(ij)$ s.t., if there are no torsion points in roots of unity of order $\leq N$, then there are none.
- **Corollary:** Can check ultrarigidity with a very simple algorithm: try all potential bad d-tuples of roots of unity.

Summary

- Symmetry-forcing has brought interesting classes of infinite frameworks within the reach of combinatorial techniques.
- In 2d, we have good characterizations in a lot of cases.
- If we don't pick the periodicity lattice in advance, the question of rigidity is still decidable

Questions

- More combinatorial characterizations of ultrarigidity?
- Nicer geometric conditions?
- How generic of a property is ultrarigidity?